Chapter 1
Basic Concept and Linearized Problem of Systems

In this chapter, we discuss the normal forms, integrability and linearized problem for the analytic autonomous differential system of two variables in a neighborhood of an elementary singular point. In addition, we give the definition of the multiplicity for a multiple singular point and study the quasi-algebraic integrals for some polynomial systems.

1.1 Basic Concept and Variable Transformation

Consider the following two-order differential equations

\[
\frac{dx}{dt} = X(x,y), \quad \frac{dy}{dt} = Y(x,y).
\]  

(1.1.1)

When \(x, y, t\) are real variables, \(X(x,y), Y(x,y)\) are real functions of \(x\) and \(y\), we say that system (1.1.1) is a real autonomous planar differential system, or \((X, Y)\) is a real planar vector field. When \(x, y, t\) are complex variables and \(X(x,y), Y(x,y)\) are complex functions of \(x\) and \(y\), we say that system (1.1.1) is a two-order complex autonomous differential system.

When the functions \(X(x,y), Y(x,y)\) are two polynomials of \(x\) and \(y\) of degree \(n\), system (1.1.1) is called a polynomial system of degree \(n\). It is often represented by \((E_n)\).

If the functions \(X(x,y), Y(x,y)\) can be expanded as a power series of \(x - x_0\) and \(y - y_0\) in a neighborhood of the point \((x_0, y_0)\) with non-zero convergent radius, then system (1.1.1) is called an analytic system in a neighborhood of \((x_0, y_0)\).

If system (1.1.1) is real and analytic in a neighborhood of \((x_0, y_0)\), then, we can see \(x, y, t\) as complex variables in this small neighborhood to extend system (1.1.1) to complex field.

We next assume that \(X(x,y), Y(x,y), F(x,y)\) is continuously differentiable in a region \(D\) of the \((x,y)\) real plane (or \((x,y)\) complex space).
Along the orbits of system (1.1.1), the total derivative of $F$ in $D$ is given by

$$\frac{dF}{dt}\bigg|_{(1.1.1)} = \frac{\partial F}{\partial x} X + \frac{\partial F}{\partial y} Y.$$ (1.1.2)

If $F(x, y)$ is not constant function and $\frac{dF}{dt}\bigg|_{(1.1.1)} \equiv 0$ in $D$, then $F$ is called a first integral of (1.1.1) in $D$.

If $M(x, y)$ is a non-zero function in $D$ and

$$\frac{\partial (MX)}{\partial x} + \frac{\partial (MY)}{\partial y} \equiv 0$$ (1.1.3)

in $D$, we say that $M$ is an integral factor of (1.1.1) in $D$. On the other hand, if $M^{-1}$ is an integral factor of (1.1.1) in $D$, we say that $M$ is an inverse integral factor of (1.1.1) in $D$.

If $X(x_0, y_0) = Y(x_0, y_0) = 0$, the point $(x_0, y_0)$ is called a singular point or equilibrium point of (1.1.1). Otherwise, $(x_0, y_0)$ is called an ordinary point of (1.1.1).

When $(x_0, y_0)$ is a singular point of (1.1.1) and $x_0, y_0$ are real, we say that $(x_0, y_0)$ is a real singular point. Otherwise, $(x_0, y_0)$ is a complex singular point.

If $(x_0, y_0)$ is a unique singular point in a neighborhood of the singular point $(x_0, y_0)$ of (1.1.1), we say that $(x_0, y_0)$ is an isolated singular point of (1.1.1). In this case, if

$$\left(\frac{\partial X}{\partial x} \frac{\partial Y}{\partial y} - \frac{\partial Y}{\partial x} \frac{\partial X}{\partial y}\right)_{x=x_0,y=y_0} \neq 0,$$ (1.1.4)

then, $(x_0, y_0)$ is called an elementary singular point. Otherwise, it called a multiple singular point.

Suppose that the functions

$$u = \varphi(x, y), \quad v = \psi(x, y)$$ (1.1.5)

are continuously differentiable in $D$ and when $(x, y) \in D$, $(u, v) \in D'$. Write that

$$J_1 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}, \quad J_2 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$ (1.1.6)

If for all $(x, y) \in D$, $J_1$ is bounded and $J_1 \neq 0$, we say that (1.1.5) is a non-singular transformation in $D$.

Clearly, if (1.1.5) is non-singular in $D$, then, for all $(x, y) \in D$, $J_1 J_2 \equiv 1$.

It is easy to prove the following two conclusions by using the chain rule.
1.2 Resultant of the Weierstrass Polynomial and Multiplicity of a Singular Point

Proposition 1.1.1. Suppose that the functions \(X(x,y), Y(x,y), M(x,y)\) are continuously differentiable in \(D\) and (1.1.5) is a non-singular transformation. Under (1.1.5), system (1.1.1) becomes

\[
\frac{du}{dt} = U(u,v), \quad \frac{dv}{dt} = V(u,v). \tag{1.1.7}
\]

Then, for all \((x,y) \in D\) and any continuously differentiable function \(F(x,y)\),

\[
\left. \frac{dF}{dt} \right|_{(1.1.1)} = \left. \frac{dF}{dt} \right|_{(1.1.7)} . \tag{1.1.8}
\]

Proposition 1.1.2. Under the conditions of Proposition 1.1.1, in addition, if the functions \(\varphi(x,y), \psi(x,y)\) are two-order differentiable in \(D\), then, for all \((x,y) \in D\),

\[
\frac{\partial(J_1 X)}{\partial x} + \frac{\partial(J_1 Y)}{\partial y} = J_1 \left( \frac{\partial U}{\partial u} + \frac{\partial V}{\partial v} \right), \tag{1.1.9}
\]

\[
\frac{\partial(J_2 U)}{\partial u} + \frac{\partial(J_2 V)}{\partial v} = J_2 \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right). \tag{1.1.10}
\]

For system

\[
\frac{dx}{dt} = M(x,y)X(x,y), \quad \frac{dy}{dt} = M(x,y)Y(x,y) \tag{1.1.11}
\]

by Proposition 1.1.2, we have

Proposition 1.1.3. Under the conditions of Propositions 1.1.1 and 1.1.2, for all continuously differentiable function \(M(x,y)\) in \((x,y) \in D\),

\[
\frac{\partial(J_1 MX)}{\partial x} + \frac{\partial(J_1 MY)}{\partial y} = J_1 \left[ \frac{\partial(MU)}{\partial u} + \frac{\partial(MV)}{\partial v} \right], \tag{1.1.12}
\]

\[
\frac{\partial(J_2 MU)}{\partial u} + \frac{\partial(J_2 MV)}{\partial v} = J_2 \left[ \frac{\partial(MX)}{\partial x} + \frac{\partial(MX)}{\partial y} \right]. \tag{1.1.13}
\]

By the above proposition, we obtain

Proposition 1.1.4. Under the conditions of Propositions 1.1.1 and 1.1.2, If \(M(x,y)\) is an integral factor of (1.1.1) in \(D\) and \(M(x,y)\) is continuously differentiable, then, \(J_2 M\) is an integral factor of (1.1.7) in \(D'\).

1.2 Resultant of the Weierstrass Polynomial and Multiplicity of a Singular Point

In this section, we first study the resultant of Weierstrass polynomials. By using their properties, we give the definition of multiplicity of singular points. For a multiple
singular point, we investigate its division and composition from some simple singular points.

Suppose that

\[ P(x, y) = \varphi_0(x)y^m + \varphi_1(x)y^{m-1} + \cdots + \varphi_m(x), \]
\[ Q(x, y) = \psi_0(x)y^n + \psi_1(x)y^{n-1} + \cdots + \psi_n(x), \]

are two polynomials of \( y \), where \( m \) and \( n \) are two positive integers, \( \varphi_k(x), \psi_k(x) \) are power series of \( x \) with non-zero convergent radius, \( x \) and \( y \) are complex variables.

In addition, \( \varphi_0(x)\psi_0(x) \) is not identically vanishing.

**Definition 1.2.1.** The following \((n + m)\)-order determinant

\[
\begin{vmatrix}
\varphi_0 & \varphi_1 & \cdots & \cdots & \varphi_m \\
\varphi_0 & \varphi_1 & \cdots & \cdots & \varphi_m \\
\varphi_0 & \varphi_1 & \cdots & \cdots & \varphi_m \\
\psi_0 & \psi_1 & \cdots & \psi_n \\
\psi_0 & \psi_1 & \cdots & \psi_n \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\psi_0 & \psi_1 & \cdots & \psi_n \\
\end{vmatrix}
\]

is called the resultant of \( P(x, y) \) and \( Q(x, y) \) with respect to \( y \).

**Definition 1.2.2.** Write that

\[ H(x, y) = y^m + h_1(x)y^{m-1} + h_2(x)y^{m-2} + \cdots + h_m(x), \]

where \( m \) is a positive integer, \( h_k(x), k = 1, \ldots, m, \) are power series of \( x \) with non-zero convergent radius. If

\[ h_1(0) = h_2(0) = \cdots = h_m(0) = 0, \]

we say that \( H(x, y) \) is a Weierstrass polynomial of degree \( m \) of \( y \).

**Definition 1.2.3.** Let \( U(x, y) \) be a power series of \( x, y \) with a non-zero convergent radius and \( U(0, 0) = 1 \). We say that \( U(x, y) \) is an unitary power series of \( x, y \).

**Definition 1.2.4.** Let \( f(z) \) be a power series of \( z \) with a non-zero convergent radius, \( q \) be a positive integer. If \( f(0) = 0 \), we say that \( x = 0 \) is an algebraic zero of the function \( f(x^\frac{1}{q}) \). If there is a positive integer \( p \), such that \( f(x) = c_px^p + o(x^p) \) and \( c_p \neq 0 \). Then, \( c_px^\frac{p}{q} \) is called the first term of \( f(x^\frac{1}{q}) \).
By the theory of the algebraic curves, we know that

**Theorem 1.2.1.** Let

\[ P(x, y) = \varphi_0(x) \prod_{k=1}^{m} (y - f_k(x)), \quad Q(x, y) = \psi_0(x) \prod_{j=1}^{n} (y - g_j(x)). \] (1.2.5)

Then,

\[ \text{Res}(P, Q, y) = \varphi_0^n(x) \psi_0^m(x) \prod_{k=1}^{m} \prod_{j=1}^{n} (f_k(x) - g_j(x)) \]

\[ = \varphi_0^n(x) \prod_{k=1}^{m} Q(x, f_k(x)) = (-1)^m \varphi_0^m(x) \prod_{j=1}^{n} P(x, g_j(x)). \] (1.2.6)

**Theorem 1.2.2.** Let \( A_k(x) \) be the algebraic cofactor of (1.2.2) with the \( k \)-row and the \((m+n)\)-column, \( B_j(x) \) be the algebraic cofactor with \((n+j)\)-row and \((m+n)\)-column, \( k = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \). Then

\[ \text{Res}(P, Q, y) = A(x, y) P(x, y) + B(x, y) Q(x, y), \] (1.2.7)

where

\[ A(x, y) = A_1(x) y^{n-1} + A_2(x) y^{n-2} + \cdots + A_n(x), \]
\[ B(x, y) = B_1(x) y^{m-1} + B_2(x) y^{m-2} + \cdots + B_m(x). \] (1.2.8)

**Theorem 1.2.3 (Weierstrass preparatory theorem).** Let \( F(x, y) \) be a power series of \( x, y \) with a non-zero convergent radius. If there exists a positive integer \( m \), such that

\[ F(0, y) = c_m y^m + \text{h.o.t.}, \quad c_m \neq 0, \] (1.2.9)

where h.o.t. stand for the high order terms. Then, there is a unique Weierstrass polynomial \( H(x, y) \) of \( y \) with the degree \( m \) and an unitary power series \( U(x, y) \), such that in a small neighborhood of the origin

\[ F(x, y) = c_m H(x, y) U(x, y). \] (1.2.10)

**Theorem 1.2.4.** Under the conditions of Theorem 1.2.3, there exist two positive number \( \sigma_1 \) and \( \sigma_2 \), such that when \( |x| < \sigma_1 \), \( F(x, y) \) as a function of \( y \), it has exactly \( m \) complex zeros \( y = f_k(x) \) inside the disk \( |y| < \sigma_2 \), where \( f_k(0) = 0 \) and \( x = 0 \) is an algebraic zero of \( f_k(x) \), \( k = 1, 2, \ldots, m \).

**Corollary 1.2.1.** If \( H(x, y) \) is a Weierstrass polynomial of \( y \) with the degree \( m \), then, there exist \( m \) functions \( f_1(x), f_2(x), \ldots, f_m(x) \), for which \( x = 0 \) is their algebraic zero, such that in a small neighborhood of the origin,

\[ H(x, y) \equiv \prod_{k=1}^{m} (y - f_k(x)). \] (1.2.11)
Corollary 1.2.2. Let $F(x, y)$ be a power series of $x, y$ with a non-zero convergent radius. If there exist an integer $m$, such that

$$F(0, 0) = \frac{\partial F(0, 0)}{\partial y} = \cdots = \frac{\partial F^{m-1}(0, 0)}{\partial y^{m-1}} = 0, \quad \frac{\partial F^m(0, 0)}{\partial y^m} \neq 0. \quad (1.2.12)$$

Then, the implicit function equation

$$F(x, y) = 0, \quad y|_{x=0} = 0 \quad (1.2.13)$$

has exactly $m$ complex solutions $y = f_k(x)$ in a small neighborhood of the origin and $x = 0$ is an algebraic zero of $f_k(x), k = 1, 2, \cdots, m$.

We know consider the multiplicity of singular point for the complex autonomous differential system:

$$\frac{dx}{dt} = F_1(x, y), \quad \frac{dy}{dt} = F_2(x, y), \quad (1.2.14)$$

where $F_1(x, y)$ and $F_2(x, y)$ are power series of $x, y$ with a non-zero convergent radius, $F_1(0, 0) = F_2(0, 0) = 0$. Suppose that $O(0, 0)$ is an isolated singular point of (1.2.14).

Without loss of generality, we assume that $F_1(0, y) \neq 0$. Hence, there is an integer $m$, such that

$$F_1(0, y) = a_m y^m + h.o.t., \quad a_m \neq 0. \quad (1.2.15)$$

By Theorem 1.2.3 and Corollary 1.2.1, in a small neighborhood of the origin, $F_1(x, y)$ has the form as follows:

$$F_1(x, y) = a_m H_1(x, y) U_1(x, y) = a_m U_1(x, y) \prod_{k=1}^{m} (y - f_k(x)), \quad (1.2.16)$$

where

$$H_1(x, y) = \prod_{k=1}^{m} (y - f_k(x)) \quad (1.2.17)$$

is a Weierstrass polynomial of $y$ of degree $m$. $x = 0$ is an algebraic zero of $f_k(x)$. $U_1(x, y)$ is a unitary power series of $x, y$.

Consider the function

$$R(x) = \prod_{k=1}^{m} F_2(x, f_k(x)). \quad (1.2.18)$$

Lemma 1.2.1. Let the origin be an isolated singular point of (1.2.14). Then there is an integer $N > 0$, such that

$$R(x) = Ax^N + o(x^N), \quad A \neq 0. \quad (1.2.19)$$
Proof. We can write that
\[ F_2(x, y) = x^s \tilde{F}_2(x, y), \]
where \( \tilde{F}_2(x, y) \) is a power series of \( x, y \) with a non-zero convergent radius and \( \tilde{F}_2(0, y) \) is not identically vanishing. Consider the following two cases.

1. Suppose that \( \tilde{F}_2(0, 0) \neq 0 \). Since \( F_2(0, 0) = 0 \), so that, \( s \) is a positive integer. (1.2.18) and (1.2.20) follow that
\[ R(x) = \tilde{F}_2^m(0, 0)x^m + o(x^m). \] (1.2.21)
By (1.2.21), when \( \tilde{F}_2(0, 0) \neq 0 \), the conclusion of Lemma 1.2.1 holds.

2. Suppose that \( \tilde{F}_2(0, 0) = 0 \). Since \( \tilde{F}_2(0, y) \) is a non-zero function, thus, there is an integer \( n > 0 \), such that
\[ \tilde{F}_2(0, y) = b_n x^n + h.o.t., \quad b_n \neq 0. \] (1.2.22)
By Theorem 1.2.3 and Corollary 1.2.1, in a small neighborhood of the origin, \( \tilde{F}_2(x, y) \) can be written as
\[ \tilde{F}_2(x, y) = b_n H_2(x, y) U_2(x, y). \] (1.2.23)
where \( H_2(x, y) \) is a Weierstrass polynomial of \( y \) with the degree \( n \) and \( U_2(x, y) \) is a unitary power series of \( x, y \). By (1.2.18), (1.2.20) and (1.2.23), we know that
\[ R(x) = b_n^m x^{sm} M(x) \prod_{k=1}^{m} H_2(x, f_k(x)), \] (1.2.24)
where
\[ M(x) = \prod_{k=1}^{m} U_2(x, f_k(x)) = 1 + o(1). \] (1.2.25)
By Theorem 1.2.1,
\[ \prod_{k=1}^{m} H_2(x, f_k(x)) = \text{Res}(H_1, H_2, y) \] (1.2.26)
is a power series of \( x \) with a non-zero convergent radius. Because the origin is an isolated singular point of (1.2.14), it follows that \( \text{Res}(H_1, H_2, y) \) is not zero function. By (1.2.24), (1.2.25) and (1.2.26), when \( \tilde{F}_2(0, 0) = 0 \), the conclusion of Lemma 1.2.1 is also holds.

Definition 1.2.5. Suppose that the origin is an isolated singular point of (1.2.14). \( F_1(x, y) \) is given by (1.2.16), where for all \( k, x = 0 \) is an algebraic zero of \( f_k(x) \) and \( U_1(x, y) \) is a unitary power series of \( x, y \), \( a_m \neq 0 \). If there is a positive integer \( N \), such that
\[ R(x) = \prod_{k=1}^{m} F_2(x, f_k(x)) = Ax^N + o(x^N), \quad A \neq 0, \] (1.2.27)
then the origin is called a \( N \)-multiple singular point of (1.2.14). \( N \) is called the multiplicity of the origin.
In the theory of algebraic curves, there is the definition of the crossing number of two curves. We see from Definition 1.2.5 that if the origin is an isolated singular point of (1.2.14) and \( F_1(x, y), F_2(x, y) \) are polynomial of \( x, y \), then, the multiplicity of the origin is the same as the crossing number of the two curves of \( F_1(x, y) = 0 \) and \( F_2(x, y) = 0 \).

How to determine the multiplicity of a singular point? The following examples give some results.

Consider the following autonomous analytic system in a neighborhood of the origin:

\[
\frac{dx}{dt} = \sum_{k=m}^{\infty} X_k(x, y) = X(x, y), \\
\frac{dy}{dt} = \sum_{k=n}^{\infty} Y_k(x, y) = Y(x, y),
\]

where \( m, n \) are integers, for all \( k, X_k(x, y), Y_k(x, y) \) are homogeneous polynomials of degree \( k \) of \( x, y \). In addition, \( X_m(x, y)Y_n(x, y) \) is not identically vanishing.

**Theorem 1.2.5.** If \( X_m(x, y) \) and \( Y_n(x, y) \) are irreducible, then the origin is a \( mn \)-multiple singular point of (1.2.28).

**Proof.** Since \( Y_n(s, 1) \) is a polynomial of degree \( n \), so that, there is a complex number \( s \) such that \( Y_n(s, 1) \neq 0 \). By the transformation

\[
\xi = x - sy, \quad \eta = y
\]

(1.2.29) becomes

\[
\frac{d\xi}{dt} = X(\xi + s\eta, \eta) - sY(\xi + s\eta, \eta) = \tilde{X}(\xi, \eta), \\
\frac{d\xi}{dt} = Y(\xi + s\eta, \eta) = \sum_{k=n}^{\infty} \tilde{Y}_k(\xi, \eta) = \tilde{Y}(\xi, \eta).
\]

(1.2.30)

Notice that \( Y(0, \eta) = Y_n(s, 1)\eta^n + o(\eta^n) \). We can write

\[
\tilde{Y}_n(\xi, \eta) = Y_n(s, 1) \prod_{k=1}^{n} (\eta - \lambda_k \xi).
\]

(1.2.31)

By Corollary 1.2.2 and (1.2.30), in a neighborhood of the origin, the implicit function equation

\[
\tilde{Y}(\xi, \eta) = 0, \quad \eta|_{\xi=0} = 0
\]

(1.2.32)

has exact \( n \) solutions \( \eta = f_k(\xi) = \lambda_k \xi + o(\xi), \quad k = 1, 2, \cdots, n \). Denote \( \tilde{X}_m(\xi, \eta) = X_m(\xi + s\eta, \eta), \) then
\[
\prod_{k=1}^{n} \tilde{X}(\xi, f_k(\xi)) = \prod_{k=1}^{n} X(\xi + sf_k(\xi), f_k(\xi)) \\
= \prod_{k=1}^{n} X_m(\xi + s\lambda_k\xi, \lambda_k\xi) + o(\xi^{mn}) \\
= \prod_{k=1}^{n} \bar{X}_m(\lambda_k\xi) + o(\xi^{mn}) \\
= \prod_{k=1}^{n} \bar{X}_m(1, \lambda_k)\xi^{mn} + o(\xi^{mn}).
\] (1.2.33)

By the irreducibility of \(X_m(x, y)\) and \(Y_n(x, y)\), we see from (1.2.31) that
\[
\prod_{k=1}^{n} \bar{X}_m(1, \lambda_k) \neq 0.
\] (1.2.34)

Thus, (1.2.33) follows the conclusion of Theorem 1.2.5.

Obviously, by this theorem, the multiplicity of an elementary singular point is 1. In addition, by Definition 1.2.5, we have

**Theorem 1.2.6.** Suppose that
\[
F_1(x, y) = ax + by + h.o.t., \quad F_2(x, y) = cx + dy + h.o.t., \quad b \neq 0.
\] (1.2.35)

If \(y = f(x)\) is the unique solution of the equation
\[
F_1(x, y) = 0, \quad y|_{x=0} = 0,
\] (1.2.36)
then, when \(F_2(x, f(x)) \equiv 0\), the origin is not a isolated singular point of (1.2.14). When \(F_2(x, f(x)) = Ax^N + o(x^N)\), where \(N\) is a positive integer, \(A \neq 0\), the origin is a \(N\)-multiple singular point of (1.2.14).

Consider the polynomial system
\[
\frac{dx}{dt} = \sum_{k=1}^{m} X_k(x, y) = \mathcal{X}_m(x, y),
\]
\[
\frac{dy}{dt} = \sum_{k=1}^{n} Y_k(x, y) = \mathcal{Y}_n(x, y),
\] (1.2.37)

where \(m, n\) are positive integers, \(X_k(x, y), Y_k(x, y)\) are homogeneous polynomials of degree \(k\). \(\mathcal{X}_m(x, y)\) and \(\mathcal{Y}_n(x, y)\) are irreducible.

By Definition 1.2.5 and Bezout theorem in the theory of algebraic curves, we obtain
Theorem 1.2.7. The sum of multiplicities of all finite singular points of (1.2.37) is less than \( mn \) or equals to \( mn \).

In addition, we have

Theorem 1.2.8. If

\[
X_m(o, y) = ay^m, \quad Y_n(0, y) = by^n, \quad ab \neq 0
\]

and \( \text{Res}(X_m, Y_m, y) = Ax^N + o(x^N), \ A \neq 0, \) then, the origin of (1.2.37) is a \( N \)-multiple singular point.

Finally, we investigate the division and composition of the singular points. Take \( m = n \). We consider the perturbed system of (1.2.37):

\[
\frac{dx}{dt} = \sum_{k=1}^{n} X_k(x, y) + \Phi_n(x, y, \varepsilon),
\]

\[
\frac{dy}{dt} = \sum_{k=1}^{n} Y_k(x, y) + \Psi_n(x, y, \varepsilon).
\]

(1.2.39)

where

\[
\Phi_n(x, y, \varepsilon) = \sum_{k+j=0}^{n} \varepsilon_{kj} x^k y^j, \quad \Psi_n(x, y, \varepsilon) = \sum_{k+j=0}^{n} \varepsilon'_{kj} x^k y^j,
\]

(1.2.40)

for all \( k, j, \varepsilon_{kj}, \varepsilon'_{kj} \) are small parameters. \( \varepsilon \) stands for a vector consisting of all \( \varepsilon_{kj}, \varepsilon'_{kj} \).

We have the following conclusions.

Theorem 1.2.9. Suppose that when \( \varepsilon = 0 \), two functions of the right hands of (1.2.39) are irreducible and the origin is a \( N \)-multiple singular point of (1.2.39)\( _{\varepsilon=0} \). Then, there exist two positive numbers \( r_0 \) and \( \varepsilon_0 \), such that when \( |\varepsilon| < \varepsilon_0 \), the sum of multiplicities of all singular points of (1.2.39) in the region \( |x| < r_0, |y| < r_0 \) is exact \( N \). In addition, the coordinates of these singular points are continuous functions of \( \varepsilon \). When \( \varepsilon \to 0 \), these singular points attend to the origin.

By choosing the parameters of \( \Phi_n(x, y, \varepsilon) \) and \( \Psi_n(x, y, \varepsilon) \), system (1.2.39) can have exactly \( N \) complex elementary singular points.

1.3 Quasi-Algebraic Integrals of Polynomial Systems

In this section, we consider the polynomial system of degree \( n \) as follows:

\[
\frac{dx}{dt} = \sum_{k=0}^{n} X_k(x, y) = \mathcal{X}_n(x, y),
\]

\[
\frac{dy}{dt} = \sum_{k=0}^{n} Y_k(x, y) = \mathcal{Y}_n(x, y).
\]

(1.3.1)
In [Darboux, 1878], the author first studied systematically the invariant algebraic curve solutions of (1.3.1) and gave a method to construct first integrals and integral factors of (1.3.1), by using finitely many invariant algebraic curve solutions.

**Definition 1.3.1.** Let \( f(x, y) \) be a nonconstant polynomial of degree \( m \). If there is a bounded function \( h(x, y) \), such that
\[
\left. \frac{df}{dt} \right|_{(1.3.1)} = \frac{\partial f}{\partial x} \mathcal{X}_n(x, y) + \frac{\partial f}{\partial y} \mathcal{Y}_n(x, y) = h(x, y)f(x, y),
\]
then, \( f = 0 \) is called an algebraic curve solution of (1.3.1). The polynomial \( f \) is called an algebraic integral of (1.3.1). The function \( h \) is called a cofactor of \( f \).

Clearly, we see from (1.3.2) that the following conclusion holds.

**Proposition 1.3.1.** If \( f \) is an algebraic integral of (1.3.1), then the cofactor \( h \) of \( f \) is a polynomial of degree at most \( n - 1 \).

In [Liu Y.R. etc, 1995], the authors developed Darboux’s results to that \( f \) is not polynomial. They defined a quasi-algebraic integral of (1.3.1). We next introduce their main conclusions.

**Definition 1.3.2.** Suppose that \( f(x, y) \) is a continuously differentiable nonconstant function in a region \( \mathcal{D} \). If there is a polynomial \( h(x, y) \) of degree at most \( n - 1 \), such that (1.3.2) holds in \( \mathcal{D} \). We say that \( f(x, y) \) is a quasi-algebraic integral of (1.3.1) in \( \mathcal{D} \).

It is easy to see that an algebraic integral must be a quasi-algebraic integral.

**Example 1.3.1.** Obviously, the quintic system
\[
\begin{align*}
\frac{dx}{dt} &= -y + x(x^2 + y^2 - 1)^2, \\
\frac{dy}{dt} &= x + y(x^2 + y^2 - 1)^2
\end{align*}
\]
has the following quasi-algebraic integrals:
\[
\begin{align*}
f_1 &= x^2 + y^2 - 1, & f_2 &= e^{\frac{1}{x^2 + y^2 - 1}}, \\
f_3 &= x^2 + y^2, & f_4 &= e^{\arctan \frac{y}{x}}, \\
f_5 &= x + iy, & f_6 &= x - iy.
\end{align*}
\]
They satisfy
\[
\left. \frac{df_k}{dt} \right|_{(1.3.3)} = h_k(x, y)f_k(x, y), \quad k = 1, 2, 3, 4, 5, 6,
\]
where for all $k = 1 - 6$, $h_k(x, y)$ are as follows:

$$
\begin{align*}
h_1 &= 2(x^2 + y^2)(x^2 + y^2 - 1), \\
h_2 &= -2(x^2 + y^2), \\
h_3 &= 2(x^2 + y^2 - 1)^2, \\
h_4 &= 1, \\
h_5 &= (x^2 + y^2 - 1)^2 + i, \\
h_6 &= (x^2 + y^2 - 1)^2 - i.
\end{align*}
$$

(1.3.6)

Because

$$
h_1 + h_2 - h_3 + 2h_4 = 0,
$$

(1.3.7)

by (1.3.5) and (1.3.7), (1.3.3) has the first integral

$$
F = f_1 f_2 f_3^{-1} f_4^2 = \text{constant},
$$

(1.3.8)

which satisfies

$$
\left. \frac{dF}{dt} \right|_{(1.3.3)} = (h_1 + h_2 - h_3 + 2h_4) F = 0.
$$

(1.3.9)

**Definition 1.3.3.** Let $f_1, f_2, \ldots, f_m$ be $m$ quasi-algebraic integrals of (1.3.1). If there exists a group of complex number $\alpha_1, \alpha_2, \ldots, \alpha_m$, such that $f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}$ is identically equal a constant, then, $f_1, f_2, \ldots, f_m$ is called dependent. Otherwise, $f_1, f_2, \ldots, f_m$ is called independent.

For example, in (1.3.4), we have that $f_5 f_6 = f_3$ and $f_5 f_6^{-1} = f_4^{2i}$, hence, $f_3, f_5, f_6$ and $f_4, f_5, f_6$ are dependent, respectively.

**Theorem 1.3.1.** The first integral $F$ and the integral factor $M$ of (1.3.1) are quasi-algebraic integrals of (1.3.1).

**Proof.** By the definition of the first integral, a first integral of (1.3.1) in a region must be a quasi-algebraic integral of (1.3.1).

Let $M$ is an integral factor of (1.3.1) in a region. Then, we have

$$
\frac{\partial(MX_n)}{\partial x} + \frac{\partial(MY_n)}{\partial y} = 0,
$$

(1.3.10)

i.e.,

$$
\frac{\partial M}{\partial x} X_n + \frac{\partial M}{\partial y} Y_n + \left( \frac{\partial X_n}{\partial x} + \frac{\partial Y_n}{\partial y} \right) M = 0.
$$

(1.3.11)

It follows that

$$
\left. \frac{dM}{dt} \right|_{(1.3.1)} = - \left( \frac{\partial X_n}{\partial x} + \frac{\partial Y_n}{\partial y} \right) M.
$$

(1.3.12)

This implies that $M$ is a quasi-algebraic integral of (1.3.1).
1.3 Quasi-Algebraic Integrals of Polynomial Systems

**Theorem 1.3.2.** Suppose that \( f_1, f_2, \cdots, f_m \) are \( m \) independent quasi-algebraic integrals of (1.3.1) satisfying

\[
\frac{df_k}{dt} \bigg|_{(1.3.1)} = h_k f_k, \quad k = 1, 2, \cdots, m. \tag{1.3.13}
\]

Then, for any group of non-zero complex numbers \( \alpha_1, \alpha_2, \cdots, \alpha_m \), the function \( f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m} \) is also a quasi-algebraic integral of (1.3.1) satisfying

\[
\frac{df}{dt} \bigg|_{(1.3.1)} = (\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_1 h_1 + \alpha_m h_m) f. \tag{1.3.14}
\]

We know from the above theorem that

**Theorem 1.3.3.** Suppose that \( f_1, f_2, \cdots, f_m \) are \( m \) independent quasi-algebraic integrals of (1.3.1) satisfying (1.3.13). If there is a group of non-zero complex numbers \( \alpha_1, \alpha_2, \cdots, \alpha_m \), such that

\[
\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_1 h_1 + \alpha_m h_m = 0. \tag{1.3.15}
\]

Then, \( f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m} \) is a first integral of (1.3.1).

By using Theorem 1.3.2 and (1.3.12), we obtain

**Theorem 1.3.4.** Suppose that \( f_1, f_2, \cdots, f_m \) are \( m \) independent quasi-algebraic integrals of (1.3.1) satisfying (1.3.13). If there is a group of non-zero complex numbers \( \alpha_1, \alpha_2, \cdots, \alpha_m \), such that

\[
\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_1 h_1 + \alpha_m h_m = - \left( \frac{\partial X_n}{\partial x} + \frac{\partial Y_n}{\partial y} \right). \tag{1.3.16}
\]

Then, \( f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m} \) are an integral factor of (1.3.1).

Because the set of all polynomials of degree \( n-1 \) forms a linear space of dimension \( n(n+1)/2 \). Every polynomial of degree \( n-1 \) is a vector of this linear space. Thus, by Theorem 1.3.2 and Theorem 1.3.3, if \( f_1, f_2, \cdots, f_m \) are \( m \) independent quasi-algebraic integrals of (1.3.1) satisfying (1.3.13), then

1. If \( h_1, h_2, \cdots, h_m \) are linear dependent, then, by using \( f_1, f_2, \cdots, f_m \), we can construct a first integral of (1.3.1);

2. If \( h_1, h_2, \cdots, h_m \) are linear independent and

\[
h_1, h_2, \cdots, h_m \frac{\partial X_n}{\partial x} + \frac{\partial Y_n}{\partial y}
\]

are linear dependent, then, by using \( f_1, f_2, \cdots, f_m \), we can construct an integral factor of (1.3.1).
For a given polynomial system, we hope to know that under what parametric conditions, it has a quasi-algebraic integral and we wish to get more quasi-algebraic integrals. In principle, if there exists an algebraic integral of (1.3.1) which satisfies (1.3.2), then, we can obtain \( f \) and \( h \), by using the method of undetermined coefficients. Letting

\[
f = \sum_{k+j=0}^{m} c_{kj} x^k y^j, \quad h = \sum_{k+j=0}^{n-1} d_{kj} x^k y^j,
\]

and substituting (1.3.17) into (1.3.2), comparing the coefficients of the corresponding terms in the two sides of obtained representation, it gives rise to a linear system of algebraic equations with respect to \( c_{kj} \), \( d_{k,j} \), \( k, j = 1, \ldots \). Solving this system, it follows \( f \) and \( h \).

Unfortunately, generally, we do not know the existence and its degree \( m \) of an algebraic integral \( f \) for a given polynomial system. This is a difficult classical problem in the theory of the planar dynamical systems.

We next discuss some special cases.

Consider the following polynomial system of degree \( n + m \):

\[
\frac{dx}{dt} = G_k(x)X_n(x, y), \quad \frac{dy}{dt} = Y_{n+m}(x, y),
\]

where \( 1 \leq k \leq m \),

\[
G_k(x) = a_0 + a_1 x + \cdots + a_k x^k, \quad a_k \neq 0
\]

Because a polynomial of degree \( k \) has exact \( k \) complex roots. We have the following conclusion.

**Proposition 1.3.2.** If \( x = x_0 \) is a simple zero of \( G_k(x) \), then, system (1.3.18) has a quasi-algebraic integral \( f = x - x_0 \). If \( x = x_0 \) is a \( j \)-multiple zero of \( G_k(x) \), then, system (1.3.18) has \( j \) independent quasi-algebraic integrals as follows:

\[
f_1 = x - x_0, \quad f_2 = e^{\frac{1}{(x-x_0)^2}}, \quad f_3 = e^{\frac{1}{(x-x_0)^3}}, \ldots, \quad f_j = e^{\frac{1}{(x-x_0)^j}}.
\]

**Proposition 1.3.3.** The system

\[
\frac{dx}{dt} = X_n(x, y), \quad \frac{dy}{dt} = Y_{n+m}(x, y)
\]

has the following \( m \) independent quasi-algebraic integrals

\[
f_1 = e^x, \quad f_2 = e^{x^2}, \ldots, \quad f_m = e^{x^m}.
\]
1.4 Cauchy Majorant and Analytic Properties in a Neighborhood of an Ordinary Point

Proposition 1.3.4. Suppose that system (1.3.1) is real and \( f = f_1 + if_2 \) is a quasi-algebraic integral of (1.3.1) satisfying

\[
\frac{df}{dt} \bigg|_{(1.3.1)} = (h_1 + ih_2)f,
\]

where \( f_1 \) and \( f_2 \) are two real non-zero functions, \( h_1 \) and \( h_2 \) are real coefficient polynomials. Then, \( \bar{f} = f_1 - if_2, f_3 = \bar{f} \bar{f} \) and \( f_4 = f^i \bar{f}^{-i} \) are quasi-algebraic integrals of (1.3.1) respectively satisfying

\[
\frac{d\bar{f}}{dt} \bigg|_{(1.3.1)} = (h_1 - ih_2)\bar{f}, \quad \frac{df_3}{dt} \bigg|_{(1.3.1)} = 2h_1f_3, \quad \frac{df_4}{dt} \bigg|_{(1.3.1)} = -2h_2f_4.
\]

Example 1.3.2. Consider the following real quadratic system

\[
\frac{dx}{dt} = -y + \delta x + lx^2 + mxy + ny^2, \quad \frac{dy}{dt} = x(1 + by).
\]

This system has the following quasi-algebraic integral

\[
f = \begin{cases} 
(1 + by) \frac{t}{b}, & \text{if } b \neq 0, \\
e^y, & \text{if } b = 0.
\end{cases}
\]

where \( e^y \) is a limit function of \( (1 + by) \frac{t}{b} \) as \( b \to 0 \).

1.4 Cauchy Majorant and Analytic Properties in a Neighborhood of an Ordinary Point

For the complex differential equations, in order to investigate the convergence of a solution of the power series, Cauchy posed the classical majorant method. It provided an important tool. In this section, we introduce this method.

Definition 1.4.1. Let \( f = \sum_{\alpha + \beta = 0}^{\infty} c_{\alpha\beta} x^\alpha y^\beta \) and \( F = \sum_{\alpha + \beta = 0}^{\infty} C_{\alpha\beta} x^\alpha y^\beta \) be two power series of \( x, y \), where \( x, y \) are complex variables and for all \( \alpha, \beta \in \mathbb{N}, c_{\alpha\beta} \) are complex coefficients, \( C_{\alpha\beta} \) are non-negative real numbers. If \( \forall \alpha, \beta \), the inequalities \(|c_{\alpha\beta}| \leq C_{\alpha\beta}\) hold, then, \( F \) is called a majorant of \( f \), denoted by \( f \prec F \) or \( F \succ f \).

Proposition 1.4.1. Suppose that \( f_1, f_2, F_1, F_2 \) are power series of \( x, y \). \( F_1, F_2 \) have non-negative real coefficients and non-zero convergent radius. If \( f_1 \prec F_1, \)
Chapter 1  Basic Concept and Linearized Problem of Systems

If \( f_2 \prec F_2 \), then, \( f_1, f_2 \) also have non-zero convergent radius and

\[
\begin{align*}
    f_1 \pm f_2 & \prec F_1 + F_2 , \quad f_1 f_2 \prec F_1 F_2 , \\
    \frac{\partial f_1}{\partial z} & \prec \frac{\partial F_1}{\partial z} , \quad \frac{\partial f_1}{\partial w} \prec \frac{\partial F_1}{\partial w} , \\
    \int f_1 dz & \prec \int F_1 dz , \quad \int f_1 dw \prec \int F_1 dw .
\end{align*}
\]  

(1.4.1)

In addition, if \( f_1(0,0) = F_1(0,0) = 0 \), then

\[
\frac{1}{1 \pm f_1} \prec \frac{1}{1 - F_1} = \sum_{k=0}^{\infty} F_1^k .
\]  

(1.4.2)

Suppose that \( f = \sum_{\alpha + \beta = 0}^{\infty} c_{\alpha \beta} x^\alpha y^\beta \) has a non-zero convergent radius. By Cauchy inequality, there are positive numbers \( M, r \), such that

\[
|c_{\alpha \beta}| \leq \frac{M}{r^{\alpha + \beta}} .
\]  

(1.4.3)

From (1.4.3) and (1.4.2), we have

**Proposition 1.4.2.** If \( f = \sum_{\alpha + \beta = 0}^{\infty} c_{\alpha \beta} x^\alpha y^\beta \) has a non-zero convergent radius, then, there exist positive numbers \( M, r \), such that

\[
f \prec \frac{M}{1 - \frac{x}{r}} \left(1 - \frac{y}{r}\right) \prec \frac{M}{1 - \frac{x + y}{r}} .
\]  

(1.4.4)

**Proposition 1.4.3.** If \( f = \sum_{\alpha + \beta = 0}^{\infty} c_{\alpha \beta} x^\alpha y^\beta \) has a non-zero convergent radius, then, there exist positive numbers \( M, r \), such that for any positive integer \( m \),

\[
\sum_{\alpha + \beta = m}^{\infty} c_{\alpha \beta} x^\alpha y^\beta \prec \frac{M(x + y)^m}{r^{m-1}(r - x - y)} .
\]  

(1.4.5)

**Proof.** By (1.4.3), for any positive integer \( k \), we have

\[
\sum_{\alpha + \beta = k} c_{\alpha \beta} x^\alpha y^\beta \prec M \sum_{\alpha + \beta = k} \frac{x^\alpha y^\beta}{r^k} \prec M \left(\frac{x + y}{r}\right)^k .
\]  

(1.4.6)

Thus,

\[
\sum_{\alpha + \beta = m}^{\infty} c_{\alpha \beta} x^\alpha y^\beta \prec M \sum_{k=m}^{\infty} \left(\frac{x + y}{r}\right)^k = \frac{M(x + y)^m}{r^{m-1}(r - x - y)} .
\]  

(1.4.7)

\[\square\]
We next discuss the analytic properties of the solutions of system (1.1.1) in a neighborhood of an ordinary point. Suppose that the right hand of (1.1.1) are analytic in a neighborhood of \((x_0, y_0)\) and \((x_0, y_0)\) is an ordinary point of (1.1.1). We can see \((x_0, y_0)\) as the origin. Consider system

\[
\begin{align*}
\frac{dx}{dt} &= a_0 + a_1 x + a_2 y + \text{h.o.t.}, \\
\frac{dy}{dt} &= b_0 + b_1 x + b_2 y + \text{h.o.t.,}
\end{align*}
\]

where

\[|a_0| + |b_0| \neq 0.\] (1.4.9)

Since the origin is an ordinary point of (1.4.8). Cauchy proved the following result.

**Theorem 1.4.1.** If \(a_0 \neq 0\), then system (1.4.8) has a unique power series solution with the initial condition \(y(0) = 0\) as follows:

\[
y = \sum_{k=1}^{\infty} c_k x^k,
\]

which has non-zero convergent radius.

By using the non-singular linear transformation

\[
u = -b_0 x + a_0 y, \quad v = -\bar{a}_0 x - \bar{b}_0 y,
\]

system (1.4.8) becomes

\[
\begin{align*}
\frac{du}{dt} &= U(u, v) = \sum_{k=0}^{\infty} \varphi_k(u) v^k, \\
\frac{dv}{dt} &= -\Delta + V(u, v) = -\Delta + \sum_{k=0}^{\infty} \psi_k(u) v^k.
\end{align*}
\]

where for all \(k\), \(\varphi_k(u), \psi_k(u)\) are power series of \(u\) and

\[
\begin{align*}
\Delta &= |a_0|^2 + |b_0|^2 > 0, \\
U(0, 0) &= \varphi_0(0) = 0, \quad V(0, 0) = \psi_0(0) = 0.
\end{align*}
\]

**Definition 1.4.2.** Let \(f = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha,\beta} x^\alpha y^\beta\) be power series of \(x, y\). If we do not consider the convergence of \(f\), then \(f\) is called a formal power series of \(x, y\).

If \(f(x, y)\) is a formal power series of \(x, y\) and \(f(0, 0) = 1\), then \(f\) is called a unitary formal power series of \(x, y\).
Let \( \{c_{\alpha_k, \beta_k}\} \) for some \( k \) be a subsequence of \( \{c_{\alpha \beta}\} \), then \( \sum c_{\alpha_k, \beta_k} x^{\alpha_k} y^{\beta_k} \) is called a subseries of \( f \).

For two formal series, when we make the algebraic operations and analytic operations of term by term, if we do not consider their convergence, these operations are called the formal operations.

**Definition 1.4.3.** Suppose that the functions of right hand of (1.1.1) are analytic in a neighborhood of the origin. If a formal series \( F(x, y) \) of \( x, y \) satisfies

\[
\frac{dF}{dt}_{(1.1.1)} = \frac{\partial F}{\partial x} X + \frac{\partial F}{\partial y} Y = 0
\]

and \( F \) is a nonconstant function in a neighborhood of the origin, then, \( F \) is called a formal first integral of (1.1.1) in a neighborhood of the origin.

If a formal series \( M(x, y) \) of \( x, y \) satisfies

\[
\frac{\partial (MX)}{\partial x} + \frac{\partial (MY)}{\partial y} = 0
\]

and \( M \) is a nonconstant function in a neighborhood of the origin, then, \( M \) is called a formal integral factor of (1.1.1) in a neighborhood of the origin.

**Lemma 1.4.1.** For system (1.4.12), one can determine term by term the formal series

\[
H(u, v) = \sum_{k=0}^{\infty} h_k(u) v^k, \quad h_0(u) = u,
\]

such that

\[
\frac{dH}{dt}_{(1.4.12)} = 0.
\]

**Proof.** Using (1.4.16) and (1.4.12) to do formal operation, we have

\[
\frac{dH}{dt}_{(1.4.12)} = \frac{\partial H}{\partial u} U + \frac{\partial H}{\partial v} (-\Delta + V)
\]

\[
= \sum_{k=0}^{\infty} h'_k(u) v^k \sum_{j=0}^{\infty} \varphi_j(u) v^j
\]

\[
+ \sum_{k=1}^{\infty} k h_k(u) v^{k-1} \left[ -\Delta + \sum_{j=0}^{\infty} \psi_j(u) v^j \right]
\]

\[
= \sum_{m=0}^{\infty} [-(m+1) (\Delta - \psi_0) h_{m+1} + g_m] v^m,
\]

where

\[
g_m = \sum_{k=0}^{m} (h'_k \varphi_{m-k} + k h_k \psi_{m-k+1})
\]
is a polynomial of $h_k, h'_k, \varphi_j, \psi_j$ with positive coefficients. (1.4.17) and (1.4.18) imply the recursion formulas of $h_m$ as follows:

$$h_{m+1} = \frac{g_m}{(m+1)(\Delta - \psi_0)}, \quad m = 0, 1, 2, \cdots.$$  (1.4.20)

Because the relationship of $h_0(u) = u$ has been given, by (1.4.20), this lemma holds.

**Lemma 1.4.2.** The function $H(u, v)$ defined by (1.4.16) has a non-zero convergent radius.

**Proof.** Since the functions $U(u, v), V(u, v)$ are analytic in a neighborhood of the origin and $U(0, 0) = V(0, 0) = 0$, by Proposition 1.4.3, there exist two positive numbers $M, r$, such that

$$U(u, v) \prec \frac{M(u+v)}{r-u-v}, \quad V(x, y) \prec \frac{M(u+v)}{r-u-v}. \quad (1.4.21)$$

Consider the majorant system

$$\frac{du}{dt} = \frac{M(u+v)}{r-u-v}, \quad \frac{dv}{dt} = -\Delta + \frac{M(u+v)}{r-u-v}. \quad (1.4.22)$$

It is easy to see that system (1.4.22) has the following formal first integral

$$\tilde{H}(u, v) = u + G(u + v), \quad (1.4.23)$$

where

$$G(w) = \frac{-M\Delta r}{(\Delta + 2M)^2} \left[ \frac{\Delta + 2M}{\Delta r} w + \ln \left( 1 - \frac{\Delta + 2M}{\Delta r} w \right) \right]$$

$$= \frac{M\Delta r}{(\Delta + 2M)^2} \sum_{k=2}^{\infty} \frac{1}{k} \left( \frac{\Delta + 2M}{\Delta r} \right)^k w^k = o(w) \quad (1.4.24)$$

is a power series of $w$ with positive coefficients, which is analytic in the disk $|w| < \Delta r/(\Delta + 2M)$. In order to prove

$$H(u, v) \prec \tilde{H}(u, v), \quad (1.4.25)$$

write that

$$\tilde{H}(u, v) = \sum_{k=0}^{\infty} \tilde{h}_k(u)v^k, \quad \frac{M(u+v)}{r-u-v} = \sum_{k=0}^{\infty} \chi_k(u)v^k, \quad (1.4.26)$$

then,

$$h_0(u) \prec \tilde{h}_0(u) = u + G(u),$$

$$\varphi_k(u) \prec \chi_k(u), \quad \psi_k(u) \prec \chi_k(u), \quad k = 0, 1, 2, \cdots. \quad (1.4.27)$$
Similar to the proof of Lemma 1.4.1, we have the recursion formulas for the computation of $\tilde{h}_k$ as follows:

$$\tilde{h}_{m+1} = \frac{\tilde{g}_m}{(m+1)(\Delta - \chi_0)}, \quad m = 0, 1, 2, \ldots,$$

where

$$\tilde{g}_m = \sum_{k=0}^{m} (\tilde{h}'_k \chi_{m-k} + k \tilde{h}_k \chi_{m-k+1}).$$

By using Proposition 1.4.1, (1.4.19), (1.4.20) (1.4.27), (1.4.28), (1.4.29) and mathematical induction, for any positive integer $m$, we obtain

$$g_m \prec \tilde{g}_m, \quad h_m \prec \tilde{h}_m.$$

It follows (1.4.25) and this lemma.

**Theorem 1.4.2.** Let

$$h^*_0(u) = u + h.o.t.$$  

be a power series which is convergence in a neighborhood of $u = 0$. One can derive successively every term of the following unique power series of $u, v$,

$$H^*(u, v) = \sum_{k=0}^{\infty} h^*_k(u)v^k,$$

with a non-zero convergent radius, such that

$$\left. \frac{dH^*}{dt} \right|_{(1.4.12)} = 0,$$

and

$$H^*(u, v) = h^*_0(H(u, v)).$$

**Proof.** Similar to the proof of Lemma 1.4.1, for any positive integer $k$, we use $h^*_k$ instead of $h_k$ of (1.4.19) and (1.4.20) to get the recursion formulas for $h^*_k(u)$. When $h^*_0(u)$ has been obtained, then, there is unique formal series $H^*(u, v)$ satisfying (1.4.33). Write that

$$\mathcal{H}(u, v) = h^*_0(H(u, v)).$$

By Lemma 1.4.1 and Lemma 1.4.2, $\mathcal{H}(u, v)$ is a power series with a non-zero convergent radius and

$$\mathcal{H}(u, 0) = h^*_0(u), \quad \left. \frac{d\mathcal{H}}{dt} \right|_{(1.4.12)} = 0.$$

By the uniqueness, we have $H^*(u, v) = \mathcal{H}(u, v)$, i.e., this theorem holds.
1.4 Cauchy Majorant and Analytic Properties in a Neighborhood of an Ordinary Point

**Theorem 1.4.3.** In a neighborhood of the origin there is a first integral of (1.4.8) as follows:

\[ F(x, y) = -b_0 x + a_0 y + \sum_{k=2}^{\infty} F_k(x, y), \]  

(1.4.37)

where \( F(x, y) \) is convergent in a small neighborhood of the origin, for every \( k \), \( F_k(x, y) \) is a homogeneous polynomial of \( x, y \). Especially,

\[ F_2(x, y) = \frac{1}{2(|a_0|^2 + |b_0|^2)^2} (\bar{a}_0 x + \bar{b}_0 y)(Ax + By), \]  

(1.4.38)

where

\[ A = 2b\bar{b}_0(a_1b_0 - a_0b_1) + \bar{a}_0(a_0a_1b_0 - a_2b_0^2 - a_0b_1 + a_0b_0b_2), \]

\[ B = 2a_0\bar{a}_0(a_2b_0 - a_0b_2) - \bar{b}_0(a_0a_1b_0 - a_2b_0^2 - a_0b_1 + a_0b_0b_2). \]  

(1.4.39)

**Proof.** Let

\[ F(x, y) = H(-b_0 x + a_0 y, -\bar{a}_0 x - \bar{b}_0 y). \]  

(1.4.40)

By Lemma 1.4.1 and Lemma 1.4.2, \( F(x, y) \) is a first integral of (1.4.8) and it has a non-zero convergent radius. Write that

\[ H(x, y) = u + (c_1 u + c_2 v)v + h.o.t., \]  

(1.4.41)

substituting (1.4.41) into (1.4.17), we can determine \( c_1, c_2 \). It follows the representation of \( F_2 \). \( \square \)

**Theorem 1.4.4.** For system (1.4.8), one can derive successively every term of the following power series

\[ T(x, y) = \frac{\bar{a}_0 x + \bar{b}_0 y}{|a_0|^2 + |b_0|^2} + h.o.t., \]  

(1.4.42)

which is convergent in a neighborhood of the origin, such that

\[ \left. \frac{dT}{dt} \right|_{(1.4.8)} = 1. \]  

(1.4.43)

**Proof.** By Lemma 1.4.1 and Lemma 1.4.1, the first integral \( H(u, v) = u + h.o.t. \) of (1.4.12) has a non-zero convergent radius. Hence, in a small neighborhood of the origin, by using the implicit function equations

\[ z = H(u, v), \quad w = v, \]  

(1.4.44)

we can uniquely solve

\[ u = \zeta(z, w) = z + h.o.t., \quad v = w, \]  

(1.4.45)
where the function $\zeta(z, w)$ has a non-zero convergent radius. By using the transformation (1.4.44), system (1.1.12) becomes

$$
\frac{dz}{dt} = 0, \quad \frac{dw}{dt} = -\Delta + V(\zeta(z, w), w).
$$

(1.4.46)

Write that

$$
-\Delta + V(\zeta(z, w), w) = \frac{1}{\Delta} + \sum_{k+j=1}^{\infty} C_{kj} z^k w^j,
$$

$$
T = \frac{w}{\Delta} + \sum_{k+j=1}^{\infty} \frac{1}{j+1} C_{kj} z^k w^{j+1}.
$$

(1.4.47)

By (1.4.11), (1.4.44) and (1.4.47), we obtain (1.4.42) and the convergence of $T(x, y)$ in a small neighborhood of the origin. By (1.4.46) and (1.4.47), we have

$$
\frac{dz}{dt} = 0, \quad \frac{dT}{dt} = 1.
$$

(1.4.48)

**Theorem 1.4.5.** For system (1.4.8), one can derive successively every term of the following unique power series of $x, y$

$$
\xi = x + \text{h.o.t.}, \quad \eta = y + \text{h.o.t.},
$$

(1.4.49)

with a non-zero convergent radius, such that, by the transformation (1.4.49), system (1.4.8) becomes the following normal form

$$
\frac{d\xi}{dt} = a_0, \quad \frac{d\eta}{dt} = b_0.
$$

(1.4.50)

**Proof.** For the function $F(x, y)$ in Theorem 1.4.3 and the function $T(x, y)$ in Theorem 1.4.4, letting

$$
\xi = a_0 T(x, y) - \frac{b_0}{|a_0|^2 + |b_0|^2} F(x, y),
$$

$$
\eta = b_0 T(x, y) + \frac{a_0}{|a_0|^2 + |b_0|^2} F(x, y),
$$

(1.4.51)

then, Theorem 1.4.3 and Theorem 1.4.4 imply (1.4.49) and (1.4.50).

Finally, we consider the following analytic system

$$
\frac{dx}{dt} = y^{n-1} \left[ a_0 + a_1 x^n + a_2 y^m + \sum_{k=2}^{\infty} X_k(x^n, y^m) \right],
$$

$$
\frac{dy}{dt} = x^{n-1} \left[ b_0 + b_1 x^n + b_2 y^m + \sum_{k=2}^{\infty} Y_k(x^n, y^m) \right],
$$

(1.4.52)
1.4 Cauchy Majorant and Analytic Properties in a Neighborhood of an Ordinary Point

where $m$ and $n$ are two positive integers, for all $k$, $X_k(u, v), Y_k(u, v)$ are homogeneous polynomials of $u, v$ with degree $k$, $b_0 \neq 0$.

The origin of (1.4.52) may be an ordinary point, an elementary singular point or a multiple singular point.

**Theorem 1.4.6.** In a neighborhood of the origin, system (1.4.52) has the following formal first integral:

$$F(x^n, y^m) = -mb_0 x^n + na_0 y^m + \sum_{k=2}^{\infty} F_k(x^n, y^m),$$

where $F(u, v)$ is analytic in a neighborhood of the origin. For all $k$, $F_k(u, v)$ are homogeneous polynomials of $u, v$.

**Proof.** By the transformation

$$u = x^n, \quad v = y^m, \quad d\tau = x^{n-1} y^{m-1} dt$$

system (1.4.52) becomes

$$\frac{du}{d\tau} = n (a_0 + a_1 u + a_2 v) + n \sum_{k=2}^{\infty} X_k(u, v),$$

$$\frac{dv}{d\tau} = m (b_0 + b_1 u + b_2 v) + m \sum_{k=2}^{\infty} Y_k(u, v).$$

Because $b_0 \neq 0$, the origin is an ordinary point of (1.4.55). Thus, Theorem 1.4.3 follows the conclusion of this theorem. \qed

For an analytic system, as a corollary of Theorem 1.4.6, we can obtain the symmetric principle to the test of center or focus in the theory of real planar dynamical systems. In fact, if $n = 2, m = 1, a_0 = 0, b_0 \neq 0$, then (1.4.52) becomes

$$\frac{dx}{dt} = a_1 x^2 + a_2 y + \sum_{k=2}^{\infty} X_k(x^2, y) = a_2 y + h.o.t.,$$

$$\frac{dy}{dt} = b_0 x + h.o.t.,$$

By the transformation

$$u = x^2, \quad v = y, \quad d\tau = x dt,$$

system (1.4.56) becomes

$$\frac{du}{d\tau} = 2a_1 u + 2a_2 v + h.o.t.,$$

$$\frac{dv}{d\tau} = b_0 + b_1 u + b_2 v + h.o.t.$$

By Theorem 1.4.3 and Theorem 1.4.6, we have
Corollary 1.4.1. In a neighborhood of the origin, system (1.4.56) has the following formal first integral:

\[ F(x, y) = -b_0 x^2 + a_2 y^2 + \sum_{k=3}^{\infty} F_k(x, y), \]  

where \( F(x, y) \) is analytic in a neighborhood of the origin and for all \( k \), \( F_k(x, y) \) are homogeneous polynomials of \( x, y \).

Suppose that (1.4.56) is a real system. Then, the vector field defined by (1.4.56) is symmetric with respect to \( y \)-axis. In this case, when \( a_2 = 0 \), the origin of (1.4.56) is a multiple singular point; When \( b_0 a_2 > 0 \), it is a saddle point. When \( b_0 a_2 < 0 \), by the symmetric principle, it is a center. Corollary 1.4.1 implies that when \( b_0 a_2 \geq 0 \) and the coefficients of the right hand of (1.4.56) are complex numbers, in a neighborhood of the origin, system (1.4.56) is integrable.

Corollary 1.4.2. For system (1.4.56), One can determine successively every term of the following convergent power series of \( x, y \)

\[ g(x, y) = y + h.o.t., \]  

such that

\[ \frac{dg}{dt} \bigg|_{(1.4.56)} = b_0 x. \]  

Proof. By Theorem 1.4.4, for system (1.4.58), in a neighborhood of the origin, there is a convergent power series

\[ T(u, v) = \frac{v}{b_0} + h.o.t., \]  

such that

\[ \frac{dT}{d\tau} \bigg|_{(1.4.58)} = 1. \]  

Let \( g(x, y) = b_0 T(x^2, y) \). Then, (1.4.57), (1.4.62) and (1.4.63) follows this lemma.

This corollary means that if (1.4.56) is a polynomial system, then, \( e^g \) is a quasi-algebraic integral of (1.4.56) in a neighborhood of the origin.

1.5 Classification of Elementary Singular Points and Linearized Problem

Suppose that system (1.1.1) is analytic in a neighborhood of the origin and the origin is an elementary singular point of (1.1.1). We consider the complex system

\[ \frac{dx}{dt} = ax + by + h.o.t., \quad \frac{dy}{dt} = cx + dy + h.o.t.. \]  

(1.5.1)
The coefficient matrix of the linearized system of (1.5.1) at the origin has the characteristic equation
\[ \lambda^2 - (a + d)\lambda + ad - bc = 0. \] (1.5.2)
Let \( \lambda_1, \lambda_2 \) be two roots of (1.5.2). If \( \lambda_1\lambda_2 = ad - bc \neq 0 \), the origin is an elementary singular point.

For an isolated singular point of the complex equations, the following result is useful.

**Theorem 1.5.1 (Briot-Bouquet theorem).** Let \( F(u, v) = Au + Bv + \text{h.o.t.} \) be a power series of \( u, v \) with a non-zero convergent radius. If \( B \) is not a positive integer, then, in a neighborhood of the origin, equation
\[ u \frac{dv}{du} = F(u, v), \quad v|_{u=0} = 0 \] (1.5.3)
has the unique solution:
\[ v = f(u) = \frac{A}{1 - B}u + \text{h.o.t.}, \] (1.5.4)
where \( f(u) \) is a power series of \( u \) with a non-zero convergent radius.

We next consider
\[ \frac{dx}{dt} = \lambda_1 x + \sum_{k=2}^{\infty} X_k(x, y), \quad \frac{dy}{dt} = \lambda_2 y + \sum_{k=2}^{\infty} Y_k(x, y), \] (1.5.5)
where the functions of right hand of (1.5.5) are analytic and \( X_2(0, 1) = A, \ Y_2(1, 0) = B \).

**Theorem 1.5.2.** If \( \lambda_1 \neq 0, \lambda_2/\lambda_1 \) is not a positive integer, then, system (1.5.5) has a solution
\[ y = \psi(x) = \frac{B}{2\lambda_1 - \lambda_2}x^2 + \text{h.o.t.}, \] (1.5.6)
satisfying \( \psi(0) = 0 \), where \( \psi(x) \) a power series of \( x \) with a non-zero convergent radius.

**Proof.** Let \( y = xv \). Then (1.5.5) becomes
\[ x \frac{dv}{dx} = -v + \frac{\lambda_2 v + \sum_{k=2}^{\infty} x^{k-1}Y_k(1, v)}{\lambda_1 + \sum_{k=2}^{\infty} x^{k-1}X_k(1, v)} = \frac{Bx + (\lambda_2 - \lambda_1)v}{\lambda_1} + \text{h.o.t.} \] (1.5.7)
(1.5.7) and Theorem 1.5.1 follows this theorem. \( \square \)
Similarly, we have

**Theorem 1.5.3.** If \( \lambda_1 \neq 0, \lambda_2/\lambda_1 \) is not a positive integer, then, system (1.5.5) has a solution

\[
x = \varphi(y) = \frac{A}{2\lambda_2 - \lambda_1} y^2 + h.o.t.,
\]

satisfying \( \varphi(0) = 0 \) where \( \varphi(y) \) is a power series of \( y \) with a non-zero convergent radius.

The above two theorems imply the following result.

**Theorem 1.5.4.** If \( \lambda_1 \lambda_2 \neq 0, \lambda_1/\lambda_2 \) and \( \lambda_2/\lambda_1 \) are not positive integers, then, system (1.5.5) has two analytic solutions \( x = \varphi(y) \) and \( y = \psi(x) \), satisfying \( \varphi(0) = \psi(0) = \varphi'(0) = \psi'(0) = 0 \). By transformation

\[
u = x - \varphi(y), \quad v = y - \psi(x)
\]

system (1.5.5) can be reduced to

\[
\frac{du}{dt} = \lambda_1 u F_1(u, v), \quad \frac{dv}{dt} = \lambda_2 v F_2(u, v).
\]

where \( F_1(u, v) \) and \( F_2(u, v) \) are two power series of \( u, v \) with non-zero convergent radius and \( F_1(0,0) = F_2(0,0) = 1 \).

**Definition 1.5.1.** If there exist two convergent power series

\[
\xi = x + \sum_{\alpha + \beta = 2}^{\infty} c_{\alpha\beta}x^\alpha y^\beta, \quad \eta = y + \sum_{\alpha + \beta = 2}^{\infty} d_{\alpha\beta}x^\alpha y^\beta,
\]

such that by transformation (1.5.11), system (1.5.1) becomes

\[
\frac{d\xi}{dt} = a\xi + b\eta, \quad \frac{d\eta}{dt} = c\xi + d\eta,
\]

Then, we say that system (1.5.1) is linearizable in a neighborhood of the origin. (1.5.11) is called a linearized transformation of (1.5.1) in a neighborhood of the origin.

**Remark 1.5.1.** Suppose that (1.5.1) is linearizable in a neighborhood of the origin and (1.5.11) is a linearized transformation of (1.5.1). The function \( F(\xi,\eta) \) is continuously differentiable and satisfies

\[
\frac{dF}{dt}\bigg|_{(1.5.12)} = 0.
\]

Let

\[
\tilde{\xi} = \xi F(\xi, \eta), \quad \tilde{\eta} = \eta F(\xi, \eta).
\]
We see from (1.5.13) and (1.5.14) that
\[
\frac{d\tilde{\xi}}{dt} = a\tilde{\xi} + b\tilde{\eta}, \quad \frac{d\tilde{\eta}}{dt} = c\tilde{\xi} + d\tilde{\eta}. \tag{1.5.15}
\]
Thus, if \(F(\xi, \eta)\) is power series of \(\xi, \eta\) with non-zero convergent radius and \(F(0,0) = 1\), then, (1.5.14) is also a linearized transformation of (1.5.1) in a neighborhood of the origin.

**Remark 1.5.2.** Suppose that (1.5.1) is linearizable in a neighborhood of the origin and (1.5.11) is a linearized transformation of (1.5.1). System (1.5.1) is a real coefficient system. Then, we can see \(x, y, t\) as real variable and write
\[
\xi = \xi_1 + i\xi_2, \quad \eta = \eta_1 + i\eta_2, \tag{1.5.16}
\]
where \(\xi_1, \eta_1, \xi_2, \eta_2\) are power series of \(x, y\) with non-zero convergent radius and
\[
\xi_1 = x + h.o.t., \quad \eta_1 = y + h.o.t.. \tag{1.5.17}
\]
Substituting (1.5.16) into (1.5.12), separating the real and imaginary parts, we have
\[
\frac{d\xi_1}{dt} = a\xi_1 + b\eta_1, \quad \frac{d\eta_1}{dt} = c\xi_1 + d\eta_1, \\
\frac{d\xi_2}{dt} = a\xi_2 + b\eta_2, \quad \frac{d\eta_2}{dt} = c\xi_2 + d\eta_2. \tag{1.5.18}
\]
In this case, (1.5.17) is a real linearized transformation of (1.5.1) in a neighborhood of the origin.

In [Qin Y.X., 1985], the author introduced the following definition.

**Definition 1.5.2.** For system (1.5.1):
1. If \(\lambda_1 \lambda_2 \neq 0, \text{Im}(\lambda_1/\lambda_2) \neq 0\), then the origin is called a focus type singular point;
2. If \(\lambda_1 \lambda_2 \neq 0, \text{Im}(\lambda_1/\lambda_2) = 0, \text{Re}(\lambda_1/\lambda_2) > 0\), then the origin is called a node type singular point;
3. If \(\lambda_1 \lambda_2 \neq 0, \text{Im}(\lambda_1/\lambda_2) = 0, \text{Re}(\lambda_1/\lambda_2) < 0\), then the origin is called a critical type singular point.

In details, we have

**Definition 1.5.3.** Suppose that the origin of (1.5.1) is a node type singular point.

If \(\lambda_1 = \lambda_2\) and \(b = c = 0\), then the origin of (1.5.1) is called a starlike node.
If \(\lambda_1 = \lambda_2\) and \(|b| + |c| \neq 0\), then the origin of (1.5.1) is called a degenerate node.
If \(\lambda_1/\lambda_2\) or \(\lambda_2/\lambda_1\) is a positive integer more than 1, then the origin of (1.5.1) is called an integer-ratio node.
If \(\lambda_1/\lambda_2\) and \(\lambda_2/\lambda_1\) are not positive integer, then the origin of (1.5.1) is called an ordinary node.
**Definition 1.5.4.** Suppose that the origin of (1.5.1) is a critical type singular point.

If \( \lambda_1/\lambda_2 = -1 \), then the origin of (1.5.1) is called a weak critical singular point.

If \( \lambda_1/\lambda_2 = -p/q \), where \( p \) and \( q \) are irreducible positive integers and \( p/q \neq -1 \), then the origin of (1.5.1) is called a \( p:q \) resonance singular point.

If \( \lambda_1/\lambda_2 \) is a negative irrational number, then the origin of (1.5.1) is called an irrational singular point.

If the origin is an elementary singular point of (1.5.1), but is not a degenerate node, then, by a linear transformation, (1.5.1) can be reduced to as follows:

\[
\begin{align*}
\frac{dx}{dt} &= \lambda_1 x + \sum_{k=2}^{\infty} X_k(x, y), \\
\frac{dy}{dt} &= \lambda_2 y + \sum_{k=2}^{\infty} Y_k(x, y),
\end{align*}
\] (1.5.19)

where \( \lambda_1 \lambda_2 \neq 0 \) and for all \( k \), \( X_k(x, y), Y_k(x, y) \) are homogeneous polynomials of \( x, y \) of degree \( k \).

By the theory of classical complex analysis, we know that

**Theorem 1.5.5.** If the origin of (1.5.19) is not an integer-ratio node, a weak critical singular point or a resonance singular point, then, one can determine successively every term of the following formal series

\[
\begin{align*}
\xi &= x + \sum_{\alpha+\beta=2}^{\infty} c_{\alpha\beta} x^\alpha y^\beta, \\
\eta &= y + \sum_{\alpha+\beta=2}^{\infty} d_{\alpha\beta} x^\alpha y^\beta,
\end{align*}
\] (1.5.20)

such that, by this formal transformation, (1.5.19) becomes the following linear system:

\[
\begin{align*}
\frac{d\xi}{dt} &= \lambda_1 \xi, \\
\frac{d\eta}{dt} &= \lambda_2 \eta.
\end{align*}
\] (1.5.21)

**Proof.** For any positive integer \( k \) more than 1, letting \( f_k(x, y), g_k(x, y) \) be two homogeneous polynomials of \( x, y \) of degree \( k \) given by

\[
\begin{align*}
f_k(x, y) &= \sum_{\alpha+\beta=k} c_{\alpha\beta} x^\alpha y^\beta, \\
g_k(x, y) &= \sum_{\alpha+\beta=k} d_{\alpha\beta} x^\alpha y^\beta.
\end{align*}
\] (1.5.22)

Clearly,

\[
\begin{align*}
\frac{d\xi}{dt} - \lambda_1 \xi &= \sum_{m=2}^{\infty} \left[ \left( \lambda_1 \frac{\partial f_m}{\partial x} x + \lambda_2 \frac{\partial f_m}{\partial y} y - \lambda_1 f_m \right) + \Phi_m(x, y) \right], \\
\frac{d\eta}{dt} - \lambda_2 \eta &= \sum_{m=2}^{\infty} \left[ \left( \lambda_1 \frac{\partial g_m}{\partial x} x + \lambda_2 \frac{\partial g_m}{\partial y} y - \lambda_2 g_m \right) + \Psi_m(x, y) \right],
\end{align*}
\] (1.5.23)
where for all \( m, \Phi_m(x, y), \Psi_m(x, y) \) are defined by the following homogeneous polynomials:

\[
\Phi_m(x, y) = X_m(x, y) + \sum_{k=2}^{m-1} \left( \frac{\partial f_k}{\partial x} X_{m-k+1} + \frac{\partial f_k}{\partial y} Y_{m-k+1} \right),
\]

\[
\Psi_m(x, y) = Y_m(x, y) + \sum_{k=2}^{m-1} \left( \frac{\partial g_k}{\partial x} X_{m-k+1} + \frac{\partial g_k}{\partial y} Y_{m-k+1} \right). \tag{1.5.24}
\]

From (1.5.22), we have

\[
\lambda_1 \frac{\partial f_m}{\partial x} x + \lambda_2 \frac{\partial f_m}{\partial y} y - \lambda_1 f_m = \sum_{\alpha + \beta = m}^{\infty} (\alpha \lambda_1 + \beta \lambda_2 - \lambda_1) c_{\alpha \beta} x^\alpha y^\beta,
\]

\[
\lambda_1 \frac{\partial g_m}{\partial x} x + \lambda_2 \frac{\partial g_m}{\partial y} y - \lambda_2 g_m = \sum_{\alpha + \beta = m}^{\infty} (\alpha \lambda_1 + \beta \lambda_2 - \lambda_2) d_{\alpha \beta} x^\alpha y^\beta. \tag{1.5.25}
\]

By the conditions of this theorem, because \( \lambda_1/\lambda_2 \) and \( \lambda_2/\lambda_1 \) are not a positive integer more than 1 and are not a negative irrational number. So that, for any positive integers \( \alpha, \beta \), when \( \alpha + \beta \geq 2 \),

\[
\alpha \lambda_1 + \beta \lambda_2 - \lambda_1 \neq 0, \quad \alpha \lambda_1 + \beta \lambda_2 - \lambda_2 \neq 0. \tag{1.5.26}
\]

Write that

\[
\Phi_m(x, y) = \sum_{\alpha + \beta = m} A_{\alpha \beta} x^\alpha y^\beta, \quad \Psi_m(x, y) = \sum_{\alpha + \beta = m} B_{\alpha \beta} x^\alpha y^\beta. \tag{1.5.27}
\]

Thus, (1.5.23), (1.5.25) and (1.5.27) imply (1.5.21) if and only if for any positive integer \( \alpha, \beta \), when \( \alpha + \beta \geq 2 \),

\[
(\alpha \lambda_1 + \beta \lambda_2 - \lambda_1) c_{\alpha \beta} = -A_{\alpha \beta}, \quad (\alpha \lambda_1 + \beta \lambda_2 - \lambda_2) d_{\alpha \beta} = -B_{\alpha \beta}. \tag{1.5.28}
\]

Obviously, (1.5.28) is just the recursion formulas to compute \( c_{\alpha \beta}, d_{\alpha \beta} \). Namely, \( c_{\alpha \beta}, d_{\alpha \beta} \) can be uniquely determined by (1.5.28).

For the convergence of the formal transformation (1.5.20), in [Qin Y.X., 1985], by using Cauchy majorant method, the author proved that if the origin of (1.5.19) is a nfocus type singular point, ordinary node or starlike node, then in (1.5.20), the power series of \( \xi, \eta \) with respect to \( x, y \) have non-zero convergent radius. Therefore, we have

**Theorem 1.5.6.** If the origin of (1.5.1) is a focus type singular point, ordinary node or starlike node, then, system (1.5.1) is linearizable in a neighborhood of the origin. In addition, the linearized transformation is unique.
If the origin of (1.5.19) is an irrational singular point, so that \( \lambda_1/\lambda_2 \) is an negative irrational number. Then, in (1.5.28), \( \alpha \lambda_1 + \beta \lambda_2 - \lambda_1 \) and \( \alpha \lambda_1 + \beta \lambda_2 - \lambda_2 \) can be taken as very small number, such that the convergence problem of the formal series (1.5.20) becomes very difficult problem.

### 1.6 Node Value and Linearized Problem of the Integer-Ratio Node

Let the origin of system (1.5.1) be an integer-ratio node. By using a suitable linear transformation, system (1.5.1) can be reduced to

\[
\frac{dx}{dt} = \lambda x + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} x^\alpha y^\beta, \quad \frac{dy}{dt} = n\lambda y + \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} x^\alpha y^\beta, \tag{1.6.1}
\]

where \( \lambda \neq 0 \) and \( n \) is an integer greater than 1. System (1.6.1) is a special case of system (1.5.19) under the condition \( \lambda_1 = \lambda, \lambda_2 = n\lambda \). For system (1.6.1), (1.5.28) becomes

\[
(\alpha + n\beta - 1)c_{\alpha\beta}\lambda = -A_{\alpha\beta}, \quad (\alpha + n\beta - n)d_{\alpha\beta}\lambda = -B_{\alpha\beta}. \tag{1.6.2}
\]

Obviously, for any natural numbers \( \alpha \) and \( \beta \), \( \alpha + \beta \geq 2 \) leads \( \alpha + n\beta - 1 \neq 0 \). \( \alpha + n\beta - n = 0 \) holds if and only if \( \alpha = n, \beta = 0 \). Hence, all \( c_{\alpha\beta}, d_{\alpha\beta} \) can be determined uniquely by (1.6.2) except \( d_{n0} \). Consequently, there is a formal transformation (1.5.20) such that system (1.6.1) becomes linear system if and only if \( B_{n0} = 0 \), and when \( B_{n0} = 0 \), \( d_{n0} \) can take any value.

By cited Theorem 2.3 in [Qin Y.X., 1985], we have the following conclusion.

**Theorem 1.6.1.** For system (1.6.1), one can find series (1.5.20) which are convergent in a neighborhood of the origin, such that system (1.6.1) reduced to the normal form

\[
\frac{d\xi}{dt} = \lambda \xi, \quad \frac{d\eta}{dt} = n\lambda \eta + \sigma \lambda \xi^n. \tag{1.6.3}
\]

In (1.6.3), \( \sigma = B_{n0}/\lambda \) is determined uniquely by the coefficients of system (1.6.1).

**Theorem 1.6.2.** If there are formal series \( \tilde{\xi} = x + \text{h.o.t.}, \tilde{\eta} = y + \text{h.o.t.} \), such that

\[
\left. \frac{d\tilde{\xi}}{dt} \right|_{(1.6.1)} = \lambda \tilde{\xi}, \quad \left. \frac{d\tilde{\eta}}{dt} \right|_{(1.6.1)} = n\lambda \tilde{\eta} + \sigma \lambda (\tilde{\xi})^n, \tag{1.6.4}
\]

then

\[
\tilde{\xi} = \xi, \quad \tilde{\eta} = \eta + C \xi^n, \tag{1.6.5}
\]

where, \( C \) is an arbitrary constant.
Proof. We take \( \tilde{\xi}, \tilde{\eta} \) as the power series of \( \xi, \eta \) of the form

\[
\tilde{\eta} = \eta + C \xi^\alpha + h(\xi, \eta), \quad h(\xi, \eta) = \sum_{k=m}^{\infty} h_k(\xi, \eta),
\]

where \( h_k(\xi, \eta) \) are homogeneous polynomials of degree \( k \) of \( \xi, \eta \), \( m \) is a positive integer. It is easy to see that all \( c_{\alpha\beta}, d_{\alpha,\beta} \) in (1.5.20) except \( d_{n0} \) are determined uniquely by (1.6.2). Thus, we have

\[
\tilde{\xi} = \xi, \quad m > n. \tag{1.6.7}
\]

From (1.6.3), (1.6.4) and (1.6.6), we obtain

\[
\frac{d\tilde{\eta}}{dt} \bigg|_{(1.6.3)} - n\lambda \tilde{\eta} - \sigma \lambda (\tilde{\xi})^n = \frac{dh}{dt} \bigg|_{(1.6.3)} - n\lambda h = 0. \tag{1.6.8}
\]

Let us prove (1.6.5) by using reductio ad absurdum, i.e., we prove that \( h \) is equivalent to zero. Suppose that

\[
h_m(\xi, \eta) = \sum_{\alpha + \beta = m} e_{\alpha\beta} \xi^\alpha \eta^\beta \tag{1.6.9}
\]

is not zero. From (1.6.8) and (1.6.9) we have

\[
0 = \frac{dh}{dt} \bigg|_{(1.6.3)} - n\lambda h = \lambda \left( \xi \frac{\partial h_m}{\partial \xi} + n\eta \frac{\partial h_m}{\partial \eta} - nh_m \right) + h.o.t.
\]

\[
= \lambda \sum_{\alpha + \beta = m} (\alpha + n\beta - n)e_{\alpha\beta} \xi^\alpha \eta^\beta + h.o.t.. \tag{1.6.10}
\]

When \( \alpha + \beta = m > n \), \( \alpha + n\beta - n \) is a positive integer. From (1.6.10), it reduces that all \( e_{\alpha\beta} \) in (1.6.9) equal zero, which contradicts with \( h_m \) is not identically zero. Hence, the conclusion of Theorem 1.6.2 holds.

System (1.6.3) has a first integral of the form

\[
\frac{\eta}{\xi^n} - \sigma \ln \xi = \text{constant}. \tag{1.6.11}
\]

Obviously, the fact of \( \sigma \) is zero or nonzero is concerned with the linearized problem system (1.6.1) and the analytic property of (1.6.11). We need to introduce the following definition given in [Liu Y.R., 2002].

**Definition 1.6.1.** \( \sigma \) is called node value of the origin of system (1.6.1).

From Theorem 1.6.1 we obtain
Theorem 1.6.3. System (1.6.1) is linearizable in a neighborhood of the origin if and only if the node value $\sigma = 0$.

Let the origin be an integer-ratio node. In order to know if system is linearizable, we need to compute node values.

Theorem 1.6.4. For every $\alpha$ and $\beta$ satisfying $2 \leq \alpha + \beta \leq n - 1$, $d_{\alpha\beta}$ defined by (1.5.20) are determined uniquely by the recurrent formula

$$d_{\alpha\beta} = \frac{1}{\lambda(n - \alpha - n\beta)} \left[ b_{\alpha\beta} + \sum_{k+j=2}^{\alpha+\beta-1} (\alpha - k + 1)a_{k,j}d_{\alpha-k+1,\beta-j} \right. \\
\left. + \sum_{k+j=2}^{\alpha+\beta-1} (\beta - j + 1)b_{k,j}d_{\alpha-k,\beta-j+1} \right]. \quad (1.6.12)$$

Furthermore, $\sigma$ is given uniquely by

$$\sigma = \frac{1}{\lambda} \left[ b_{n0} + \sum_{k=2}^{n-1} (n - k + 1)a_{k0}d_{n-k+1,0} + b_{k0}d_{n-k,1} \right]. \quad (1.6.13)$$

Proof. From (1.5.20) and (1.6.3), we have

$$0 = \left. \frac{d\eta}{dt} \right|_{(1.6.1)} - n\lambda\eta$$

$$= \sum_{\alpha+\beta=2}^{\infty} \alpha d_{\alpha\beta} x^{\alpha-1} y^\beta \left( \lambda x + \sum_{k+j=2}^{\infty} a_{k,j} x^k y^j \right)$$

$$+ \left( 1 + \sum_{\alpha+\beta=2}^{\infty} \beta d_{\alpha\beta} x^{\alpha} y^{\beta-1} \right) \left( n\lambda y + \sum_{k+j=2}^{\infty} b_{k,j} x^k y^j \right)$$

$$- n\lambda \left( y + \sum_{\alpha+\beta=2}^{\infty} d_{\alpha\beta} x^\alpha y^\beta \right). \quad (1.6.14)$$

It implies that

$$0 = \lambda \sum_{\alpha+\beta=2}^{\infty} (\alpha + n\beta - n)d_{\alpha\beta} x^{\alpha} y^\beta + \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} x^\alpha y^\beta$$

$$+ \sum_{\alpha+\beta=2}^{\infty} \sum_{k+j=2}^{\infty} \alpha d_{\alpha\beta} a_{k,j} x^{\alpha+k-1} y^{\beta+j}$$

$$+ \sum_{\alpha+\beta=2}^{\infty} \sum_{k+j=2}^{\infty} \beta d_{\alpha\beta} b_{k,j} x^{\alpha+k} y^{\beta+j-1}. \quad (1.6.15)$$
Thus, we have

\[ 0 = \lambda \sum_{\alpha + \beta = 2}^{\infty} (\alpha + n\beta - n) d_{\alpha\beta} x^\alpha y^\beta + \sum_{\alpha + \beta = 2}^{\infty} b_{\alpha\beta} x^\alpha y^\beta \]
\[ + \sum_{\alpha + \beta = 2}^{\infty} \sum_{k+j=2}^{\infty} (\alpha - k + 1) a_{kj} d_{\alpha-k+1,\beta-j+1} x^\alpha y^\beta \]
\[ + \sum_{\alpha + \beta = 2}^{\infty} \sum_{k+j=2}^{\infty} (\beta - j + 1) b_{kj} d_{\alpha-k,\beta-j+1} x^\alpha y^\beta. \quad (1.6.16) \]

Because \( \xi = x + h.o.t. \) and when \( \alpha + \beta \geq 2, \alpha + n\beta - n = 0 \) if and only if \( \alpha = n, \beta = 0. \) (1.6.16) follows the conclusion of Theorem 1.6.4.

From the recursive formulas (1.6.12) and (1.6.13) we have

**Theorem 1.6.5.** When \( n = 2, 3 \) and 4, the node values of the origin of system (1.6.1) are as follows

\[ \sigma|_{n=2} = \frac{1}{\lambda} b_{20}, \]
\[ \sigma|_{n=3} = \frac{1}{\lambda^2} [(2a_{20} - b_{11})b_{20} + b_{30}\lambda], \]
\[ \sigma|_{n=4} = \frac{1}{4\lambda^3} [b_{20}(12a_{20}^2 - 10a_{20}b_{11} + 2b_{11}^2 - 2a_{11}b_{20} + b_{02}b_{20}) + 2(2a_{30}b_{20} - b_{20}b_{21} + 6a_{20}b_{30} - 2b_{11}b_{30})\lambda + 4b_{40}\lambda^2]. \quad (1.6.17) \]

**Theorem 1.6.6.** If
\[ b_{20} = b_{30} = \cdots = b_{n-1,0} = 0, \quad (1.6.18) \]
then the node value of the origin of system (1.6.1) is
\[ \sigma = \frac{1}{\lambda} b_{n0}. \quad (1.6.19) \]

**Proof.** Theorem 1.6.4 follows that if (1.6.18) holds, then \( d_{20} = 0 \) and when \( \alpha = 3, 4, \cdots, n - 1, \) we have
\[ d_{\alpha0} = \frac{1}{\lambda(n - \alpha)} \sum_{k=2}^{\alpha-1} (\alpha - k + 1) a_{k0} d_{\alpha-k+1,0}. \quad (1.6.20) \]

By using the mathematical induction, we obtain
\[ d_{20} = d_{30} = \cdots = d_{n-1,0} = 0. \quad (1.6.21) \]

By (1.6.18), (1.6.21) and (1.6.13), the conclusion of Theorem 1.6.6 holds.
Chapter 1  Basic Concept and Linearized Problem of Systems

Theorem 1.6.6 tells us that if \( b_{20} = b_{30} = \cdots = b_{n0} = 0 \), then the node value of the origin of system (1.6.1) is zero.

**Corollary 1.6.1.** If \( y = 0 \) is a solution of system (1.6.1), then at the origin, the node value \( \sigma = 0 \).

Notice that in some special cases, other singular points can become an integer-ratio nodes. Therefore, at these singular points, the integrability and linearized problem of systems can be solved by computing node values in an integer-ratio nodes. For example, we have

**Theorem 1.6.7.** System

\[
\frac{dz}{dT} = z + 2b_3 z^3 w + a_2 z^2 w^2 + a_1 z w^3, \\
\frac{dw}{dT} = -w - b_1 w^4 - 2a_2 z^3 w - b_3 z^2 w^2
\]

is linearizable in a neighborhood of the origin.

**Proof.** By the transformation

\[
z_1 = zw^2, \quad w_1 = w^3
\]

system (1.6.22) can become a special quadratic system

\[
\frac{dz_1}{dT} = -z_1 - 3a_2 z_1^2 + (a_1 - 2b_1) z_1 w_1, \\
\frac{dw_1}{dT} = -3w_1 - 3b_3 z_1^2 - 6a_2 z_1 w_1 - 3b_1 w_1^2.
\]

The origin of system (1.6.24) is an integer-ratio node with \( n = 3 \). Theorem 1.6.5 implies that the node value \( \sigma = 0 \). Notice that \( z_1 = 0 \) is a solution of system (1.6.24), Theorem 1.6.1 follows that there are two convergent power series

\[
\xi = z_1 f_1(z_1, w_1), \quad \eta = w_1 + \sum_{k=2}^{\infty} \eta_k(z_1, w_1),
\]

in a neighborhood of the origin, where \( f_1(0, 0) = 1 \), and \( \eta_k(z_1, w_1) \) are homogeneous polynomials of degree \( k \) of \( z_1, w_1 \), such that system (1.6.24) becomes

\[
\frac{d\xi}{dT} = -\xi, \quad \frac{d\eta}{dT} = -3\eta.
\]

Let

\[
z_2 = \xi \eta^{-\frac{2}{3}}, \quad w_2 = \eta^\frac{1}{3},
\]

in a neighborhood of the origin, where \( f_1(0, 0) = 1 \), and \( \eta_k(z_1, w_1) \) are homogeneous polynomials of degree \( k \) of \( z_1, w_1 \), such that system (1.6.24) becomes

\[
\frac{d\xi}{dT} = -\xi, \quad \frac{d\eta}{dT} = -3\eta.
\]

Let

\[
z_2 = \xi \eta^{-\frac{2}{3}}, \quad w_2 = \eta^\frac{1}{3},
\]
1.7 Linearized Problem of the Degenerate Node

then from (1.6.26) we have

\[ \frac{dz_2}{dT} = z_2, \quad \frac{dw_2}{dT} = -w_2. \] (1.6.28)

We next prove that \( z_2, w_2 \) are power series in \( z, w \). Write that

\[ f_2(z, w) = 1 + \sum_{k=2}^{\infty} w^{2k-3} \eta_k(z, w). \] (1.6.29)

From (1.6.23) and (1.6.25), we have

\[ \xi = z w^2 f_1(z w^2, w^3), \quad \eta = w^3 f_2(z, w). \] (1.6.30)

Hence (1.6.27) and (1.6.30) follows that

\[ z_2 = z f_1(z w^2, w^3) f_2^2(z, w) = z + h.o.t., \]
\[ w_2 = w f_2^3(z, w) = w + h.o.t.. \] (1.6.31)

This means that \( z_2, w_2 \) are power series of \( z, w \) having nonzero radius of convergence. So from (1.6.28), it is obtained that system (1.6.22) is linearizable in a neighborhood of the origin.

Similarly, we have

\[ \frac{dx}{dt} = \lambda x + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha,\beta} x^\alpha y^\beta, \quad \frac{dy}{dt} = \mu x + \lambda y + \sum_{\alpha+\beta=2}^{\infty} b_{\alpha,\beta} x^\alpha y^\beta, \] (1.7.1)

where \( \lambda \mu \neq 0 \).

In the classical complex analytic theory, the linearized problem of the degenerate node is still an open problem. In this section, we discuss this problem.
Lemma 1.7.1. In a neighborhood of the origin, there is a convergent power series solution of system (1.7.1) as follows:

\[ x = \varphi(y) = \frac{a_{02}}{\lambda} y^2 + \text{h.o.t.} \tag{1.7.2} \]

where all coefficients of power series of \( \varphi(y) \) can be determined uniquely by the coefficients of (1.7.1).

Proof. Let \( x = yv \), from (1.7.1) we have

\[ y \frac{dv}{dy} = \frac{\lambda v + \sum_{\alpha+\beta=2}^\infty a_{\alpha\beta} v^\alpha y^{\alpha+\beta-1}}{\mu v + \lambda + \sum_{\alpha+\beta=2}^\infty b_{\alpha\beta} v^\alpha y^{\alpha+\beta-1}} - v \]

\[ = \frac{a_{02}}{\lambda} y + \text{h.o.t.} \tag{1.7.3} \]

According to Theorem 1.5.1, equation (1.7.3) has a unique and convergent power series solution in a neighborhood of the origin

\[ v = v(y) = \frac{a_{02}}{\lambda} y + \text{h.o.t.} \tag{1.7.4} \]

which follows Lemma 1.7.1. \qed

Let \( x = \varphi(y) \) given by (1.7.2) be a convergent power series solution of system (1.7.1) in a neighborhood of the origin, then by the transformation

\[ u = x - \varphi(y), \quad v = y, \tag{1.7.5} \]

system (1.7.1) becomes the following analytic system:

\[ \frac{du}{dt} = \lambda u \left( 1 + \sum_{\alpha+\beta=1}^\infty a'_{\alpha\beta} u^\alpha v^\beta \right), \]

\[ \frac{dv}{dt} = \mu u + \lambda v + \sum_{\alpha+\beta=2}^\infty b'_{\alpha\beta} u^\alpha v^\beta. \tag{1.7.6} \]

Letting

\[ u = w^2, \quad v = v. \tag{1.7.7} \]

System (1.7.6) changes to

\[ \frac{dw}{dt} = \frac{\lambda}{2} w \left( 1 + \sum_{\alpha+\beta=1}^\infty a'_{\alpha\beta} w^{2\alpha} v^\beta \right), \]

\[ \frac{dv}{dt} = \mu w^2 + \lambda v + \sum_{\alpha+\beta=2}^\infty b'_{\alpha\beta} w^{2\alpha} v^\beta. \tag{1.7.8} \]
where the origin of system (1.7.8) is an integer-ratio node with \( n = 2 \). By Theorem 1.6.5, the node value \( \sigma = 2\mu/\lambda \). From Theorem 1.6.1 and Theorem 1.6.2, we have

**Lemma 1.7.2.** There are two power series of \( w \) and \( v \)

\[
\begin{align*}
  f(w, v) &= w + \text{h.o.t.}, \quad g(w, v) = v + \text{h.o.t.}, \\
  \frac{df}{dt} &= \frac{\lambda}{2} f, \quad \frac{dg}{dt} = \lambda g + \mu f^2.
\end{align*}
\] (1.7.9)

having a nonzero convergent radius, such that system (1.7.8) becomes

\[
\begin{align*}
  \frac{df}{dt} &= \frac{\lambda}{2} f, \quad \frac{dg}{dt} = \lambda g + \mu f^2.
\end{align*}
\] (1.7.10)

Moreover, if there exist two formal series of \( w \) and \( v \) of the form

\[
\begin{align*}
  \tilde{f} &= w + \text{h.o.t.}, \quad \tilde{g} = v + \text{h.o.t.},
\end{align*}
\] (1.7.11)

such that system (1.7.8) changes to

\[
\begin{align*}
  \frac{d\tilde{f}}{dt} &= \frac{\lambda}{2} \tilde{f}, \quad \frac{d\tilde{g}}{dt} = \lambda \tilde{g} + \mu \tilde{f}^2.
\end{align*}
\] (1.7.12)

Then

\[
\begin{align*}
  \tilde{f} &= f, \quad \tilde{g} = g + Cf^2,
\end{align*}
\] (1.7.13)

where \( C \) is a constant.

**Remark 1.7.1.** Lemma 1.7.2 implies that we can assume that the coefficient of \( w^2 \) in the power series of the \( g \) given by (1.7.9) is zero.

**Lemma 1.7.3.** The function \( f = f(w, v) \) given by (1.7.9) is an odd function of \( w \), i.e.,

\[
\begin{align*}
  f(w, v) &= w h(w^2, v),
\end{align*}
\] (1.7.14)

where \( h(u, v) \) is a power series of \( u \) and \( v \) with nonzero convergent radius and \( h(0, 0) = 1 \).

**Proof.** Write that

\[
\begin{align*}
  f(w, v) &= f_1(w, v) + f_2(w, v),
\end{align*}
\] (1.7.15)

where

\[
\begin{align*}
  f_1(w, v) &= \frac{f(w, v) - f(-w, v)}{2}, \\
  f_2(w, v) &= \frac{f(w, v) + f(-w, v)}{2}.
\end{align*}
\] (1.7.16)

Clearly, \( f_1 \) is an odd function of \( w \) and \( f_2 \) is an even function of \( w \). We see from (1.7.8) that

\[
\frac{df_1}{dt} \bigg|_{(1.7.8)} = \frac{df_1}{dw} \frac{dw}{dt} + \frac{df_1}{dv} \frac{dv}{dt}
\] (1.7.17)
is an odd function of \( w \) and

\[
\frac{df_2}{dt} \bigg|_{(1.7.8)} = \frac{df_2}{dw} \frac{dw}{dt} + \frac{df_2}{dv} \frac{dv}{dt} \tag{1.7.18}
\]

is an even function of \( w \). (1.7.10) and (1.7.15) follows that

\[
\frac{df_1}{dt} \bigg|_{(1.7.8)} + \frac{df_2}{dt} \bigg|_{(1.7.8)} = \frac{\lambda}{2} f_1 + \frac{\lambda}{2} f_2. \tag{1.7.19}
\]

Comparing the functions in the right and left sides of (1.7.19), we have

\[
\frac{df_1}{dt} \bigg|_{(1.7.8)} = \frac{\lambda}{2} f_1, \quad \frac{df_2}{dt} \bigg|_{(1.7.8)} = \frac{\lambda}{2} f_2. \tag{1.7.20}
\]

Because the function \( f \) in Lemma 1.7.2 is unique and \( f_1 = w + \text{h.o.t.} \). Therefore, we obtain \( f = f_1 \). It give rise to this lemma.

Similarly, we have

**Lemma 1.7.4.** The function \( g(w, v) \) in Lemma 1.7.2 is an even function of \( w \).

**Theorem 1.7.1.** There are two power series of \( x \) and \( y \) with a nonzero convergent radius of the form

\[
\xi = x + \text{h.o.t.}, \quad \eta = y + \text{h.o.t.}, \tag{1.7.21}
\]

such that system (1.7.1) becomes the following linear system:

\[
\frac{d\xi}{dt} = \lambda \xi, \quad \frac{d\eta}{dt} = \mu \xi + \lambda \eta. \tag{1.7.22}
\]

In addition, if there are another two formal series of \( x \) and \( y \)

\[
\tilde{\xi} = x + \text{h.o.t.}, \quad \tilde{\eta} = y + \text{h.o.t.}, \tag{1.7.23}
\]

such that system (1.7.1) reduces to

\[
\frac{d\tilde{\xi}}{dt} = \lambda \tilde{\xi}, \quad \frac{d\tilde{\eta}}{dt} = \mu \tilde{\xi} + \lambda \tilde{\eta}, \tag{1.7.24}
\]

then

\[
\tilde{\xi} = \xi, \quad \tilde{\eta} = \eta + C \xi, \tag{1.7.25}
\]

where \( C \) is a constant.

**Proof.** By Lemma 1.7.4 and Remark 1.7.1, the function \( g(w, v) \) in Lemma 1.7.2 can be written as

\[
g(w, v) = \eta(w^2, v), \tag{1.7.26}
\]
where $\eta(u, v)$ is a power series of $u$ and $v$ having a nonzero convergent radius:

$$\eta(u, v) = v + \sum_{\alpha+\beta=2}^{\infty} e_{\alpha\beta} u^\alpha v^\beta. \quad (1.7.27)$$

Let

$$\xi(u, v) = uh^2(u, v). \quad (1.7.28)$$

From (1.7.14) and (1.7.28) we have

$$\xi(w^2, v) = f^2(w, v). \quad (1.7.29)$$

Thus, (1.7.7), (1.7.10), (1.7.26) and (1.7.29) follow that

$$\left. \frac{d\xi}{dt}\right|_{(1.7.6)} = \lambda \xi, \quad \left. \frac{d\eta}{dt}\right|_{(1.7.6)} = \mu \xi + \lambda \eta. \quad (1.7.30)$$

(1.7.5), (1.7.30) and Lemma 1.7.2 implies the result of Theorem 1.7.1.

1.8 Integrability and Linearized Problem of Weak Critical Singular Point

Let the origin of system (1.5.1) be a weak critical singular point. By using a suitable linear transformation, system (1.5.1) can become the following second-order complex differential autonomous system

$$\frac{dx}{dt} = -y + \sum_{k=2}^{\infty} X_k(x, y) = X(x, y),$$

$$\frac{dy}{dt} = x + \sum_{k=2}^{\infty} Y_k(x, y) = Y(x, y), \quad (1.8.1)$$

which is analytic in a neighborhood of the origin, where $X_k(x, y), Y_k(x, y)$ are polynomials of degree $k$ of $x$ and $y$:

$$X_k(x, y) = \sum_{\alpha+\beta=k} A_{\alpha\beta} x^\alpha y^\beta, \quad Y_k(x, y) = \sum_{\alpha+\beta=k} B_{\alpha\beta} x^\alpha y^\beta. \quad (1.8.2)$$

Making the transformation

$$z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}, \quad (1.8.3)$$

system (1.8.1) becomes

$$\frac{dz}{dT} = z + \sum_{k=2}^{\infty} Z_k(z, w) = Z(z, w),$$

$$\frac{dw}{dT} = -w - \sum_{k=2}^{\infty} W_k(z, w) = -W(z, w), \quad (1.8.4)$$
where

\[ Z_k = Y_k - iX_k = \sum_{\alpha + \beta = k} a_{\alpha \beta} z^\alpha w^\beta, \]
\[ W_k = Y_k + iX_k = \sum_{\alpha + \beta = k} b_{\alpha \beta} w^\alpha z^\beta \]

are homogeneous polynomials of degree \( k \) of \( z \) and \( w \) \((k = 2, 3, \ldots)\), \( z, w, T \) are independent complex variables, \( a_{\alpha \beta}, b_{\alpha \beta} \) are independent complex constants.

We call that system (1.8.1) is the associated system of (1.8.4) and vice versa. We see from (1.8.5) that \( \forall (\alpha, \beta) \), \( A_{\alpha, \beta}, B_{\alpha, \beta} \) are real coefficients if and only if \( \forall (\alpha, \beta) \), \( b_{\alpha \beta} = \bar{a}_{\alpha \beta} \).

If \( \forall (\alpha, \beta) \), \( A_{\alpha, \beta}, B_{\alpha, \beta} \) are real coefficients and \( x, y, t \) are all real variables, then system (1.8.1) is a real planar differential autonomous system, for which the origin is a center or a focus. While if \( \forall (\alpha, \beta) \), \( a_{\alpha \beta}, b_{\alpha \beta} \) are real coefficients and \( z, w, T \) are all real variables, then system (1.8.4) is a real planar differential autonomous system, for which the origin is a weak saddle. The monograph [Amelikin etc, 1982] proved that

**Theorem 1.8.1.** For any given \( \tilde{c}_{k+1, k} \) and \( \tilde{d}_{k+1, k} \), \( k = 1, 2, \ldots \), one can determine successively other \( \tilde{c}_{k, j} \) and \( \tilde{d}_{k, j} \) and derive uniquely the formal series

\[ \tilde{\xi} = z + \sum_{k+j=2}^{\infty} \tilde{c}_{kj} z^k w^j, \]
\[ \tilde{\eta} = w + \sum_{k+j=2}^{\infty} \tilde{d}_{kj} w^k z^j, \]  

such that by formal variable transformation (1.8.6), system (1.8.4) reduces to the following normal form

\[ \frac{d\tilde{\xi}}{dT} = \tilde{\xi} + \tilde{\xi} \sum_{k=1}^{\infty} \tilde{p}_k (\tilde{\xi} \tilde{\eta})^k, \]
\[ \frac{d\tilde{\eta}}{dT} = -\tilde{\eta} - \tilde{\eta} \sum_{k=1}^{\infty} \tilde{q}_k (\tilde{\xi} \tilde{\eta})^k. \]  

**Proof.** We denote

\[ \tilde{\xi} = \sum_{k=1}^{\infty} f_k(z, w), \quad \tilde{\eta} = \sum_{k=1}^{\infty} g_k(z, w), \]  

where \( f_1 = z, g_1 = w, f_k(z, w), g_k(z, w) \) are homogeneous polynomials of degree \( k \) of \( z, w \). Write that

\[ f_k(z, w) = \sum_{\alpha + \beta = k} f_{\alpha \beta} z^\alpha w^\beta, \quad g_k(z, w) = \sum_{\alpha + \beta = k} g_{\alpha \beta} z^\alpha w^\beta. \]
where \( \Phi_k(z, w) \), \( \Psi_k(z, w) \) are homogeneous polynomials of degree \( k \) of \( z, w \). From (1.8.4) and (1.8.8), we have

\[
\frac{d\xi}{dT} - \xi = \sum_{m=2}^{\infty} \left[ \left( \frac{\partial f_m}{\partial z} z - \frac{\partial f_m}{\partial w} w - f_m \right) + F_m \right],
\]

\[
\frac{d\tilde{\eta}}{dT} + \tilde{\eta} = \sum_{m=2}^{\infty} \left[ \left( \frac{\partial g_m}{\partial z} z - \frac{\partial g_m}{\partial w} w + g_m \right) + G_m \right],
\]

where

\[
F_m = Z_m + \sum_{j=2}^{m-1} \left( \frac{\partial f_j}{\partial z} Z_{m-j+1} - \frac{\partial f_j}{\partial w} W_{m-j+1} \right),
\]

\[
G_m = W_m + \sum_{j=2}^{m-1} \left( \frac{\partial g_j}{\partial w} W_{m-j+1} - \frac{\partial g_j}{\partial z} Z_{m-j+1} \right)
\]

are homogeneous polynomials of degree \( m \) of \( z, w \). From (1.8.9), (1.8.10) and (1.8.7), we obtain

\[
\sum_{m=2}^{\infty} \left( \frac{\partial f_m}{\partial z} z - \frac{\partial f_m}{\partial w} w - f_m \right) = \sum_{m=2}^{\infty} \left( \Phi_m - F_m \right) + \sum_{k=1}^{\infty} \tilde{p}_k z^{k+1} w^k,
\]

\[
\sum_{m=2}^{\infty} \left( \frac{\partial g_m}{\partial w} w - \frac{\partial g_m}{\partial z} z - g_m \right) = \sum_{m=2}^{\infty} \left( \Psi_m - G_m \right) + \sum_{k=1}^{\infty} \tilde{q}_k w^{k+1} z^k,
\]

where

\[
\frac{\partial f_m}{\partial z} z - \frac{\partial f_m}{\partial w} w - f_m = \sum_{\alpha + \beta = m} (\alpha - \beta - 1) \tilde{c}_{\alpha\beta} z^\alpha w^\beta,
\]

\[
\frac{\partial g_m}{\partial w} w - \frac{\partial g_m}{\partial z} z - g_m = \sum_{\alpha + \beta = m} (\alpha - \beta - 1) \tilde{d}_{\alpha\beta} w^\alpha z^\beta.
\]

For \( F_m, G_m, \Phi_m, \Psi_m \) in (1.8.12), we see from (1.8.8) (1.8.9) and (1.8.11) that for any positive integer \( k \), \( \Phi_{2k}, \Phi_{2k+1} \) and \( \Psi_{2k+1} \) are polynomials of \( f_1, f_2, \cdots, f_{2k-1}, g_1, g_2, \cdots, g_{2k-1}, p_1, p_2, \cdots, p_{k-1}, q_1, q_2, \cdots, q_{k-1} \), which have positive rational coefficients. In addition, for any \( m \), \( F_m, G_m \) only depend on \( f_1, f_2, \cdots, f_{m-1}, g_1, g_2, \cdots, g_{m-1} \). Write that
\[ \Phi_m - F_m = \sum_{\alpha + \beta = m} C_{\alpha \beta} z^\alpha w^\beta, \quad \Psi_m - G_m = \sum_{\alpha + \beta = m} D_{\alpha \beta} w^\alpha z^\beta. \] (1.8.14)

From (1.8.13) and (1.8.14), we know that (1.8.12) holds if and only if for any positive integer \( k \),
\[
\sum_{\alpha + \beta = 2k} (\alpha - \beta - 1) \tilde{c}_{\alpha \beta} z^\alpha w^\beta = \sum_{\alpha + \beta = 2k} C_{\alpha \beta} z^\alpha w^\beta, \\
\sum_{\alpha + \beta = 2k} (\alpha - \beta - 1) \tilde{d}_{\alpha \beta} w^\alpha z^\beta = \sum_{\alpha + \beta = 2k} D_{\alpha \beta} w^\alpha z^\beta. \] (1.8.15)

and
\[
\sum_{\alpha + \beta = 2k + 1} (\alpha - \beta - 1) \tilde{c}_{\alpha \beta} z^\alpha w^\beta = \tilde{p}_k z^{k+1} w^k + \sum_{\alpha + \beta = 2k + 1} C_{\alpha \beta} z^\alpha w^\beta, \\
\sum_{\alpha + \beta = 2k + 1} (\alpha - \beta - 1) \tilde{d}_{\alpha \beta} w^\alpha z^\beta = \tilde{q}_k w^{k+1} z^k + \sum_{\alpha + \beta = 2k + 1} D_{\alpha \beta} w^\alpha z^\beta. \] (1.8.16)

Because in (1.8.16), all coefficients of \( \tilde{c}_{k+1,k}, \tilde{d}_{k+1,k} \) are zeros. Hence, all \( \tilde{c}_{k+1,k} \) and \( \tilde{d}_{k+1,k} \) can be given as arbitrary constants. By (1.8.15) and (1.8.16), for any two natural numbers \( \alpha, \beta \) satisfying \( \alpha + \beta \geq 2 \) and \( \alpha - \beta - 1 \neq 0 \), \( \tilde{c}_{\alpha \beta}, \tilde{d}_{\alpha \beta} \) can be uniquely determined by the recursive formulas
\[ \tilde{c}_{\alpha \beta} = \frac{C_{\alpha \beta}}{\alpha - \beta - 1}, \quad \tilde{d}_{\alpha \beta} = \frac{D_{\alpha \beta}}{\alpha - \beta - 1}. \] (1.8.17)

In addition, \( \tilde{p}_k, \tilde{q}_k \) can be derived uniquely by the recursive formulas
\[ \tilde{p}_k = -C_{k+1,k}, \quad \tilde{q}_k = -D_{k+1,k}. \] (1.8.18)

This completes the proof of this theorem. \( \square \)

**Definition 1.8.1.** Suppose that by means of formal transformation (1.8.6), system (1.8.4) can be reduced to the normal form (1.8.7). Then, the transformation (1.8.6) is called a normal transformation in a neighborhood of the origin of system (1.8.4). System (1.8.7) is called a normal form corresponding to the transformation (1.8.6).

Let (1.8.6) be a normal transformation in a neighborhood of the origin of system (1.8.4) and \( \tilde{c}_{k+1,k} = \tilde{d}_{k+1,k} = 0, k = 1, 2, \ldots \). Then (1.8.6) is called a standard normal transformation in a neighborhood of the origin of system (1.8.4), which is written by
\[ \xi = z + \sum_{k+j=2}^\infty c_{kj} z^k w^j = \xi(z, w), \quad \eta = w + \sum_{k+j=2}^\infty d_{kj} w^k z^j = \eta(z, w). \] (1.8.19)
The normal form derived by standard normal transformation is called standard normal form, which is written by

\[
\begin{align*}
\frac{d\xi}{dT} &= \xi + \xi \sum_{k=1}^{\infty} p_k(\xi \eta)^k, \\
\frac{d\eta}{dT} &= -\eta - \eta \sum_{k=1}^{\infty} q_k(\xi \eta)^k.
\end{align*}
\] (1.8.20)

From Theorem 1.8.1 and its proof, we have

**Corollary 1.8.1.** Let

\[
\begin{align*}
\xi^* &= z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \\
\eta^* &= w + \sum_{k+j=2}^{\infty} d_{kj} w^k z^j
\end{align*}
\] (1.8.21)

and (1.8.6) be two standard normal transformations in a neighborhood of the origin of system (1.8.4). If for any positive integer \( k \), we have \( c_{k+1,k}^* = \tilde{c}_{k+1,k}, d_{k+1,k}^* = \tilde{d}_{k+1,k} \). Then \( \xi^* = \tilde{\xi}, \eta^* = \tilde{\eta} \).

**Remark 1.8.1.** From Corollary 1.8.1 and the proof of Theorem 1.8.1, we know that the standard normal transformation is unique in a neighborhood of the origin of system (1.8.4). Moreover, all \( c_{kj}, d_{kj}, p_k \) and \( q_k \) in (1.8.19) and (1.8.20) are polynomials of \( a_{\alpha \beta} \)'s, \( b_{\alpha \beta} \)'s. Their coefficients are all rational numbers.

[Amelikin etc, 1982] proved that

**Theorem 1.8.2.** If for any positive integer \( k \), we have \( p_k = q_k \). Then the formal series of \( \xi, \eta \) in the standard normal transformation have nonzero convergent radius.

**Theorem 1.8.3.** Let \( H = \xi \eta \) and \( F(H), G(H) \) be any unit formal power series of \( H \). Then,

\[
\tilde{\xi} = \xi F(H), \quad \tilde{\eta} = \eta G(H)
\] (1.8.22)

is a normal transformation in a neighborhood of the origin of system (1.8.4).

**Proof.** From equations (1.8.20) and (1.8.22), we have

\[
\begin{align*}
\frac{d\tilde{\xi}}{dT} &= \tilde{\xi} \Phi(H), \\
\frac{d\tilde{\eta}}{dT} &= -\tilde{\eta} \Psi(H),
\end{align*}
\] (1.8.23)

where \( \Phi(H), \Psi(H) \) are the following unit formal power series of \( H \):

\[
\begin{align*}
\Phi(H) &= 1 + \sum_{k=1}^{\infty} p_k H^k + \frac{F'(H)}{F(H)} \sum_{k=1}^{\infty} (p_k - q_k) H^{k+1}, \\
\Psi(H) &= 1 + \sum_{k=1}^{\infty} q_k H^k - \frac{G'(H)}{G(H)} \sum_{k=1}^{\infty} (p_k - q_k) H^{k+1}.
\end{align*}
\] (1.8.24)
Denote that 
\[ \tilde{H} = \tilde{\xi} \tilde{\eta} = HF(H)G(H) = H + h.o.t. \]  
(1.8.25)

Thus, \( H \) can be written as the formal series of \( \tilde{H} \)
\[ H = \tilde{H} + o(\tilde{H}). \]  
(1.8.26)

(1.8.23) and (1.8.26) follow the assertion of this theorem. \( \square \)

**Theorem 1.8.4.** Let (1.8.6) be a normal transformation in a neighborhood of the origin of system (1.8.4). Then, there exists unit formal series of \( H \) as follows:
\[ F(H) = 1 + \sum_{k=1}^{\infty} A_k H^k, \quad G(H) = 1 + \sum_{k=1}^{\infty} B_k H^k, \]  
(1.8.27)
such that \( \tilde{\xi} = \xi F(H), \tilde{\eta} = \eta G(H) \), where \( A_k, B_k \) are the given constant coefficients.

**Proof.** Denote that
\[ f = \xi F(H) = z + \sum_{k+j=2}^{\infty} c'_{kj} z^k w^j, \]
\[ g = \eta G(H) = w + \sum_{k+j=2}^{\infty} d'_{kj} w^k z^j. \]  
(1.8.28)

Since the functions of \( \xi, \eta \) in the standard normal transformation are determined uniquely, we only need to find \( A_k, B_k \) of \( f \) and \( g \) \((k = 1, 2, \cdots)\). From (1.8.27) and (1.8.28), \( f \) and \( g \) can be written as
\[ f(z, w) = z + \sum_{k=1}^{\infty} A_k z^{k+1} w^k + \sum_{k=2}^{\infty} f_k(z, w), \]
\[ g(z, w) = w + \sum_{k=1}^{\infty} B_k w^{k+1} z^k + \sum_{k=2}^{\infty} g_k(z, w), \]  
(1.8.29)

where \( f_k(z, w), g_k(z, w) \) are homogeneous polynomials of degree \( k \) of \( z, w \). For any positive integer \( k \), \( f_{2k+1} \) only depend on \( A_1, A_2, \cdots, A_{k-1} \), while \( g_{2k+1} \) only depend on \( B_1, B_2, \cdots, B_{k-1} \). From (1.8.28) and (1.8.29), we can take appropriately \( A_k, B_k \), such that for any positive integer \( k \), \( c'_{k+1,k} = \tilde{c}_{k+1,k}, d'_{k+1,k} = \tilde{d}_{k+1,k} \) hold. By Corollary 1.8.1, we obtain the conclusion of this theorem. \( \square \)

From equation (1.8.20) and Proposition 1.1.2, we obtain the following three important formulas

**Theorem 1.8.5.** Denote that
\[ \mu_k = p_k - q_k, \quad \tau_k = p_k + q_k, \quad k = 1, 2, \cdots. \]  
(1.8.30)
For system (1.8.4), we have

\[
\frac{dH}{dT} = \sum_{k=1}^{\infty} \mu_k H^{k+1},
\]
\[
\frac{d\Omega}{dT} = \frac{1}{2i} \left( 2 + \sum_{k=1}^{\infty} \tau_k H^k \right)
\]  
(1.8.31)

and

\[
\frac{\partial}{\partial z}(JZ) - \frac{\partial}{\partial w}(JW) = J \sum_{k=1}^{\infty} (k+1)\mu_k (\xi \eta)^k,
\]  
(1.8.32)

where

\[
H = \xi \eta, \quad \Omega = \frac{1}{2i} \ln \frac{\xi}{\eta}, \quad J(z, w) = \begin{vmatrix} \frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial w} \\ \frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial w} \end{vmatrix}.
\]  
(1.8.33)

In [Liu Y.R. et al, 1989] and [Liu Y.R. et al, 2003a], we introduced the following definition.

**Definition 1.8.2.** Let \( \mu_0 = \tau_0 = 0 \). For any positive integers \( k, \mu_k = p_k - q_k \) is called the \( k \)-th singular point value of the origin of system (1.8.4), while \( \tau_k = p_k + q_k \) is called the \( k \)-th period constant of the origin of system (1.8.4).

If there exists a positive integer \( m \), such that \( \mu_0 = \mu_1 = \cdots = \mu_{m-1} = 0, \mu_m \neq 0 \), then the origin of is called a \( m \)-order weak critical singular point of system (1.8.4). If for all \( k \), we have \( \mu_k = 0 \). Then the origin of system (1.8.4) is called a complex center.

**Theorem 1.8.6.** Let (1.8.6) be any normal transformation in a neighborhood of the origin of system (1.8.4). Denote that \( \tilde{H} = \xi \tilde{\eta} \). When the origin of system (1.8.4) is a \( m \)-order weak critical singular point, we have

\[
\frac{d\tilde{H}}{dT} = \mu_m \tilde{H}^{m+1} + \text{h.o.t.}
\]  
(1.8.34)

When the origin of system (1.8.4) is a complex center, we have

\[
\frac{d\tilde{H}}{dT} = 0.
\]  
(1.8.35)

**Proof.** Let (1.8.6) be any normal transformation in a neighborhood of the origin of system (1.8.4). By Theorem 1.8.4, there exist two unit formal series \( F(H), G(H) \) of \( H \), such that

\[
\tilde{H} = HF(H)G(H) = H + \text{h.o.t.}
\]  
(1.8.36)
From (1.8.36) and Theorem 1.8.5, we have
\[
\frac{d\tilde{H}}{dT} = (FG + HF'G + HFG') \frac{dH}{dT} = (FG + HF'G + HFG') \sum_{k=1}^{\infty} \mu_k H^{k+1},
\] (1.8.37)
where \(FG + HF'G + HFG'\) is a unit formal series of \(H\). By using (1.8.36), we can represent \(H\) as a formal series of \(\tilde{H}\):
\[
H = \tilde{H} + \text{h.o.t.}
\] (1.8.38)
Hence, (1.8.37) and (1.8.38) follows the conclusion of this theorem.

**Lemma 1.8.1.** Let \(F\) be a formal first integral in a neighborhood of the origin of system (1.8.4). Then, \(F\) can be written as a formal power series of \(\xi, \eta\) as follows:
\[
F = C_{mm}(\xi\eta)^m + \sum_{\alpha + \beta = 2m+1}^{\infty} C_{\alpha\beta} \xi^\alpha \eta^\beta, \quad C_{mm} \neq 0,
\] (1.8.39)
where \(m\) is a positive integer.

**Proof.** Solving \(z\) and \(w\) from (1.8.19), we obtain
\[
z = z(\xi, \eta) = \xi + \text{h.o.t.}, \quad w = w(\xi, \eta) = \eta + \text{h.o.t.}
\] (1.8.40)
Hence, \(F\) can be written as the following formal power series of \(\xi, \eta\):
\[
F = \sum_{k=n}^{\infty} F_n(\xi, \eta) = \sum_{\alpha + \beta = n}^{\infty} C_{\alpha\beta} \xi^\alpha \eta^\beta,
\] (1.8.41)
where \(n\) is a positive integer. \(F_k(\xi, \eta)\) is a homogeneous polynomial of degree \(k\) of \(\xi, \eta\). \(F_n\) is a non-zero polynomial. From (1.8.41) and (1.8.20), we have
\[
0 = \frac{dF}{dT} = \frac{\partial F}{\partial \xi} \frac{d\xi}{dT} + \frac{\partial F}{\partial \eta} \frac{d\eta}{dT}
\]
\[
= \frac{\partial F_n}{\partial \xi} \xi - \frac{\partial F_n}{\partial \eta} \eta + \text{h.o.t.}
\]
\[
= \sum_{\alpha + \beta = n} (\alpha - \beta) C_{\alpha\beta} \xi^\alpha \eta^\beta + \text{h.o.t.}
\] (1.8.42)
It implies that
\[
\sum_{\alpha + \beta = n} (\alpha - \beta) C_{\alpha\beta} \xi^\alpha \eta^\beta = 0.
\] (1.8.43)
Since \(F_n\) is a non-zero polynomial, we see from (1.8.43) that \(n = 2m, H_{2m} = C_{mm} \xi^m \eta^m, C_{mm} \neq 0\). Thus, the assertion of this lemma holds. \(\square\)
1.8 Integrability and Linearized Problem of Weak Critical Singular Point

**Theorem 1.8.7.** System (1.8.4) has a formal first integral in a neighborhood of the origin if and only if all singular point values of the origin are zero.

*Proof.* First, we prove the sufficiency of theorem. If all singular point values are zeros, then by Theorem 1.8.2, the power series of $\xi, \eta$ have a nonzero convergent radius. Theorem 1.8.5 implies that $H = \xi \eta$ is a first integral in a neighborhood of the origin, which is a power series of $z, w$ with a a nonzero convergent radius.

Second, we prove the necessity of theorem. Suppose that system (1.8.4) has a formal first integral $F$ in a neighborhood of the origin. By Lemma 1.8.1, $F$ can be written as the form of (1.8.41). From (1.8.40) and (1.8.20), we have

$$
\frac{dF}{dT} = \left( mC_{mm}^{m} \eta^{m} + \sum_{\alpha + \beta = 2m+1}^{\infty} \alpha C_{\alpha \beta} \xi^{\alpha} \eta^{\beta} \right) \left( 1 + \sum_{k=1}^{\infty} p_{k} \xi^{k} \eta^{k} \right)
- \left( mC_{mm}^{m} \xi^{m} \eta^{m} + \sum_{\alpha + \beta = 2m+1}^{\infty} \beta C_{\alpha \beta} \xi^{\alpha} \eta^{\beta} \right) \left( 1 + \sum_{k=1}^{\infty} q_{k} \xi^{k} \eta^{k} \right).
$$

(1.8.44)

It can be represented by a formal power series of $\xi, \eta$ as follows:

$$
\frac{dF}{dT} = \sum_{\alpha + \beta = 2m}^{\infty} D_{\alpha \beta} \xi^{\alpha} \eta^{\beta}.
$$

(1.8.45)

Since $F$ is a formal first integral in a neighborhood of the origin for system (1.8.4), therefore, all $D_{\alpha \beta}$ must be zeros. From (1.8.44) and (1.8.45), we have

$$
0 = \sum_{k=m}^{\infty} D_{kk} \xi^{k} \eta^{k}
= \left( mC_{mm}^{m} \eta^{m} + \sum_{j=m+1}^{\infty} j C_{jj} \xi^{j} \eta^{j} \right) \left( 1 + \sum_{k=1}^{\infty} p_{k} \xi^{k} \eta^{k} \right)
- \left( mC_{mm}^{m} \xi^{m} \eta^{m} + \sum_{j=m+1}^{\infty} j C_{jj} \xi^{j} \eta^{j} \right) \left( 1 + \sum_{k=1}^{\infty} q_{k} \xi^{k} \eta^{k} \right)
= \left( mC_{mm}^{m} \xi^{m} \eta^{m} + \sum_{j=m+1}^{\infty} j C_{jj} \xi^{j} \eta^{j} \right) \sum_{k=1}^{\infty} \mu_{k} \xi^{k} \eta^{k} = 0.
$$

(1.8.46)

Because of $C_{mm} \neq 0$. (1.8.46) follows that

$$
\sum_{k=1}^{\infty} \mu_{k} \xi^{k} \eta^{k} = 0.
$$

(1.8.47)

It means that for all $k$, $\mu_{k} = 0$. \qed
Theorem 1.8.8. If the origin of system (1.8.4) is a complex center, then, in a neighborhood of the origin, any formal first integral $F$ of system (1.8.4) can be represented by

$$F = F(H),$$  \hspace{1cm} (1.8.48)

where $F(H)$ is a formal series of $H$.

Proof. Let the origin of system (1.8.4) be a complex center. Then, $H(z, w) = \xi \eta$ is an analytic first integral in a neighborhood of the origin. Suppose that $F$ is a first integral in a neighborhood of the origin of system (1.8.4), which is represented as a formal series of $\xi, \eta$:

$$F = \sum_{\alpha+\beta=1}^{\infty} C_{\alpha\beta} \xi^\alpha \eta^\beta. \hspace{1cm} (1.8.49)$$

Write that

$$F^* = \sum_{k=1}^{\infty} C_{kk} (\xi \eta)^k, \quad \tilde{F} = F - F^*. \hspace{1cm} (1.8.50)$$

Clearly, $F^*$ is also a formal first integral in a neighborhood of the origin of system (1.8.4). Lemma 1.8.1 follows that $\tilde{F}$ is not a formal first integral in a neighborhood of the origin of system (1.8.4). Since $\tilde{F}$ is the difference of two formal first integrals. So that, $\tilde{F} \equiv 0$. \qed

Theorem 1.8.8 gives rise to the following conclusion.

**Theorem 1.8.9.** If the origin of system (1.8.4) is a complex center, then in a neighborhood of the origin, any analytic first integral of system (1.8.4) can be written as a power series of $H$ with a nonzero convergent radius.

**Theorem 1.8.10.** The origin of system (1.8.4) is a complex center if and only if there exists an analytic integrating factor $M(z, w)$ in a neighborhood of the origin with $M(0, 0) \neq 0$.

Proof. The sufficiency of the conclusion is obvious. We prove the necessity. If the origin of system (1.8.4) is a complex center, Theorem 1.8.2 and Theorem 1.8.5 tell us that the Jacobian determinant $J(z, w)$ of $\xi, \eta$ with respect to $z, w$ is an analytic integral factor in a neighborhood of the origin and $J(0, 0) = 1$. \qed

**Theorem 1.8.11.** In a neighborhood of the origin, system (1.8.4) is linearizable if and only if

$$p_k = q_k = 0, \quad k = 1, 2, \ldots. \hspace{1cm} (1.8.51)$$

Proof. If (1.8.51) holds, then system (1.8.20) is just a linear system.
Suppose that system (1.8.4) can be linearized in a neighborhood of the origin. Thus, there exists a normal transformation (1.8.6) in a neighborhood of origin, such that system (1.8.4) is reduced to the linear system

\[
\frac{d\tilde{\xi}}{dT} = \tilde{\xi}, \quad \frac{d\tilde{\eta}}{dT} = -\tilde{\eta}.
\]  

(1.8.52)

Denote that

\[
\tilde{H} = \tilde{\xi}\tilde{\eta}, \quad \tilde{\Omega} = \frac{1}{2i} \ln \frac{\tilde{\xi}}{\tilde{\eta}}.
\]  

(1.8.53)

From (1.8.52), we have

\[
\frac{d\tilde{H}}{dT} \equiv 0, \quad \frac{d\tilde{\Omega}}{dT} \equiv -i.
\]  

(1.8.54)

It follows that \(\tilde{H}\) is a first integral in a neighborhood of the origin of system (1.8.4). Hence, by Theorem 1.8.7, we obtain

\[
\mu_k = 0, \quad k = 1, 2, \ldots.
\]  

(1.8.55)

From Theorem 1.8.4, there are two unit formal series \(F(H)\) and \(G(H)\) of \(H\), such that

\[
\tilde{\xi} = \xi F(H), \quad \tilde{\eta} = \eta G(H).
\]  

(1.8.56)

From (1.8.33), (1.8.53) and (1.8.56), we have

\[
\tilde{\Omega} - \Omega = \frac{1}{2i} \ln \frac{F(H)}{G(H)}.
\]  

(1.8.57)

Because the right side of (1.8.57) is a formal power series of \(H\), from (1.8.31), (1.8.55) and (1.8.57), we obtain

\[
\frac{d\tilde{\Omega}}{dT} = \frac{d\Omega}{dT} = \frac{1}{2i} \left(2 + \sum_{k=1}^{\infty} \tau_k H^k\right).
\]  

(1.8.58)

From (1.8.54) and (1.8.58), we have

\[
\tau_k = 0, \quad k = 1, 2, \ldots.
\]  

(1.8.59)

Thus, (1.8.30), (1.8.55) and (1.8.59) give rise to (1.8.51).

\[\square\]

**Theorem 1.8.12.** In system (1.8.4), if for all \(\alpha\) and \(\beta\), the relationships \(b_{\alpha\beta} = \bar{a}_{\alpha\beta}\) hold. Then in (1.8.19) and (1.8.20), we have that \(\forall k, j, c_{kj} = \bar{d}_{kj}, p_k = \bar{q}_k\).

**Proof.** The relationships \(\forall (\alpha, \beta), b_{\alpha\beta} = \bar{a}_{\alpha\beta}\) imply that \(A_{\alpha\beta}, B_{\alpha\beta}\) are real numbers in (1.8.2). Let \(x, y, t\) be real variables. Then (1.8.1) is real planar differential system. From (1.8.3), we have

\[
\bar{z} = w, \quad \bar{w} = z, \quad T^* = -T.
\]  

(1.8.60)
By (1.8.19) and (1.8.60), we obtain
\[
\bar{\eta} = z + \sum_{k+j=2}^{\infty} \bar{d}_{kj} z^k w^j, \quad \bar{\xi} = w + \sum_{k+j=2}^{\infty} \bar{c}_{kj} w^k z^j.
\] (1.8.61)

Denote that
\[
\xi^* = \bar{\eta}, \quad \eta^* = \bar{\xi}.
\] (1.8.62)

Making the conjugated transformation on the two sides of (1.8.20), from (1.8.60) and (1.8.62), we have
\[
\frac{d\xi^*}{dT^*} = \xi^* + \sum_{k=1}^{\infty} \tilde{q}_k (\xi^* \eta^*)^k, \quad \frac{d\eta^*}{dT^*} = -\eta^* - \sum_{k=1}^{\infty} \tilde{p}_k (\xi^* \eta^*)^k.
\] (1.8.63)

(1.8.61) and (1.8.63) follows that (1.8.62) is a standard normal transformation in a neighborhood of the origin of system (1.8.4). The uniqueness of the standard normal transformation gives that
\[
\eta = \bar{\xi}, \quad \tilde{q}_k = \tilde{p}_k, \quad k = 1, 2, \ldots.
\] (1.8.64)

It follows the conclusion of this theorem.

For system (1.8.1), consider the normal transformation
\[
u = \frac{\xi(x + iy, x - iy) + \eta(x + iy, x - iy)}{2} = x + \sum_{k+j=2}^{\infty} c'_{kj} x^k y^j,
\]
\[v = \frac{\xi(x + iy, x - iy) - \eta(x + iy, x - iy)}{2i} = y + \sum_{k+j=2}^{\infty} d'_{kj} x^k y^j.
\] (1.8.65)

Theorem 1.8.12 implies that if all coefficients on the right side of system (1.8.1) are real numbers, then \(u, v\) are power series of \(x, y\) having real coefficients. (1.8.20) and (1.8.3) follow the following conclusion given by [Amelikin etc, 1982].

**Theorem 1.8.13.** By using formal transformation (1.8.65), complex autonomous differential system (1.8.1) can become a normal form as follows:
\[
\frac{du}{dt} = -v + \frac{1}{2} \sum_{k=1}^{\infty} (\sigma_k u - \tau_k v)(u^2 + v^2)^k = U(u, v),
\]
\[
\frac{dv}{dt} = u + \frac{1}{2} \sum_{k=1}^{\infty} (\tau_k u + \sigma_k v)(u^2 + v^2)^k = V(u, v),
\] (1.8.66)

where
\[
\sigma_k = i(p_k - q_k), \quad \tau_k = p_k + q_k
\] (1.8.67)
and all \(\sigma_k, \tau_k, c'_{kj}\) and \(d'_{kj}\) are polynomials of \(A_{\alpha\beta}, B_{\alpha\beta}\) with rational coefficients.
Proposition 1.1.2 and Theorem 1.8.13 imply the following three important formulas.

**Theorem 1.8.14.** For system (1.8.1), we have

\[
\frac{d\mathcal{H}}{dt} = \sum_{k=1}^{\infty} \sigma_k \mathcal{H}^{k+1}, \quad \frac{d\omega}{dt} = 1 + \frac{1}{2} \sum_{k=1}^{\infty} \tau_k \mathcal{H}^k
\]  

(1.8.68)

and

\[
\frac{\partial(JX)}{\partial x} + \frac{\partial(JY)}{\partial y} = J \left( \frac{\partial U}{\partial u} + \frac{\partial V}{\partial v} \right) = J \sum_{k=1}^{\infty} (k+1) \sigma_k \mathcal{H}^k,
\]  

(1.8.69)

where

\[
\mathcal{H} = u^2 + v^2, \quad \omega = \arctan \frac{v}{u}, \quad J = \begin{vmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{vmatrix}.
\]  

(1.8.70)

Theorem 1.8.7 gives the following conclusion.

**Theorem 1.8.15.** System (1.8.4) has a formal first integral in a neighborhood of the origin if and only if all \( \sigma_k = 0 \).

From Theorem 1.8.8 and Theorem 1.8.9, we have

**Theorem 1.8.16.** If the origin of system (1.8.1) is a complex center, then any first integral \( \mathcal{F} \) in a neighborhood of the origin of system (1.8.1) can be written by

\[
\mathcal{F} = \mathcal{F}(\mathcal{H}), \quad \text{where } \mathcal{F}(\mathcal{H}) \text{ is a formal series of } \mathcal{H}. \quad \text{In addition, any analytic first integral in a neighborhood of the origin of system (1.8.1) can be represented as a power series of } \mathcal{H} \text{ with a nonzero convergent radius.}
\]  

(1.8.71)

Theorem 1.8.10 derives the following conclusion.

**Theorem 1.8.17.** The origin of system (1.8.1) is a complex center if and only if there exists an analytic integral factor \( M(x, y) \) in a neighborhood of the origin and \( M(0,0) \neq 0 \).

By Theorem 1.8.11, we have

**Theorem 1.8.18.** System (1.8.1) is linearizable in a neighborhood of the origin if and only if for all positive integer \( k \), \( \sigma_k = 0 \), \( \tau_k = 0 \).

**Definition 1.8.3.** Suppose that the functions of the right side of system (1.8.1) satisfy

\[
X(x,-y) = -X(x,y), \quad Y(x,-y) = Y(x,y).
\]  

(1.8.72)
We say that the functions of the right side of system (1.8.1) are symmetric with respect to the coordinate $x$.

Suppose that the functions of the right side of system (1.8.1) satisfy
\[
X(-x, y) = X(x, y), \quad Y(-x, y) = -Y(x, y), \tag{1.8.73}
\]
we say that the functions of the right side of system (1.8.1) are symmetric with respect to the coordinate $y$.

If one of (1.8.72) and (1.8.73) satisfies, we say that system (1.8.1) is a symmetric system with respect to a coordinate.

From Corollary 1.4.1, we have

**Theorem 1.8.19 (The symmetric principle).** Suppose that (1.8.1) is a symmetric system with respect to a coordinate. Then, it has a analytic first integral in a neighborhood of the origin.

Since the coefficients of the right side of system (1.8.1) can be complex, Theorem 1.8.19 expands the symmetric principle for the center-focus problem in real planar differential autonomous systems.

**Theorem 1.8.20 (The anti-symmetric principle).** Suppose that the origin of system (1.8.1) is a complex center. Then, there exist two power series $u, v$ of $x, y$:
\[
u = x + h.o.t., \quad v = y + h.o.t.. \tag{1.8.74}
\]
with a nonzero convergent radius, such that by transformation (1.8.74), system (1.8.1) becomes a symmetric system.

**Proof.** Since the origin of system (1.8.1) is a complex center, by Theorem 1.8.2, the functions of (1.8.65) are power series of $x, y$ with nonzero convergent radius. From Theorem 1.8.13 and Theorem 1.8.15, we see that by the transformation (1.8.65), system (1.8.1) can become the following symmetric system:
\[
\begin{align*}
\frac{du}{dt} &= -v - \frac{1}{2} v \sum_{k=1}^{\infty} \tau_k (u^2 + v^2)^k, \\
\frac{dv}{dt} &= u + \frac{1}{2} u \sum_{k=1}^{\infty} \tau_k (u^2 + v^2)^k. \tag{1.8.75}
\end{align*}
\]

By using the above two theorems, we obtain a method to check if the origin is a complex center. In fact, for a given system of the form (1.8.1), if we find a suitable transformation to make this system become a symmetric system, then the origin of the system is a complex center.
Example 1.8.1. Consider the real planar differential system

\[
\begin{align*}
\frac{dx}{dt} &= -y + x^2, \\
\frac{dy}{dt} &= x + 2x^3 - 5ax^8 - 2(1 - 4ax^5)xy - 4ax^2y^3 + ay^4.
\end{align*}
\] (1.8.76)

By the transformation \(u = x, v = y - x^2\), system (1.8.76) becomes

\[
\begin{align*}
\frac{du}{dt} &= -v, \\
\frac{dv}{dt} &= u - 6au^4v^2 + av^4.
\end{align*}
\] (1.8.77)

Letting \(\xi = v^2\), the above system reduces to the Riccati equation

\[
\frac{d\xi}{du} = -2(u - 6au^4\xi + a\xi^2). 
\] (1.8.78)

The functions of the right side of system (1.8.77) is symmetric with respect to the variable \(v\). Therefore, Theorem 1.8.19 follows that the origin of system (1.8.76) is a center.

We now consider the existence of integrating factor in a neighborhood of the origin when the origin of system (1.8.20) is a \(m\)-order weak critical singular point. It is easy to show that the following conclusion holds.

Theorem 1.8.21. Let the origin of system (1.8.20) be a \(m\)-order weak critical singular point. Then, in a neighborhood of the origin, system (1.8.20) has the following integrating factor:

\[
M(\xi, \eta) = \frac{1}{H^{m+1}} \left( 1 + \sum_{k=1}^{\infty} \frac{\mu_{m+k}}{\mu_m} H^k \right) = \frac{1}{H^{m+1}} (1 + h.o.t.),
\] (1.8.79)

where \(H = \xi \eta\).

From Proposition 1.1.4 and Theorem 1.8.21, we have

Theorem 1.8.22. Let the origin of system (1.8.4) be a \(m\)-order weak critical singular point. Then, in a neighborhood of the origin, system (1.8.4) has the following integrating factor:

\[
\mathcal{M}(z, w) = J\mathcal{M} = \frac{1}{(zw + h.o.t.)^{m+1}},
\] (1.8.80)

where \(H\) and \(J\) are given by (1.8.33), \(M\) is given by (1.8.79).

Similarly, we have
Theorem 1.8.23. Let the origin of system (1.8.66) be a $m$-order weak critical singular point. Then, in a neighborhood of the origin, system (1.8.66) has the following integrating factor:

$$M^*(u, v) = \frac{1}{\mathcal{H}^{m+1}} \left( \frac{1}{1 + \sum_{k=1}^{\infty} \frac{\sigma_{m+k}}{\sigma_m} \mathcal{H}^k} \right) = \frac{1}{\mathcal{H}^{m+1}} (1 + \text{h.o.t.}). \quad (1.8.81)$$

where $\mathcal{H} = u^2 + v^2$.

Theorem 1.8.24. Let the origin of system (1.8.1) be a $m$-order weak focus. Then, in a neighborhood of the origin, system (1.8.1) has the following integrating factor:

$$\mathcal{M}^*(x, y) = \mathcal{J} M^* = \frac{1}{\mathcal{H}^{m+1}} (1 + \text{h.o.t.}) = \frac{1}{(x^2 + y^2 + \text{h.o.t.})^{m+1}}, \quad (1.8.82)$$

where $\mathcal{H}$ and $\mathcal{J}$ are given by (1.8.70), $M^*$ is given by (1.8.81).

Theorem 1.8.25. Suppose that in a neighborhood of the origin, system (1.8.1) has an integrating factor $\tilde{M}^*(x, y)$ with the form $f^s(x, y)G(x, y)$ and $s + 1$ is not a negative integer, where

$$f(x, y) = x^2 + y^2 + \text{h.o.t.}, \quad G(x, y) = 1 + \text{h.o.t.} \quad (1.8.83)$$

are two formal series of $x, y$. Then, the origin of (1.8.1) is a complex center.

Proof. We use reductio ad absurdum. Suppose that the origin of system (1.8.1) is not a complex center but a $m$ order weak focus. Then, Theorem 1.8.24 follows that there is a first integral of (1.8.1)

$$F(x, y) = \frac{\tilde{M}^*(x, y)}{M^*(x, y)} = f^s(x, y)G(x, y)(x^2 + y^2 + \text{h.o.t.})^{m+1} \quad (1.8.84)$$

in a neighborhood of the origin. From (1.8.83) and (1.8.84), $F(r \cos \theta, r \sin \theta)$ has the form

$$F(r \cos \theta, r \sin \theta) = r^{2(s+m+1)} \left[ 1 + \sum_{k=1}^{\infty} \zeta_k(\theta) r^k \right], \quad (1.8.85)$$

where for all $k$, $\zeta_k(\theta)$ are polynomials of $\cos \theta, \sin \theta (k = 1, 2, \cdots)$. Because $s + 1$ is not a negative integer, hence, $s+m+1 \neq 0$. (1.8.85) implies that the origin of system (1.8.1) is a complex center which is in contradiction to the original hypothesis. \( \square \)

We next consider the case of $s + 1$ is a negative integer. Let $s + 1 = -k$, where $k$ is a positive integer. Then, $\tilde{M}^*(x, y)$ in Theorem 1.8.25 has the form:

$$\tilde{M}^*(x, y) = \frac{G(x, y)}{f^{k+1}(x, y)}. \quad (1.8.86)$$
Theorem 1.8.26. Suppose that system \( (1.8.1) \) has an integrating factor \( \tilde{M}^*(x, y) \) with the form \( (1.8.86) \) in a neighborhood of the origin. If the origin is not a \( k \) order weak focus, then the origin of \( (1.8.1) \) is a complex center.

Proof. We use reductio ad absurdum. Suppose that the origin of system \( (1.8.1) \) is not a complex center but a \( m \) order weak focus, where \( k \neq m \). Then, Theorem 1.8.24 follows that in a neighborhood of the origin of system \( (1.8.1) \), there is the following first integral

\[
F(x, y) = \frac{\tilde{M}^*(x, y)}{M^*(x, y)} = \frac{G(x, y)(x^2 + y^2 + h.o.t.)^{m+1}}{f^{k+1}(x, y)}. \tag{1.8.87}
\]

Since \( k \neq m \), \( (1.8.87) \) implies the origin of system \( (1.8.1) \) is a complex center. It is in contradiction to the original hypothesis.

Similarly, we have

Theorem 1.8.27. If system \( (1.8.4) \) has an integrating factor \( \tilde{M}(z, w) \) of the form \( f^s(z, w)G(z, w) \) in a neighborhood of the origin, where \( s + 1 \) is not a negative integer and

\[
f(z, w) = zw + h.o.t., \quad G(z, w) = 1 + h.o.t. \tag{1.8.88}
\]

are two formal series of \( z, w \). Then the origin of \( (1.8.4) \) is a complex center.

If \( s + 1 = -k \), where \( k \) is a positive integer. Then \( \tilde{M}(z, w) \) given by Theorem 1.8.27 becomes

\[
\tilde{M}(z, w) = \frac{G(z, w)}{f^{k+1}(z, w)}. \tag{1.8.89}
\]

Theorem 1.8.28. Suppose that system \( (1.8.4) \) has an integrating factor \( \tilde{M}(z, w) \) of the form \( (1.8.89) \) in a neighborhood of the origin. If the origin of \( (1.8.4) \) is not a \( k \)-order weak critical singular point. Then, the origin of \( (1.8.4) \) is a complex center.

Example 1.8.2. System

\[
\frac{dz}{dT} = z + 3zw(az^2 + bw^2), \quad \frac{dw}{dT} = -w \tag{1.8.90}
\]

has an integrating factor

\[
\tilde{M}(z, w) = \frac{1}{(zw)^3e^{2bw^3}}. \tag{1.8.91}
\]

This is a singular factor. However, we have the first two singular point values \( \mu_1 = \mu_2 = 0 \) of the origin of system \( (1.8.90) \). The origin is not a 2-order weak critical singular point. By Theorem 1.8.28, the origin is a complex center.

Example 1.8.2 tells us that Theorem 1.8.25 \( \sim \) Theorem 1.8.28 are useful for solving the center problem.
**Definition 1.8.4.** If there are a constant $\gamma \neq 0$ and three formal series of $(z^*, w^*)$:

$$
\varphi(z^*, w^*) = \gamma w^* + \text{h.o.t.}, \quad \psi(z^*, w^*) = \frac{1}{\gamma} z^* + \text{h.o.t.}, \quad G(z^*, w^*) = 1 + \text{h.o.t.},
$$

(1.8.92)

such that by using the transformation

$$
z = \varphi(z^*, w^*), \quad w = \psi(z^*, w^*)
$$

(1.8.93)

system (1.8.4) becomes

$$
\frac{dz^*}{dT} = -Z(z^*, w^*)G(z^*, w^*), \quad \frac{dw^*}{dT} = W(z^*, w^*)G(z^*, w^*).
$$

(1.8.94)

Then, system (1.8.4) is called generalized time-reversible system.

Let the origin of system (1.8.4) is a complex center. Then, in a neighborhood of the origin, the standard normal form of (1.8.4) has the form

$$
\frac{d\xi}{dT} = \xi \left[ 1 + \frac{1}{2} \sum_{k=1}^{\infty} \tau_k (\xi\eta)^k \right] = \Phi(\xi, \eta),
$$

$$
\frac{d\eta}{dT} = -\eta \left[ 1 + \frac{1}{2} \sum_{k=1}^{\infty} \tau_k (\xi\eta)^k \right] = -\Psi(\xi, \eta).
$$

(1.8.95)

By using the transformation $\xi = \eta^*, \eta = \xi^*$, system (1.8.94) can become the following system:

$$
\frac{d\xi^*}{dT} = -\Phi(\xi^*, \eta^*), \quad \frac{d\eta^*}{dT} = \Psi(\xi^*, \eta^*),
$$

(1.8.96)

It implies that system (1.8.95) is generalized time-reversible system.

**Theorem 1.8.29.** If the origin of system (1.8.4) is a complex center, then, by using a suitable analytic transformation, system (1.8.4) can become a generalized time-reversible system.

The following is the converse theorem of Theorem 1.8.29.

**Theorem 1.8.30.** If system (1.8.4) is a generalized time-reversible system, then the origin of system (1.8.4) is a complex center.

**Proof.** We use reductio ad absurdum. Suppose that system (1.8.4) is a generalized time-reversible system and its origin is not a complex center. Then, there is a positive integer $m$, such that $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = 0, \mu_m \neq 0$. Thus, there is a polynomial $F(z, w) = zw + \text{h.o.t.}$, such that

$$
\left. \frac{dF(z, w)}{dT} \right|_{(1.8.4)} = \frac{\partial F(z, w)}{\partial z} Z(z, w) - \frac{\partial F(z, w)}{\partial w} W(z, w)
$$

$$
= \mu_m (zw)^{m+1} + \text{h.o.t.}.
$$

(1.8.97)
By using (1.8.93), in a neighborhood of the origin, we can solve \( z^* \) and \( w^* \) as follows:

\[
z^* = \varphi^*(z, w) = \gamma w + h.o.t., \quad w^* = \psi^*(z, w) = \frac{1}{\gamma} z + h.o.t. \tag{1.8.98}
\]

Let

\[
F^*(z, w) = F(\varphi^*(z, w), \psi^*(z, w)) = zw + h.o.t. \tag{1.8.99}
\]

We see from (1.8.97) that

\[
\left. \frac{dF^*(z, w)}{dT} \right|_{(1.8.4)} = \left. \frac{dF(z^*, w^*)}{dT} \right|_{(1.8.94)} = -G(z^*, w^*) \left[ \frac{\partial F(z^*, w^*)}{\partial z^*} Z(z^*, w^*) - \frac{\partial F(z^*, w^*)}{\partial w^*} W(z^*, w^*) \right] \]

\[
= -\mu_m (zw)^{m+1} + h.o.t. \tag{1.8.100}
\]

(1.8.97) and (1.8.100) follows that \( \mu_m = 0 \). It gives the conclusion of this theorem.

**Example 1.8.3.** Consider the following system

\[
\begin{align*}
\frac{dz}{dT} &= Z(z, w) \\
&= z + 9(7 - 8\lambda)[3(1 + 4\lambda)z^3 + 9\lambda w^2 + (7 - 8\lambda)w^2 z - 3(2 - \lambda)w^3]z, \\
\frac{dw}{dT} &= -W(z, w) \\
&= -w + 9(7 - 8\lambda)[3(1 + 4\lambda)w^3 + (4 + 7\lambda)w^2 z - 9wz^2 + 3(2 - \lambda)z^3]w. \tag{1.8.101}
\end{align*}
\]

This system has a algebraic integral

\[
f(z, w) = 1 + 27(1 + 4\lambda)(7 - 8\lambda)(w + z)^2(z - w). \tag{1.8.102}
\]

Let

\[
z^* = \frac{w}{f^{\frac{1}{2}}(z, w)}, \quad w^* = \frac{z}{f^{\frac{1}{2}}(z, w)}. \tag{1.8.103}
\]

Then

\[
z = \frac{w^*}{f^{\frac{1}{2}}(z^*, w^*)}, \quad w = \frac{z^*}{f^{\frac{1}{2}}(z^*, w^*)}. \tag{1.8.104}
\]

By transformation (1.8.103), system (1.8.101) becomes

\[
\begin{align*}
\frac{dz^*}{dT} &= -\frac{Z(z^*, w^*)}{f(z^*, w^*)}, \quad \frac{dw^*}{dT} = \frac{W(z^*, w^*)}{f(z^*, w^*)}. \tag{1.8.105}
\end{align*}
\]

Thus, system (1.8.101) is a generalized time-reversible system. By Theorem 1.8.30, the origin of system (1.8.101) is a complex center.
1.9 Integrability and Linearized Problem of the Resonant Singular Point

Let the origin of the system (1.5.1) be a $p, q$ resonant singular point. By using a suitable linear transformation, system (1.5.1) become

$$
\frac{dz}{dT} = pz + \sum_{k=2}^{\infty} Z_k(z, w) = Z(z, w),
$$

$$
\frac{dw}{dT} = -qw - \sum_{k=2}^{\infty} W_k(z, w) = -W(z, w),
$$

(1.9.1)

where $p, q$ are two irreducible integers, $Z(z, w), W(z, w)$ are two power series of $z, w$ having nonzero convergent radius. For all $k$, $Z_k(z, w), W_k(z, w)$ are homogeneous polynomials of degree $k$ of $z, w$:

$$
Z_k(z, w) = \sum_{\alpha + \beta = k} a_{\alpha \beta} z^\alpha w^\beta, \quad W_k(z, w) = \sum_{\alpha + \beta = k} b_{\alpha \beta} w^\alpha z^\beta.
$$

(1.9.2)

We now cite the definition of a normal form of system (1.9.1) given by [Christopher ect, 2003].

**Definition 1.9.1.** Suppose that there are two formal series of $z, w$

$$
\tilde{\xi} = z + h.o.t., \quad \tilde{\eta} = w + h.o.t., \quad (1.9.3)
$$

such that by transformation (1.9.3), system (1.9.1) reduces to the form:

$$
\frac{d\tilde{\xi}}{dT} = p\tilde{\xi} \left[ 1 + \sum_{k=1}^{\infty} \tilde{p}_k(\tilde{\xi}^q \tilde{\eta}^p)^k \right],
$$

$$
\frac{d\tilde{\eta}}{dT} = -q\tilde{\eta} \left[ 1 + \sum_{k=1}^{\infty} \tilde{q}_k(\tilde{\xi}^q \tilde{\eta}^p)^k \right].
$$

(1.9.4)

Then, we say that (1.9.3) is a normal transformation in a neighborhood of the origin of the system (1.9.1). System (1.9.4) is a normal form corresponding to the transformation (1.9.3).

A resonant singular point can be transformed to a weak critical singular point by a suitable transformation. Actually, from Theorem 1.5.4, there are two power series $\varphi(w), \psi(z)$ with nonzero convergent radius, satisfying $\varphi(0) = \psi(0) = \varphi'(0) = \psi'(0) = 0$, such that by the transformation

$$
u = z - \varphi(w), \quad v = w - \psi(z), \quad (1.9.5)$$

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system (1.9.1) becomes
\[
\frac{du}{dT} = p \ u \ U(u, v) = p \left[ 1 + \sum_{k=1}^{\infty} U_k(u, v) \right], \\
\frac{dv}{dT} = -q \ v \ V(u, v) = -q \left[ 1 + \sum_{k=1}^{\infty} V_k(u, v) \right], \tag{1.9.6}
\]
where \( U(u, v), V(u, v) \) are two power series with nonzero convergent radius of \( u, v \).

Suppose that \( z = \varphi(w), w = \psi(z) \) are analytic solutions of the system (1.9.1), passing through the origin. By using the transformation
\[
u = x^p, \quad v = y^q \tag{1.9.7}
\]
system (1.9.6) becomes the following special system having weak critical singular point \( O(0,0) \):
\[
\frac{dx}{dT} = xU(x^p, y^q) = x \left[ 1 + \sum_{k=1}^{\infty} U_k(x^p, y^q) \right], \\
\frac{dy}{dT} = -yV(x^p, y^q) = -y \left[ 1 + \sum_{k=1}^{\infty} V_k(x^p, y^q) \right]. \tag{1.9.8}
\]
Since \( x = 0 \) and \( y = 0 \) are two solutions of system (1.9.8), hence, any normal transformation in a neighborhood of the origin of system (1.9.8) has the form
\[
\chi^* = x \left( 1 + \sum_{\alpha + \beta = 1}^{\infty} c^*_{\alpha \beta} x^\alpha y^\beta \right), \quad \zeta^* = y \left( 1 + \sum_{\alpha + \beta = 1}^{\infty} d^*_{\alpha \beta} y^\alpha x^\beta \right). \tag{1.9.9}
\]
By Theorem 1.8.1, for the coefficients of the formal series (1.9.9), first, \( c^*_{kk}, d^*_{kk} \) are taken as any constant numbers. Then, the other coefficients can be determined uniquely. Corresponding to (1.9.9), the normal form of system (1.9.8) is as follows:
\[
\frac{d\chi^*}{dT} = \chi^* \left[ 1 + \sum_{k=1}^{\infty} p_k^*(\chi^* \zeta^*)^k \right], \\
\frac{d\zeta^*}{dT} = -\zeta^* \left[ 1 + \sum_{k=1}^{\infty} q_k^*(\chi^* \zeta^*)^k \right]. \tag{1.9.10}
\]
Compare with system (1.8.4), the right sides functions of the system (1.9.8) have the following properties:

(1) \( x = 0 \) and \( y = 0 \) are two solutions of (1.9.8).
(2) \( U(x^p, y^q), V(x^p, y^q) \) are two power series of \( x^p, y^q \).

These properties of (1.9.8) make it have a particular normal transformation and a special normal form.

**Definition 1.9.2.** Let (1.9.9) be a normal transformation of system (1.9.8) in a neighborhood of the origin satisfying for \( m/(pq) \) are not positive integers, \( c_m^{*} = d_m^{*} = 0 \). We say that (1.9.9) is a \( p, q \) resonant normal transformation. The corresponding normal form is called the \( p, q \) resonant normal form.

Obviously, the standard normal transformation in a neighborhood of the origin of system (1.9.8) is \( p, q \) resonant.

**Theorem 1.9.1.** For all given \( \tilde{c}_{kq}, \tilde{d}_{kp} \) and \( \tilde{d}_{kp}, \tilde{c}_{kq} \) \((k = 1, 2, \cdots)\), one can derive uniquely and successively the terms of the formal series

\[
\tilde{\chi} = x \left( 1 + \sum_{\alpha + \beta = 1}^{\infty} \tilde{c}_{\alpha \beta} x^{\alpha p} y^{\beta q} \right) = x \tilde{f}(x^p, y^q),
\]

\[
\tilde{\zeta} = y \left( 1 + \sum_{\alpha + \beta = 1}^{\infty} \tilde{d}_{\alpha \beta} y^{\alpha q} x^{\beta p} \right) = y \tilde{g}(x^p, y^q),
\]

(1.9.11)

such that by transformation (1.9.11), system (1.9.8) becomes the following normal form

\[
\frac{d\tilde{\chi}}{dT} = \tilde{\chi} \left[ 1 + \sum_{k=1}^{\infty} \tilde{p}_k (\tilde{\chi} \tilde{\zeta})^{kpq} \right],
\]

\[
\frac{d\tilde{\zeta}}{dT} = -\tilde{\zeta} \left[ 1 + \sum_{k=1}^{\infty} \tilde{q}_k (\tilde{\chi} \tilde{\zeta})^{kpq} \right].
\]

(1.9.12)

**Proof.** Write that

\[
\tilde{f}(u, v) = 1 + \sum_{m=1}^{\infty} \tilde{f}_m(u, v),
\]

\[
\tilde{g}(u, v) = 1 + \sum_{m=1}^{\infty} \tilde{g}_m(u, v),
\]

(1.9.13)

where \( \tilde{f}_m(u, v), \tilde{g}_m(u, v) \) are homogeneous polynomials of degree \( m \) of \( u, v \):

\[
\tilde{f}_m(u, v) = \sum_{\alpha + \beta = m} \tilde{c}_{\alpha \beta} u^{\alpha} v^{\beta}, \quad \tilde{g}_m(u, v) = \sum_{\alpha + \beta = m} \tilde{d}_{\alpha \beta} v^{\alpha} u^{\beta}.
\]

(1.9.14)
From (1.9.8), (1.9.11) and (1.9.13), we have

\[
\frac{d\tilde{x}}{dT} - \tilde{x} = x \sum_{m=1}^{\infty} \left[ \tilde{f}_m + pu \frac{\partial \tilde{f}_m}{\partial u} - qv \frac{\partial \tilde{f}_m}{\partial v} + F_m(u, v) \right], \\
\frac{d\tilde{\zeta}}{dT} + \tilde{\zeta} = y \sum_{m=1}^{\infty} \left[ \tilde{g}_m - qv \frac{\partial \tilde{g}_m}{\partial v} - G_m(u, v) \right],
\]

(1.9.15)

where \(F_m(u, v), G_m(u, v)\) are homogeneous polynomials of degree \(m\) of \(u, v:\)

\[
F_m = U_m + \sum_{k=1}^{m-1} \left[ \left( \tilde{f}_k + pu \frac{\partial \tilde{f}_k}{\partial u} \right) U_{m-k} - \frac{\partial \tilde{f}_k}{\partial v} V_{m-k} \right], \\
G_m = V_m + \sum_{k=1}^{m-1} \left[ \left( \tilde{g}_k + qv \frac{\partial \tilde{g}_k}{\partial v} \right) V_{m-k} - \frac{\partial \tilde{g}_k}{\partial u} U_{m-k} \right].
\]

(1.9.16)

By using (1.9.11), we obtain

\[
\tilde{x} \sum_{k=1}^{\infty} \tilde{p}_k (\tilde{x})^{k_{pq}} = x \sum_{k=1}^{\infty} \tilde{p}_k u^{k_{pq}} v^{k_{pq}} \tilde{f}^{k_{pq}+1} (u, v) \tilde{g}^{k_{pq}} (u, v), \\
\tilde{\zeta} \sum_{k=1}^{\infty} \tilde{q}_k (\tilde{\zeta})^{k_{pq}} = y \sum_{k=1}^{\infty} \tilde{q}_k u^{k_{pq}} v^{k_{pq}} \tilde{g}^{k_{pq}+1} (u, v) \tilde{f}^{k_{pq}} (u, v).
\]

(1.9.17)

Let

\[
\tilde{x} \sum_{k=1}^{\infty} \tilde{p}_k (\tilde{x})^{k_{pq}} = x \left[ \sum_{k=1}^{\infty} \tilde{p}_k u^{k_{pq}} v^{k_{pq}} + \sum_{m=1}^{\infty} \Phi_m(u, v) \right], \\
\tilde{\zeta} \sum_{k=1}^{\infty} \tilde{q}_k (\tilde{\zeta})^{k_{pq}} = y \left[ \sum_{k=1}^{\infty} \tilde{q}_k u^{k_{pq}} v^{k_{pq}} + \sum_{m=1}^{\infty} \Psi_m(u, v) \right].
\]

(1.9.18)

Then, by (1.9.13), (1.9.17) and (1.9.18), we see that for any integer \(m, \Phi_m, \Psi_m\) only depend on \(\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_{m-1}\) and \(\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_{m-1}\). For any positive integer \(k\), when \((k-1)(p+q) < m \leq k(p+q)\), \(\Phi_m, \Psi_m\) only depend on \(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_{k-1}\) and \(\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_{k-1}\). We know from (1.9.15) and (1.9.18) that (1.9.12) holds if and only if

\[
\sum_{m=1}^{\infty} \left( pu \frac{\partial \tilde{f}_m}{\partial u} - qv \frac{\partial \tilde{f}_m}{\partial v} \right) = \sum_{m=1}^{\infty} (\Phi_m - F_m) + \sum_{k=1}^{\infty} \tilde{p}_k u^{k_{pq}} v^{k_{pq}}, \\
\sum_{m=1}^{\infty} \left( qv \frac{\partial \tilde{g}_m}{\partial v} - pu \frac{\partial \tilde{g}_m}{\partial u} \right) = \sum_{m=1}^{\infty} (\Psi_m - G_m) + \sum_{k=1}^{\infty} \tilde{q}_k u^{k_{pq}} v^{k_{pq}}.
\]

(1.9.19)
From (1.9.14), for any positive integer \( m \), we have

\[
p_u \frac{\partial \tilde{f}_m}{\partial u} - q_v \frac{\partial \tilde{f}_m}{\partial v} = \sum_{\alpha+\beta=m} (\alpha p - \beta q) \tilde{c}_{\alpha\beta} u^\alpha v^\beta, \]

\[
q_v \frac{\partial \tilde{g}_m}{\partial v} - p_u \frac{\partial \tilde{g}_m}{\partial u} = \sum_{\alpha+\beta=m} (\beta q - \alpha p) \tilde{d}_{\beta\alpha} u^\alpha v^\beta. \tag{1.9.20}
\]

Denote that

\[
\Phi_m - F_m = \sum_{\alpha+\beta=m} C_{\alpha\beta} u^\alpha v^\beta, \quad \Psi_m - G_m = \sum_{\alpha+\beta=m} D_{\alpha\beta} u^\alpha v^\beta. \tag{1.9.21}
\]

Then, from (1.9.19), (1.9.20) and (1.9.21) we get

\[
\sum_{m=1}^{\infty} \sum_{\alpha+\beta=m} (\alpha p - \beta q) \tilde{c}_{\alpha\beta} u^\alpha v^\beta = \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m} C_{\alpha\beta} u^\alpha v^\beta + \sum_{k=1}^{\infty} \tilde{p}_k u^k q^k v^k,
\]

\[
\sum_{m=1}^{\infty} \sum_{\alpha+\beta=m} (\beta q - \alpha p) \tilde{d}_{\beta\alpha} u^\alpha v^\beta = \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m} D_{\alpha\beta} u^\alpha v^\beta + \sum_{k=1}^{\infty} \tilde{q}_k u^k q^k v^k. \tag{1.9.22}
\]

Since \( p \) and \( q \) are two irreducible integers, hence for any natural numbers \( \alpha, \beta \), when \( \alpha + \beta \geq 1, \alpha p - \beta q = 0 \) if and only if there exists a positive integer \( k \), such that \( \alpha = kp, \beta = kq \). Thus, (1.9.22) follows that for any natural numbers \( \alpha, \beta \), when \( \alpha p - \beta q \neq 0 \), all \( \tilde{c}_{\alpha\beta}, \tilde{d}_{\beta\alpha} \) are determined uniquely by the following recursive formulas

\[
\tilde{c}_{\alpha\beta} = \frac{1}{\alpha p - \beta q} C_{\alpha\beta}, \quad \tilde{d}_{\beta\alpha} = \frac{1}{\beta q - \alpha p} D_{\beta\alpha}. \tag{1.9.23}
\]

Moreover, for any positive integer \( k \), \( \tilde{p}_k, \tilde{q}_k \) are determined uniquely by the following recursive formulas

\[
\tilde{p}_k = -C_{kq, kp}, \quad \tilde{q}_k = -D_{kp, kq}. \tag{1.9.24}
\]

Because all coefficients of \( \tilde{c}_{kq,kp} \) and \( \tilde{d}_{kp,kq} \) in (1.9.22) are zeros. So that, \( \tilde{c}_{kq,kp} \) and \( \tilde{d}_{kp,kq} \) can be given as arbitrary constants in advance. \( \square \)

From Theorem 1.9.1 and Corollary 1.8.1 we have

**Theorem 1.9.2.** In a neighborhood of the origin, any \( p, q \) resonant normal transformation of system (1.9.8) has the form of (1.9.11). The corresponding normal form has the form of (1.9.12).

In a neighborhood of the origin, the standard normal transformation and the standard normal form of the origin of system (1.9.8) can be written respectively as follows:
\[ \chi = x \left( 1 + \sum_{\alpha + \beta = 1}^{\infty} c_{\alpha\beta} x^{\alpha p} y^{\beta q} \right) = x f(x^p, y^q), \]
\[ \zeta = y \left( 1 + \sum_{\alpha + \beta = 1}^{\infty} d_{\alpha\beta} y^{\alpha p} x^{\beta q} \right) = y g(x^p, y^q), \] (1.9.25)

\[ \frac{d\chi}{dT} = \chi \left[ 1 + \sum_{k=1}^{\infty} p_k (\chi \zeta)^{kpq} \right], \]
\[ \frac{d\zeta}{dT} = -\zeta \left[ 1 + \sum_{k=1}^{\infty} q_k (\chi \zeta)^{kpq} \right], \] (1.9.26)

where
\[ c_{kq}, k_p = d_{kp}, k_q = 0, \quad k = 1, 2, \ldots. \] (1.9.27)

**Theorem 1.9.3.** Let (1.9.11) be a \( p, q \) resonant normal transformation in a neighborhood of the origin of system (1.9.8) and its corresponding normal form be (1.9.12). Then,
\[ \tilde{\xi} = (z - \varphi) \tilde{f}(z - \varphi, w - \psi), \quad \tilde{\eta} = (w - \psi) \tilde{g}(z - \varphi, w - \psi) \] (1.9.28)
is a resonant normal transformation in a neighborhood of the origin of system (1.9.1). By transformation (1.9.28), system (1.9.1) can be reduced to (1.9.4).

**Proof.** From (1.9.5), (1.9.7), (1.9.11) and (1.9.28), we have
\[ \tilde{\xi} = \tilde{\chi}^p, \quad \tilde{\eta} = \tilde{\zeta}^q. \] (1.9.29)
(1.9.12) and (1.9.29) follows (1.9.4). \( \square \)

**Theorem 1.9.4.** Let (1.9.3) be a normal transformation in a neighborhood of the origin of system (1.9.1) and corresponding normal form be (1.9.4). Then, there exist two unit formal series \( \tilde{f}(u, v), \tilde{g}(u, v) \) of \( u, v \), such that \( \tilde{\xi}, \tilde{\eta} \) can be expressed as the form of (1.9.28). By using transformation
\[ \tilde{\chi} = x \tilde{f}(x^p, y^q), \quad \tilde{\zeta} = y \tilde{g}(x^p, y^q) \] (1.9.30)
system (1.9.8) becomes the \( p, q \) resonance normal form (1.9.12).

**Proof.** Let (1.9.3) be a normal transformation in a neighborhood of the origin of system (1.9.1). Then, by transformation (1.9.5), \( \tilde{\xi} \) and \( \tilde{\eta} \) can be represented as two formal series of \( u, v \). Since \( u = 0 \) and \( v = 0 \) are two solutions of system (1.9.6) hence, there are two unit formal series \( \tilde{F}(u, v), \tilde{G}(u, v) \) of \( u, v \), such that,
\[ \tilde{\xi} = u \tilde{F}(u, v), \quad \tilde{\eta} = v \tilde{G}(u, v). \] (1.9.31)
Denote that
\[
\tilde{f}(u, v) = \tilde{F}_p(u, v), \quad \tilde{g}(u, v) = \tilde{G}_q(u, v),
\]
where the functions of the right hands take their principal values. Then, \(\tilde{f}(u, v), \tilde{g}(u, v)\) are unit formal series of \(u, v\). From (1.9.5), (1.9.31) and (1.9.32) we obtain the representations (1.9.28) of \(\tilde{\xi}, \tilde{\eta}\).

From (1.9.5), (1.9.7), (1.9.28) and (1.9.30), we have (1.9.29). (1.9.4) and (1.9.29) follows (1.9.12).

**Remark 1.9.1.** Theorem (1.9.3) and theorem (1.9.4) imply that in a neighborhood of the origin, the \(p, q\) resonance normal transformation (1.9.11) of system (1.9.8) and the following normal transformation of system (1.9.1)

\[
\begin{align*}
\tilde{\xi} &= (z - \varphi) \left[ 1 + \sum_{\alpha+\beta=1}^{\infty} \tilde{c}_{\alpha\beta}(z - \varphi)^\alpha(w - \psi)^\beta \right] = z + h.o.t., \\
\tilde{\eta} &= (w - \psi) \left[ 1 + \sum_{\alpha+\beta=1}^{\infty} \tilde{d}_{\alpha\beta}(w - \psi)^\alpha(z - \varphi)^\beta \right] = w + h.o.t.
\end{align*}
\]

have the one-to-one correspondence relation. Moreover, (1.9.5) and (1.9.7) imply (1.9.29).

**Definition 1.9.3.** We say that in a neighborhood of the origin,

\[
\begin{align*}
\xi &= (z - \varphi) \left[ 1 + \sum_{\alpha+\beta=1}^{\infty} c_{\alpha\beta}(z - \varphi)^\alpha(w - \psi)^\beta \right] = z + h.o.t., \\
\eta &= (w - \psi) \left[ 1 + \sum_{\alpha+\beta=1}^{\infty} d_{\alpha\beta}(w - \psi)^\alpha(z - \varphi)^\beta \right] = w + h.o.t.
\end{align*}
\]

is the standard normal transformation of system (1.9.1), where \(c_{kq}, k_p = d_{kp}, k_q = 0, k = 1, 2, \cdots\). Corresponding to transformation (1.9.34), system

\[
\begin{align*}
\frac{d\xi}{dT} &= p\xi \left[ 1 + \sum_{k=1}^{\infty} p_k (\xi^q \eta^p)^k \right] = \Phi(\xi, \eta), \\
\frac{d\eta}{dT} &= -q\eta \left[ 1 + \sum_{k=1}^{\infty} q_k (\xi^q \eta^p)^k \right] = -\Psi(\xi, \eta)
\end{align*}
\]

is called the standard normal form in a neighborhood of the origin of system (1.9.1).

From Remark 1.9.1 and Theorem 1.8.2, we have
Theorem 1.9.5. In a neighborhood of the origin, the standard normal transformation of system (1.9.1) and the standard normal transformation of system (1.9.8) have the following relation

\[ \xi = \chi^p, \quad \eta = \zeta^q. \]  

(1.9.36)

Moreover, if for all \( k \), \( p_k = q_k \), then \( \xi, \eta \) are two power series of \( z, w \) having nonzero convergent radius.

Theorem 1.9.6. Let

\[ \mu_k = p_k - q_k, \quad \tau_k = p_k + q_k, \quad k = 1, 2, \ldots. \]  

(1.9.37)

For system (1.9.1), we have

\[ \frac{dH}{dT} = pq \sum_{k=1}^{\infty} \mu_k H^{k+1}, \]

\[ \frac{d\Omega}{dT} = \frac{pq}{2i} \left( 2 + \sum_{k=1}^{\infty} \tau_k H^k \right) \]  

(1.9.38)

and

\[ \frac{\partial}{\partial z}(MZ) - \frac{\partial}{\partial w}(MW) = pqM \sum_{k=1}^{\infty} (k + 1)\mu_k (H)^k, \]  

(1.9.39)

where

\[ H = \xi^q \eta^p, \quad \Omega = \frac{1}{2i} \ln \frac{\xi^q}{\eta^p}, \quad M = \xi^{q-1} \eta^{p-1} \left| \begin{array}{cc} \frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial w} \\ \frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial w} \end{array} \right|. \]  

(1.9.40)

Proof. By using (1.9.35) and (1.9.40) to do computations directly, we obtain (1.9.38). Let \( M = \xi^{q-1} \eta^{p-1} J \), where \( J \) is the Jacobian of \( \xi, \eta \) with respect to \( z, w \). Then, Proposition 1.1.3 follows that

\[ \frac{\partial}{\partial z}(MZ) - \frac{\partial}{\partial w}(MW) = \frac{\partial}{\partial z} \left( \xi^{p-1} \eta^{q-1} JZ \right) - \frac{\partial}{\partial w} \left( \xi^{p-1} \eta^{q-1} JW \right) \]

\[ = J \left[ \frac{\partial}{\partial \xi} \left( \xi^{p-1} \eta^{q-1} \Phi \right) - \frac{\partial}{\partial \eta} \left( \xi^{p-1} \eta^{q-1} \Psi \right) \right]. \]  

(1.9.41)

(1.9.41) implies (1.9.39). \( \square \)

Similar to Theorem 1.8.3, we have
Theorem 1.9.7. Let $F(H)$ and $G(H)$ be two unit formal power series of $H$. Then,
\[ \tilde{\xi} = \xi F(H), \quad \tilde{\eta} = \eta G(H) \] (1.9.42)
gives the normal transformation in a neighborhood of the origin of system (1.9.1).

Theorem 1.9.8. If (1.9.3) is a normal transformation in a neighborhood of the origin of system (1.9.1), then there exist two units of formal series of $H$ of the form
\[ F(H) = 1 + \sum_{k=1}^{\infty} A_k H^k, \quad G(H) = 1 + \sum_{k=1}^{\infty} B_k H^k, \] (1.9.43)
such that $\tilde{\xi} = \xi F(H), \tilde{\eta} = \eta G(H)$.

Proof. Let (1.9.3) be a normal form in a neighborhood of the origin of system (1.9.1). Then system (1.9.1) becomes the normal form (1.9.4) by transformation (1.9.3). By Theorem 1.9.4, there are two unit formal series $\tilde{f}(u, v), \tilde{g}(u, v)$ of $u, v$, such that (1.9.28) holds. System (1.9.8) becomes the normal form (1.9.12) by transformation (1.9.30). From Theorem 1.8.4, $\tilde{\chi}, \tilde{\zeta}$ can be written as the following formal series of $\chi, \zeta$:
\[ \tilde{\chi} = \chi \left[ 1 + \sum_{m=1}^{\infty} \tilde{A}_m (\chi \zeta)^m \right], \quad \tilde{\zeta} = \zeta \left[ 1 + \sum_{m=1}^{\infty} \tilde{B}_m (\chi \zeta)^m \right]. \] (1.9.44)

From (1.9.25) and (1.9.44), we have
\[ \tilde{\chi} = xf \left[ 1 + \sum_{m=1}^{\infty} \tilde{A}_m (xy)^m (fg)^m \right], \]
\[ \tilde{\zeta} = yg \left[ 1 + \sum_{m=1}^{\infty} \tilde{B}_m (xy)^m (fg)^m \right], \] (1.9.45)

where $f = f(x^p, y^q), g = g(x^p, y^q)$. By (1.9.30), $\tilde{\chi}/x = \tilde{f}(x^p, y^q)$ and $\tilde{\zeta}/y = \tilde{g}(x^p, y^q)$ are two formal series of $x^p$ and $y^q$. Thus, when $m/(pq)$ is not a positive integer, we have $\tilde{A}_m = \tilde{B}_m = 0$. Now (1.9.45) can be become
\[ \tilde{\chi} = \chi \left[ 1 + \sum_{k=1}^{\infty} \tilde{A}_{kpq} (\chi \zeta)^{kpq} \right], \quad \tilde{\zeta} = \zeta \left[ 1 + \sum_{k=1}^{\infty} \tilde{B}_{kpq} (\chi \zeta)^{kpq} \right]. \] (1.9.46)

(1.9.29), (1.9.36) and (1.9.46) follow that
\[ \tilde{\xi} = \xi \left[ 1 + \sum_{k=1}^{\infty} \tilde{A}_{kpq} (\zeta^q \eta^p)^k \right]^p, \quad \tilde{\eta} = \eta \left[ 1 + \sum_{k=1}^{\infty} \tilde{B}_{kpq} (\zeta^q \eta^p)^k \right]^q. \] (1.9.47)

This gives the conclusion.
In [Xiao P., 2005], the author gave the following definition.

**Definition 1.9.4.** For any positive integer \( k \), \( \mu_k = p_k - q_k \) is called the \( k \)-th resonant singular point value of the origin of system (1.9.1) and \( \tau_k = p_k + q_k \) is called the \( k \)-th resonant period constant of the origin of system (1.9.1).

Define that \( \mu_0 = 0 \). If there is a positive integer \( k \), such that \( \mu_0 = \mu_1 = \cdots = \mu_{k-1} = 0 \), but \( \mu_k \neq 0 \), then the origin is called the \( k \)-order resonant singular point;

If for any positive integer \( k \), there are \( \mu_k = 0 \), then the origin is called a complex resonant center.

**Remark 1.9.2.** The \( k \)-th resonant singular point value is the \( k \)-th saddle quantity given by [Christopher etc, 2003]

**Remark 1.9.3.** Theorem 1.9.5 and Theorem 1.9.6 imply that if the origin of system (1.9.1) is a complex resonant center, then \( H = \xi^p \eta^q \) is an analytic first integral of system (1.9.1), and \( H \) a is power series in \( z, w \) having nonzero convergent radius.

Similar to the proofs of Theorem 1.8.7~ Theorem 1.8.11, we have the following results.

**Theorem 1.9.9.** System (1.9.1) has an analytic first integral in a neighborhood of the origin if and only if all resonant singular point values of the origin are zeros.

**Theorem 1.9.10.** System (1.9.1) in a neighborhood of the origin is linearizable if and only if for all \( k \), \( p_k = 0 \) and \( q_k = 0 \).

**Theorem 1.9.11.** If the origin of system (1.9.1) is a complex resonant center, then any first integral in a neighborhood of the origin of system (1.9.1) can be expressed as a formal series of \( H \). In addition, any analytic first integral in a neighborhood of the origin of system (1.9.1) can be expressed as power series of \( H \) with a nonzero convergent radius.

Because system (1.9.8) can be reduced to system (1.9.26) by using standard normal transformation (1.9.25). Therefore, we have

**Theorem 1.9.12.** The origin of system (1.9.1) is a complex resonant center if and only if in a neighborhood of the origin there is an analytic integral factor:

\[
M(z, w) = z^{q-1}w^{p-1} + h.o.t..
\]  \hspace{1cm} (1.9.48)

**Remark 1.9.4.** In [Simon etc 2000], the conditions of Theorem 1.9.9 are taken as the definition of the integrability. While the conditions given by Theorem 1.9.10 are taken as the definition of the linearizable systems in a neighborhood of the origin of system (1.9.1).
Theorem 1.9.13. For any positive integer \( k \), the \( k \)-th resonant singular value and resonant period constant of the origin of system (1.9.1) are the \( kpq \)-th singular value and the \( kpq \)-th period constant of the origin of system (1.9.8), respectively. In addition, if \( m/(pq) \) isn’t a positive integer, then the \( m \)-th singular value and the \( m \)-th period constant of the origin of (1.9.8) are zeros.

Finally, we have

Theorem 1.9.14. For system (1.9.1), if the origin is a \( m \)-order resonant singular point, then its standard normal form (1.9.35) has the following integrating factor in a neighborhood of the origin:

\[
M = \frac{1}{\xi \eta H^m \left( 1 + \sum_{k=1}^{\infty} \frac{\mu_{m+k}}{\mu_m} H^k \right)}.
\]  

(1.9.49)

Moreover, system (1.9.1) has an integrating factor \( J M \) in a neighborhood of the origin, where \( H = \xi^q \eta^p \), \( J = 1 + \text{h.o.t.} \) is the Jacobian of \( \xi, \eta \) with respect to \( z, w \).

Bibliographical Notes

The materials of this chapter are taken from [Amelikin etc, 1982; Qin Y.X., 1985; Griffiths, 1985; Liu Y.R. etc, 1989; Liu Y.R. etc, 1995; Shen L.R., 1998; Liu Y.R., 1999; Simon etc, 2000; Xiao P., 2005; Christopher etc, 2003].

For the linearized problem, a great of number of papers had been published. For instance, see [Zhu D.M., 1987; Chavarriga etc, 1996; Schlomiuk, 1993a; Lloyd etc, 1996; Chavarriga etc, 1997; Chen X.W. etc, 2008; Llibre etc, 2009a; Llibre etc, 2009c; Zhang Q. etc; 2011; Giné ect, 2011] et al.