## Contents

Preface xi

**Chapter 1 Finite State Space, a Trial Run**

1.1 An Extrinsic Perspective 1  
1.1.1. The Structure of \( \Theta_n \) 1  
1.1.2. Back to \( M_1(\mathbb{Z}_n) \) 4  
1.2 A More Intrinsic Approach 5  
1.2.1. The Semigroup Structure on \( M_1(\mathbb{Z}_n) \) 5  
1.2.2. Infinitely Divisible Flows 6  
1.2.3. An Intrinsic Description of \( T_{\delta_x}(M_1(\mathbb{Z}_n)) \) 8  
1.2.4. An Intrinsic Approach to (1.1.6) 9  
1.2.5. Exercises 9  
1.3 Vector Fields and Integral Curves on \( M_1(\mathbb{Z}_n) \) 10  
1.3.1. Affine and Translation Invariant Vector Fields 10  
1.3.2. Existence of an Integral Curve 11  
1.3.3. Uniqueness for Affine Vector Fields 12  
1.3.4. The Markov Property and Kolmogorov’s Equations 14  
1.3.5. Exercises 16  
1.4 Pathspace Realization 17  
1.4.1. Kolmogorov’s Approach 18  
1.4.2. Lévy Processes on \( \mathbb{Z}_n \) 21  
1.4.3. Exercises 24  
1.5 Itô’s Idea 26  
1.5.1. Itô’s Construction 26  
1.5.2. Exercises 31  
1.6 Another Approach 32  
1.6.1. Itô’s Approximation Scheme 33  
1.6.2. Exercises 34  

**Chapter 2 Moving to Euclidean Space, the Real Thing** 35  
2.1 Tangent Vectors to \( M_1(\mathbb{R}^n) \) 35  
2.1.1. Differentiable Curves on \( M_1(\mathbb{R}^n) \) 35  
2.1.2. Infinitely Divisible Flows on \( M_1(\mathbb{R}^n) \) 36  
2.1.3. The Tangent Space at \( \delta_x \) 44  
2.1.4. The Tangent Space at General \( \mu \in M_1(\mathbb{R}^n) \) 46  
2.1.5. Exercises 48
Contents

2.2 Vector Fields and Integral Curves on $\mathbf{M}_1(\mathbb{R}^n)$

2.2.1. Existence of Integral Curves

2.2.2. Uniqueness for Affine Vector Fields

2.2.3. The Markov Property and Kolmogorov’s Equations

2.2.4. Exercises

2.3 Pathspace Realization, Preliminary Version

2.3.1. Kolmogorov’s Construction

2.3.2. Path Regularity

2.3.3. Exercises

2.4 The Structure of Lévy Processes on $\mathbb{R}^n$

2.4.1. Construction

2.4.2. Structure

2.4.3. Exercises

Chapter 3 Itô’s Approach in the Euclidean Setting

3.1 Itô’s Basic Construction

3.1.1. Transforming Lévy Processes

3.1.2. Hypotheses and Goals

3.1.3. Important Preliminary Observations

3.1.4. The Proof of Convergence

3.1.5. Verifying the Martingale Property in (G2)

3.1.6. Exercises

3.2 When Does Itô’s Theory Work?

3.2.1. The Diffusion Coefficients

3.2.2. The Lévy Measure

3.2.3. Exercises

3.3 Some Examples to Keep in Mind

3.3.1. The Ornstein–Uhlenbeck Process

3.3.2. Bachelier’s Model

3.3.3. A Geometric Example

3.3.4. Exercises

Chapter 4 Further Considerations

4.1 Continuity, Measurability, and the Markov Property

4.1.1. Continuity and Measurability

4.1.2. The Markov Property

4.1.3. Exercises

4.2 Differentiability

4.2.1. First Derivatives

4.2.2. Second Derivatives and Uniqueness
## Chapter 5 Itô’s Theory of Stochastic Integration

5.1 Brownian Stochastic Integrals
   5.1.1. A Review of the Paley–Wiener Integral 126
   5.1.2. Itô’s Extension 128
   5.1.3. Stopping Stochastic Integrals and a Further Extension 132
   5.1.4. Exercises 134

5.2 Itô’s Integral Applied to Itô’s Construction Method 137
   5.2.1. Existence and Uniqueness 137
   5.2.2. Subordination 142
   5.2.3. Exercises 144

5.3 Itô’s Formula 144
   5.3.1. Exercises 150

## Chapter 6 Applications of Stochastic Integration to Brownian Motion

6.1 Tanaka’s Formula for Local Time
   6.1.1. Tanaka’s Construction 152
   6.1.2. Some Properties of Local Time 156
   6.1.3. Exercises 160

6.2 An Extension of the Cameron–Martin Formula
   6.2.1. Introduction of a Random Drift 161
   6.2.2. An Application to Pinned Brownian Motion 167
   6.2.3. Exercises 171

6.3 Homogeneous Chaos
   6.3.1. Multiple Stochastic Integrals 175
   6.3.2. The Spaces of Homogeneous Chaos 177
   6.3.3. Exercises 181

## Chapter 7 The Kunita–Watanabe Extension

7.1 Doob–Meyer for Continuous Martingales
   7.1.1. Uniqueness 190
   7.1.2. Existence 192
   7.1.3. Exercises 194

7.2 Kunita–Watanabe Stochastic Integration
   7.2.1. The Hilbert Structure of $\mathcal{M}_{10c}(\mathbb{P}; \mathbb{R})$ 196
   7.2.2. The Kunita–Watanabe Stochastic Integral 198
   7.2.3. General Itô’s Formula 201
   7.2.4. Exercises 203

7.3 Representations of Continuous Martingales
   7.3.1. Representation via Random Time Change 206
   7.3.2. Representation via Stochastic Integration 209
   7.3.3. Skorohod’s Representation Theorem 213
   7.3.4. Exercises 217