Bilinear characterizations of companion matrices

Abstract: Companion matrices of the second type are characterized by properties that involve bilinear maps.

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1 Introduction

Let $K$ be a field. The matrix


is the second companion matrix or (in the terminology of [1]) the companion matrix of the second type associated with

$$p = [p_0, p_1, \ldots, p_{n-1}]^T \in K^n.$$ 

It is well known (see e.g. [5] for references) that companion matrices are important in linear algebra, numerical analysis and applications, e.g. in systems and control theory and signal processing. In this paper we focus on bilinear properties of companion matrices that play a role in the single-input case of sensor-only fault detection and identification [4], [2].

Let $A \in K^{n \times n}$, and $b = [b_0, b_1, \ldots, b_{n-1}]^T \in K^n$, $g \in K^n$. Set $e_{n-1} = [0, \ldots, 0, 1]^T \in K^n$ such that $F_p e_{n-1} = p$. The maps

$$h : K^n \times K^n \to K^n, \ (b, g) \mapsto h(A; b, g) = [b, A, \ldots, A^{n-2}b]g,$$

and

$$u : K^n \times K^n \to K^n, \ (b, g) \mapsto u(A; b, g) = \begin{bmatrix} e_{n-1}^T(b_0I_n + b_1A + b_2A^2 + \cdots + b_{n-1}A^{n-1}) \\ e_{n-1}^T(b_1I_n + b_2A + \cdots + b_{n-1}A^{n-2}) \\ \vdots \\ e_{n-1}^T(b_{n-2}I_n + b_{n-1}A) \\ e_{n-1}^T(b_{n-1}I_n) \end{bmatrix} g$$

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We shall use the following notation. The vectors
\[ e_0 = [1, 0, \ldots, 0]^T, \ldots, e_{n-1} = [0, \ldots, 0, 1]^T, \]
are the unit vectors of \( K^n \). The characteristic polynomial of a matrix \( A \in K^{n \times n} \) will be denoted by \( \chi_A(z) \). If \( g \in K^n \) then the Krylov matrix
\[ R(A, g) = [g, Ag, \ldots, A^{n-1}g] \in K^{n \times n} \]
is the reachability matrix (see e.g. [9]) of the pair \( (A, g) \). Note that
\[ R(A, b)g = h(A; b, g). \]

We recall [9] that \( A \) is similar to a companion matrix if and only if \( R(A, g) \) has full rank for some \( g \in K^n \). With a vector \( b = [b_0, b_1, \ldots, b_{n-1}]^T \in K^n \) we associate the polynomial \( b(z) = b_0 + b_1z + \cdots + b_{n-1}z^{n-1} \). Thus \( b(A) = \sum_{i=0}^{n-1} b_i A^i \). In particular, \( e_0(z) = 1, \ldots, e_{n-1}(z) = z^{n-1} \). Moreover, if \( A \in K^{n \times n} \) then
\[ R(A, g)b = b(A)g. \]  
(2.1)
The following lemma [6] characterizes companion matrices in terms of reachability matrices. To make our note self-contained we include a proof.

**Lemma 2.1.** For a matrix \( A \in K^{n \times n} \) the following statements are equivalent.

(i) \( A = F_p \) for some \( p \in K^n \).
(ii) \( R(A, g) = g(A) \) for all \( g \in K^n \).
(iii) \( R(A, e_0) = e_0(A) = I_n \).

**Proof.** It is obvious that \( A \) is a companion matrix of the form (1.1) if and only if
\[ A[e_0, e_1, \ldots, e_{n-2}] = [e_1, e_2, \ldots, e_{n-1}]. \]  
(2.2)

(i) ⇒ (ii): We have to show that \( R(F_p, g) = g(F_p) \) holds for all \( g = \sum_{i=0}^{n-1} g_i e_i \). From
\[ e_i = F_p^i e_0, \quad i = 0, \ldots, n-1, \]  
(2.3)
follows \( R(F_p, e_0) = I_n \). Therefore
\[ R(F_p, e_i) = F_p^i R(F_p, e_0) \]
is implied \( R(F_p, e_i) = F_p^i \). Hence
\[ R(F_p, g) = \sum_{i=0}^{n-1} g_i R(F_p, e_i) = \sum_{i=0}^{n-1} g_i F_p^i = g(F_p). \]
The implication (ii) ⇒ (iii) is obvious.

(iii) ⇒ (i): From \( R(A, e_0) = I_n \) follows (2.2). Therefore, \( A \) is a companion matrix.

It was shown in [3] and [4, Proposition A.2] that matrices in second companion form satisfy an identity with “curiously commuting vectors”, stated in (2.4) below. We note that the identity (2.4) is equivalent to the symmetry of the map \( h(A; b, g) \).

### 2 The main results

We shall use the following notation. The vectors
\[ e_0 = [1, 0, \ldots, 0]^T, \ldots, e_{n-1} = [0, \ldots, 0, 1]^T, \]
are bilinear. It is the purpose of this note to show that a matrix \( A \) is a second companion matrix if and only if the maps \( h(A; b, g) \) and \( u(A; b, g) \), respectively, are symmetric. These results will be proved in Section 2. In Section 3 we deal with block companion matrices and we describe extensions of results of Section 2 to matrix polynomials.
Theorem 2.2. Let \( A \in K^{n \times n} \). We have \( A = F_p \) for some \( p \in K^n \) if and only if
\[
[g, Ag, \ldots, A^{n-1}g]b = [b, Ab, \ldots, A^{n-1}b]g.
\]

(2.4)

Proof. According to Lemma 2.1 we have \( A = F_p \) if and only if \( b(A) = R(A, b) \) for all \( b \in K^n \), that is,
\[
b(A)g = R(A, b)g \text{ for all } b, g \in K^n.
\]

(2.5)

Then (2.1) implies that (2.5) is equivalent to \( R(A, b)g = R(A, g)b \) for all \( b, g \in K^n \). \( \square \)

We now deal with the map \( u(A; b, g) \).

Theorem 2.3. Let \( A \in K^{n \times n} \). The following statements are equivalent.

(i) \( A = F_p \) for some \( p \in K^n \).

(ii) We have
\[
\begin{bmatrix}
e^T_{n-1}A^{n-1} \\
e^T_{n-2}A^{n-2} \\
\vdots \\
e^T_{n-1}A \\
e^T_{n-1}
\end{bmatrix} =
\begin{bmatrix}
1 & e^T_{n-1}Ae_{n-1} & \cdots & e^T_{n-1}A^{n-1}e_{n-1} \\
& 1 & \cdots & e^T_{n-1}A^{n-2}e_{n-1} \\
& & \ddots & \vdots \\
& & & 1 \\
& & & e^T_{n-1}Ae_{n-1} \\
& & & 1
\end{bmatrix}.
\]

(2.6)

(iii) The bilinear map \( u(A; b, g) \) in (1.2) satisfies \( u(A; b, g) = u(A; g, b) \) for all \( b, g \in K^n \).

Proof. Define
\[
L(A, g) =
\begin{bmatrix}
e^T_{n-1}g & e^T_{n-1}Ag & \cdots & e^T_{n-1}A^{n-1}g \\
e^T_{n-1}g & e^T_{n-1}Ag & \cdots & e^T_{n-1}A^{n-2}g \\
& \ddots & \ddots & \vdots \\
& & \ddots & e^T_{n-1}g \\
& & & e^T_{n-1}g
\end{bmatrix}
\]

and
\[
Q(A, g) =
\begin{bmatrix}
e^T_{n-1}(g_0I_n + g_1A + g_2A^2 + \cdots + g_{n-1}A^{n-1}) \\
e^T_{n-1}(g_1I_n + g_2A + \cdots + g_{n-1}A^{n-2}) \\
& \vdots \\
e^T_{n-1}(g_{n-2}I_n + g_{n-1}A) \\
e^T_{n-1}(g_{n-1}I_n)
\end{bmatrix}.
\]

Then \( u(A; b, g) \) in (1.2) can be written as
\[
u(A; b, g) = L(A, g)b.
\]

We have \( u(A; g, b) = Q(A, g)b \). Hence (iii) holds if and only if
\[
L(A, g) = Q(A, g) \text{ for all } g \in K^n.
\]

(2.7)

The maps \( g \mapsto L(A, g) \) and \( g \mapsto Q(A, g) \) are linear. Hence (2.7) is equivalent to
\[
L(A, e_i) = Q(A, e_i), \ \ i = 0, \ldots, n - 1.
\]

(2.8)

In the case \( i = n - 1 \) the corresponding matrices are
\[
Q(A, e_{n-1}) =
\begin{bmatrix}
e^T_{n-1}A^{n-1} \\
e^T_{n-1}A^{n-2} \\
& \ddots \\
e^T_{n-1}A \\
e^T_{n-1}
\end{bmatrix}.
\]
and

\[
L(A, e_{n-1}) = \begin{bmatrix}
1 & e_{n-1}^T A e_{n-1} & \ldots & e_{n-1}^T A^{n-2} e_{n-1} \\
& 1 & e_{n-1}^T A e_{n-1} & \ldots & e_{n-1}^T A^{n-2} e_{n-1} \\
& & \ddots & \ddots & \vdots \\
& & & 1 & e_{n-1}^T A e_{n-1}
\end{bmatrix}.
\]

(i) ⇒ (iii): Suppose \( A = F_p \). Then \( A^r e_i = A^{s+i} e_0 \) if \( 0 \leq i \leq n - 1 \) and \( s \in \mathbb{N}_0 \). We have

\[
L(A, g)e_r = [e_{n-1}^T A g, \ldots, e_{n-1}^T A g, 0, \ldots, 0]^T.
\]

Thus

\[
L(A, e_i)e_r = [e_{n-1}^T A^{i+r} e_0, \ldots, e_{n-1}^T A^{i-1} e_0, e_{n-1}^T A^i e_0, 0, \ldots, 0]^T.
\]

From

\[
Q(A, e_i) = \begin{bmatrix}
e_{n-1}^T A^r \\
\vdots \\
e_{n-1}^T A \\
e_{n-1} \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

we obtain \( Q(A, e_i)e_r = L(A, e_i)e_r, r = 0, \ldots, n - 1 \). Hence (2.8) is satisfied, which is equivalent to (iii).

(iii) ⇒ (ii): We have seen that (iii) is equivalent to (2.8). Choosing \( i = n - 1 \) in (2.8) yields \( Q(A, e_{n-1}) = L(A, e_{n-1}) \), that is (2.6).

(ii) ⇒ (i): Suppose (2.6) holds. Let

\[
A^k = [a^{(k)}_{ij}]_{i,j=0}^{n-1}, \quad k = 0, 1, \ldots, n - 1.
\]

We show that the rows of \( A \) are the rows of a companion matrix. The proof is by induction. The induction hypothesis is

\[
e_{n-v}^T A = e_{n-v-1}^T + a_{n-v,n-1} e_{n-1}^T = [0, \ldots, 0, 1, 0, \ldots, 0, a_{n-v,n-1}], \quad 1 \leq v < n - 1.
\]

From (2.6) we obtain

\[
e_{n-1}^T A = [0, \ldots, 0, 1, e_{n-1}^T A e_{n-1}] = [0, \ldots, 0, 1, 0, a_{n-1,n-1}].
\]

Hence (2.9) is satisfied for \( v = 1 \). Assume that (2.9) is valid for \( v = 1, \ldots, k \). Then \( A = \begin{bmatrix} W_k \\ H_k \end{bmatrix} \) with

\[
\begin{bmatrix} e_{n-k}^T \\ \vdots \\ e_{n-2}^T \\ e_{n-1}^T \end{bmatrix} A = H_k = \begin{bmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0 & a_{n-k,n-1} \\
\vdots & & & \ddots & & & \vdots & \vdots \\
0 & \ldots & 0 & 0 & 1 & 0 & a_{n-2,n-1} \\
0 & \ldots & 0 & 0 & 0 & 1 & a_{n-1,n-1} & a_{n-1,n-1}
\end{bmatrix}_{k \times n}.
\]

Set

\[
e_{n-1-k} A = [\tilde{a}_0, \ldots, \tilde{a}_{n-2}, \tilde{a}_{n-1}].
\]
From (2.6) and (2.10) follows
\[ e_{n-1}^T A^{k+1} = (e_{n-1}^T A^k) A = [0, \ldots, 0, 1, a_{n-1,n-1}, a_{n-1,n-1}^{(2)}, \ldots, a_{n-1,n-1}^{(k)}] \]
\[
\begin{bmatrix}
\tilde{a}_0, \ldots, \tilde{a}_{n-k-2}, \tilde{a}_{n-k-1} & \ldots & \tilde{a}_{n-2}, \tilde{a}_{n-1} \\
0 & 0 & 1 & 0 & \ldots & 0 & a_{n-2,n-1}^{(k)} \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
0 & \ldots & 0 & 1 & \ldots & 0 & a_{n-2,n-1} \\
0 & \ldots & 0 & 0 & \ldots & 0 & 1 & a_{n-1,n-1}^{(k+1)}
\end{bmatrix}
\]
\[
[\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_{n-k-2}, \tilde{a}_{n-k-1} + a_{n-1,n-1}, \tilde{a}_{n-k} + a_{n-1,n-1}^{(2)}, \ldots, \tilde{a}_{n-2} + a_{n-1,n-1}^{(k)}, a_{n-1,n-1}^{(k+1)}].
\]

The assumption (2.6) implies
\[ e_{n-1}^T A^{k+1} = [0, \ldots, 0, 1, a_{n-1,n-1}, a_{n-1,n-1}^{(2)}, \ldots, a_{n-1,n-1}^{(k)}] \]

Hence \([\tilde{a}_0, \ldots, \tilde{a}_{n-k-1}] = [0, \ldots, 0], and \ a_{n-k-2} = 1. Moreover
\[
[\tilde{a}_{n-k-1} + a_{n-1,n-1}, \tilde{a}_{n-k} + a_{n-1,n-1}^{(2)}, \ldots, \tilde{a}_{n-2} + a_{n-1,n-1}^{(k)}] = [a_{n-1,n-1}, a_{n-1,n-1}^{(2)}, \ldots, a_{n-1,n-1}^{(k)}]
\]
yields \([\tilde{a}_{n-k-1}, \tilde{a}_{n-k}, \ldots, \tilde{a}_{n-2}] = [0, \ldots, 0]. This proves (2.9) in the case \(v = k + 1, and we obtain A = H_n = F_a \]
with \(a = [a_{0,0}, \ldots, a_{n-1,n-1}]^T.\)

It was proved in [4] that the map \((g, b) \mapsto u(A; g, b)\) is symmetric if \(A = F_p). The proof of [4, Proposition A.5] with the identities
\[
e_{n-1}^T \left( \sum_{j=k}^{n-1} b_j A^{j-k} \right) g = e_{n-1}^T \left( \sum_{j=k}^{n-1} g_j A^{j-k} \right) b, \quad k = 0, 1, \ldots, n - 1,
\]
is rather involved. We remark that
\[
L(F_p, e_{n-1}) = \begin{bmatrix}
1 & p_{n-1} & p_{n-2} & \ldots & p_1 \\
1 & p_{n-1} & \ldots & p_2 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & & & & 1
\end{bmatrix}
\]

The “crossover” identity (2.12) below appears in [4, Proposition A.3].

**Theorem 2.4.** Suppose \(A \in K^{n \times n}\) and
\[
\chi_A(z) = z^n - \sum_{i=0}^{n-1} p_i z^i = z^n - p(z). \quad (2.11)
\]

Let \(g = [g_0, \ldots, g_{n-1}]^T \in K^n, b = [b_0, \ldots, b_{n-1}]^T \in K^n, and g_n, b_n \in K. The following statements are equivalent.

(i) The identity
\[
\sum_{k=0}^{n} A^k g_k (b + b_n p) = \sum_{k=0}^{n} A^k b_k (g + g_n p) \quad (2.12)
\]
holds for all \(b, b_n, g, g_n.\)

(ii) The matrix \(A\) is a second companion matrix such that \(A = F_p.\)

**Proof.** (i) \(\Rightarrow\) (ii): Note that (2.12) is equivalent to
\[
R(A, b) g + b_n R(A, p) g + g_n A^n b = R(A, g) b + g_n R(A, p) b + b_n A^n g. \quad (2.13)
\]
If we choose \(b_n = g_n = 0\) in (2.13) then we obtain \(R(A, b) g = R(A, g) b. Hence A = F_q for some q \in K^n (by Theorem 2.2). Then (2.11) implies \(q = p.\)

(ii) \(\Rightarrow\) (i): Assume \(A = F_p. Then R(A, b) g = R(A, g) b\) for all \(b, g \in K^n. We also have \(A^n = p(A)\) and therefore \(A^n g = p(A) g = R(A, p) g. Hence (2.13) holds for all b, g, b_n, g_n.\)
3 Block companion matrices

In this section we extend results of the preceding section to block companion matrices. Let \( P_i \in K^{nt}, i = 0, \ldots, n - 1 \), be the entries of the block matrix

\[
P = \begin{bmatrix}
P_0 \\
\vdots \\
P_{n-1}
\end{bmatrix}_{nt \times nt}
\]

and the coefficients of the monic polynomial

\[
\tilde{p}(z) = z^nI_t - (z^{n-1}P_{n-1} + \cdots + P_0).
\]

A linearization \([7]\) of \( \tilde{p}(z) \) gives rise to the block companion matrix of the second type

\[
F_p = \begin{bmatrix}
0 & \cdots & P_0 \\
I_t & 0 & \cdots & P_1 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & I_t & P_{n-1}
\end{bmatrix} \in K^{nt \times nt}.
\]

Our main tool is the Kronecker product. Let \( S = (s_{ij}) \in K^{t \times m} \) and \( T \in K^{pr \times t} \) then \( S \otimes T = (s_{ij}T) \in K^{tp \times mr} \). Define

\[
E_0 = e_0 \otimes I_t = [I_t, 0, \ldots, 0]^T, \ldots, E_{n-1} = e_{n-1} \otimes I_t = [0, \ldots, 0, I_t]^T.
\]

To the block matrix

\[
B = \begin{bmatrix}
B_0 \\
B_1 \\
\vdots \\
B_{n-1}
\end{bmatrix} = \sum_{i=0}^{n-1} E_i B_i \in K^{nt \times m}
\]

with \( B_j \in K^{t \times m}, j = 0, \ldots, n - 1 \), we associate the matrix polynomial

\[
B(z) = B_0 + zB_1 + \cdots + z^{n-1}B_{n-1} \in K^{t \times m}[z].
\]

Let \( A \in K^{nt \times nt} \). We define

\[
B(A) = (I_n \otimes B_0) + (I_n \otimes B_1) + \cdots + A^{n-1} (I_n \otimes B_{n-1}) \in K^{nt \times nm}.
\]  

(3.1)

In (3.1) we have an example of an operator substitution (see [8]). The matrix

\[
R(A, B) = [B, AB, \ldots, A^{n-1}B] \in K^{nt \times nm}
\]

is the \( nt \times t \) matrices

\[
B = \begin{bmatrix}
B_0 \\
B_1 \\
\vdots \\
B_{n-1}
\end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix}
G_0 \\
G_1 \\
\vdots \\
G_{n-1}
\end{bmatrix}
\]

are blockwise commuting if \( B_i G_j = G_j B_i, i, j = 0, \ldots, n - 1 \). Adapting arguments of the preceding section to Kronecker products we prove the following result.

**Theorem 3.1.** Let \( A \in K^{nt \times nt} \). The following statements are equivalent.
(i) $A = F_P$ for some $P \in K^{nt}$.
(ii) $R(A, E_0) = I_{nt}$.
(iii) $R(A, G) = G(A)$ for all $G \in K^{nt \times m}$.
(iv) The identity

\[ [B, AB, \ldots, A^{n-1}B]G = [G, AG, \ldots, A^{n-1}G]B \]  

holds for all blockwise commuting matrices $B, G \in K^{nt \times t}$.

**Proof.** We proceed along the lines suggested by a referee.

(ii) $\Rightarrow$ (i): The implication is obvious.
(i) $\Rightarrow$ (iii): For $A \in K^{nt \times nt}$ we have

\[ R(A, G) = \sum_{i=0}^{n-1} R(A, E_i)(I_n \otimes G_i). \]

If $A = F_P$ then $R(A, E_i) = F^i_P$, and we obtain $R(A, G) = G(A)$.

(iii) $\Rightarrow$ (iv): If $B$ and $G$ are blockwise commuting then

\[ B(A)G = \sum_{i,j=0}^{n-1} A^i(I_n \otimes G_i)(I_n \otimes B_j) = G(A)B. \]

Thus (iii) implies $R(A, B)G = R(A, G)B$, which is the target identity (3.2).

(iv) $\Rightarrow$ (ii): Choose $G = E_0$ in (3.2). Then it follows from $R(A, B)E_0 = B$ that $R(A, E_0)B = B$ holds for all $B \in K^{nt \times t}$. Hence $R(A, E_0) = I_{nt}$.

\[ \square \]

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**References**