Chapter 5
Theory of Center-Focus and Bifurcation of Limit Cycles at Infinity of a Class of Systems

We have mentioned in Preface of this book that a real planar polynomial vector field \( V \) can be compactified on the sphere. The vector field \( p(V) \) restricted to the upper hemisphere completed with the equator \( \Gamma_\infty \) is called \textit{Poincaré compactification} of a polynomial vector field. If a real polynomial vector field has no real singular point in the equator \( \Gamma_\infty \) of the \textit{Poincaré disc} and \( \Gamma_\infty \) can be seen a trajectory, all trajectories in a inner neighborhood of \( \Gamma_\infty \) are spirals or closed orbits, then \( \Gamma_\infty \) is called the equator cycle of the vector field. \( \Gamma_\infty \) can be become a point by using the Bendixson reciprocal radius transformation. This point is called infinity of the system.

In this chapter, we discuss the center-focus problem of infinity (i.e., to distinguish when the trajectories in a inner neighborhood of \( \Gamma_\infty \) are either closed orbits or spirals) and the bifurcation of limit cycles at infinity for a class of systems.

5.1 Definition of the Focal Values of Infinity

Consider the following real planar polynomial system of degree \((2n + 1)\):

\[
\frac{dx}{dt} = \sum_{k=0}^{2n+1} X_k(x, y), \quad \frac{dy}{dt} = \sum_{k=0}^{2n+1} Y_k(x, y),
\]

(5.1.1)

where \( n \) is a positive integer and \( X_k(x, y), Y_k(x, y) \) are homogeneous polynomials of degree \( k \) in \( x, y \) of the form

\[
k(x, y) = \sum_{\alpha+\beta=k} A_{\alpha\beta} x^\alpha y^\beta,
\]

\[
Y_k(x, y) = \sum_{\alpha+\beta=k} B_{\alpha\beta} x^\alpha y^\beta.
\]

(5.1.2)

Suppose that the function \( xy^{2n+1}(x, y) - y^{2n+1}(x, y) \) is not identically zero. Then, system (5.1.1) only has finite real or complex singular points in \( \Gamma_\infty \). It has
no real singular point in $\Gamma_\infty$ if and only if $xY_{2n+1}(x, y) - yX_{2n+1}(x, y)$ is a positive (or negative) definite function in the real field. This function can be expressed as a product of linear terms in the complex field as follows:

$$xY_{2n+1}(x, y) - yX_{2n+1}(x, y) = \prod_{k=1}^{2n+2} (\alpha_k x + \beta_k y).$$

(5.1.3)

On the Poincaré disk, all infinite singular points (real and complex) of system (5.1.1) are the intersection points of the straight line $\alpha_k x + \beta_k y = 0$ and the unit circle $x^2 + y^2 = 1$, $k = 1, 2, \ldots, 2n + 2$.

Without loss of the generality, we assume that $I(x, y)$ is positive definite (otherwise, we can take a transformation $t \rightarrow -t$), then, there exists a positive numbers $d$, such that

$$xY_{2n+1}(x, y) - yX_{2n+1}(x, y) \geq d(x^2 + y^2)^{n+1}.$$  

(5.1.4)

By using the transformation

$$x = \frac{\cos \theta}{r}, \quad y = \frac{\sin \theta}{r},$$

(5.1.5)

system (5.1.1) becomes

$$\frac{dr}{dt} = -\frac{1}{r^{2n-1}} \sum_{k=0}^{2n+1} \varphi_{2n+2-k}(\theta) r^k,$$

$$\frac{d\theta}{dt} = \frac{1}{r^{2n}} \sum_{k=0}^{2n+1} \psi_{2n+2-k}(\theta) r^k.$$  

(5.1.6)

Thus, we have

$$\frac{dr}{d\theta} = -r \frac{\varphi_{2n+2}(\theta) + \sum_{k=1}^{2n+1} \varphi_{2n+2-k}(\theta) r^k}{\psi_{2n+2}(\theta) + \sum_{k=1}^{2n+1} \psi_{2n+2-k}(\theta) r^k},$$

(5.1.7)

where $\varphi_k(\theta)$, $\psi_k(\theta)$ are given by (2.1.5). Especially,

$$\varphi_{2n+2}(\theta) = \cos \theta X_{2n+1}(\cos \theta, \sin \theta) + \sin \theta Y_{2n+1}(\cos \theta, \sin \theta),$$

$$\psi_{2n+2}(\theta) = \cos \theta Y_{2n+1}(\cos \theta, \sin \theta) - \sin \theta X_{2n+1}(\cos \theta, \sin \theta).$$  

(5.1.8)

(5.1.4) and (5.1.8) follow that

$$\psi_{2n+2}(\theta) \geq d > 0.$$  

(5.1.9)
Since for all $k$, $\varphi_k(\theta)$ and $\psi_k(\theta)$ are homogeneous polynomials of degree $k$ in $(\cos \theta, \sin \theta)$, we have

$$\varphi_k(\theta + \pi) = (-1)^k \varphi_k(\theta), \quad \psi_k(\theta + \pi) = (-1)^k \psi_k(\theta). \quad (5.1.10)$$

It implies that equation (5.1.7) is the specific form of the equation (2.1.7).

For a sufficiently small constant $h$, we write the solution of (5.1.7) with the initial condition $r|_{\theta=0} = h$ as

$$r = \bar{r}(\theta, h) = \sum_{k=1}^{\infty} \nu_k(\theta) h^k. \quad (5.1.11)$$

From (5.1.7) and (5.1.11), we obtain

$$\nu_1(\theta) = \exp \int_{0}^{\theta} \frac{-\varphi_{2n+2}(\vartheta)}{\psi_{2n+2}(\vartheta)} d\vartheta. \quad (5.1.12)$$

By Corollary 2.1.1, if $\nu_1(2\pi) = 1$, then the first positive integer $k$ satisfying $\nu_k(2\pi) \neq 0$ is an odd number.

**Definition 5.1.1.** For any positive integer $k$, $\nu_{2k+1}(2\pi)$ is called the $k$-th focal value at infinity of system (5.1.1)

**Definition 5.1.2.** For system (5.1.1):

1. If $\nu_1(2\pi) \neq 1$ and when $\nu_1(2\pi) < 1$ ($> 1$), infinity is called a stable (an unstable) rough focus;

2. If $\nu_1(2\pi) = 1$ and there exists a positive integer $k$, such that $\nu_2(2\pi) = \nu_3(2\pi) = \cdots = \nu_{2k}(2\pi) = 0$ and $\nu_{2k+1}(2\pi) \neq 0$, then when $\nu_{2k+1}(2\pi) < 0$ ($> 0$), infinity is called a stable (an unstable) weak focus;

3. If $\nu_1(2\pi) = 1$ and for any positive integer $k$, we have $\nu_{2k+1}(2\pi) = 0$, then infinity is called a center.

From Corollary 2.1.1 and the geometric properties of the Poincaré successor function $\Delta(h) = \bar{r}(2\pi, h) - h$, we obtain

**Theorem 5.1.1.** If infinity is a stable (an unstable) focus of system (5.1.1), then $\Gamma_\infty$ is an internal stable (an internal unstable) limit cycle.

If infinity is a center, then there exists a family of closed orbits of system (5.1.1) in a inner neighborhood of the equator $\Gamma_\infty$.

For a given polynomial system, to solve the center-focus problem of infinity, it depends on the computations of the focal values of infinity. In next sections, we discuss this difficult problem.
5.2 Conversion of Questions

First, we consider a special case of system (5.1.1). Letting

\[ X_{2n+1}(x, y) = (\delta x - y)(x^2 + y^2)^n, \]
\[ Y_{2n+1}(x, y) = (x + \delta y)(x^2 + y^2)^n, \]  

(5.2.1)

then system (5.1.1) becomes

\[ \frac{dx}{dt} = (\delta x - y)(x^2 + y^2)^n + \sum_{\alpha + \beta = 0}^{2n} A_{\alpha \beta} x^\alpha y^\beta, \]
\[ \frac{dy}{dt} = (x + \delta y)(x^2 + y^2)^n + \sum_{\alpha + \beta = 0}^{2n} B_{\alpha \beta} x^\alpha y^\beta. \]  

(5.2.2)

For system (5.2.2), (5.1.8) reduces to

\[ \varphi_{2n+2}(\theta) \equiv \delta, \quad \psi_{2n+2}(\theta) \equiv 1. \]  

(5.2.3)

Thus, (5.1.7) becomes

\[ \frac{dr}{d\theta} = -r \left( \delta + \sum_{k=1}^{2n+1} \varphi_{2n+2-k}(\theta)r^k \right) \frac{1 + \sum_{k=1}^{2n+1} \psi_{2n+2-k}(\theta)r^k}{r}. \]  

(5.2.4)

It is easy to prove that

**Proposition 5.2.1.** For system (5.2.2), we have \( \nu_1(\theta) = e^{-\delta \theta} \) and when \( \delta > 0 \) \((< 0)\), infinity is a stable \((\text{an unstable})\) focus.

From Lemma 2.1.2, we obtain

**Proposition 5.2.2.** If \( \delta = 0 \), then for system (5.2.2), all \( \nu_k(\theta) \) are polynomials in \( \theta, \sin \theta, \cos \theta \), and their coefficients are polynomials in \( A_{\alpha \beta}, B_{\alpha \beta} \). Especially, for all \( k \), \( \nu_k(\pi), \nu_k(2\pi) \) are polynomials in \( A_{\alpha \beta}, B_{\alpha \beta} \).

Notice that infinity of system (5.2.2) can be changed to the origin by using a suitable transformation. In fact, by the transformation

\[ x = \frac{u}{u^2 + v^2}, \quad y = \frac{v}{u^2 + v^2}, \quad \frac{dt}{d\tau} = (u^2 + v^2)^{2n} \]  

(5.2.5)
system (5.2.2) becomes
\[
\frac{du}{d\tau} = - (\delta u + v)(u^2 + v^2)^n + \sum_{k=0}^{2n} (u^2 + v^2)^{2n-k} [(u^2 - v^2)X_k(u, v) + 2uvY_k(u, v)],
\]
\[
\frac{dv}{d\tau} = (u - \delta v)(u^2 + v^2)^n + \sum_{k=0}^{2n} (u^2 + v^2)^{2n-k} [(u^2 - v^2)Y_k(u, v) - 2uvX_k(u, v)].
\]
(5.2.6)

In the transformation (5.2.5),
\[
x = \frac{u}{u^2 + v^2}, \quad y = \frac{v}{u^2 + v^2}
\]
is called Bendixson reciprocal radius transformation. Making the polar coordinate transformation \( u = r \cos \theta, \ v = r \sin \theta, \) the transformation (5.2.7) becomes the transformation (5.1.5).

The transformation (5.2.5) makes infinity of system (5.2.2) become the origin of system (5.2.6). Thus, the studies of the center-focus problem and the bifurcation of limit cycles of infinity of system (5.2.2) can be changed to the studies of the corresponding problems for the origin of system (5.2.6). Since the origin of system (5.2.6) is a higher-order singular point (or degenerate singular point), it leads to some difficult problems. We discuss them in Section 6.

If for all \( k \in \{n+1, n+2, \cdots, 2n\}, \) we have \( X_k(x, y) = Y_k(x, y) = 0, \) then system (5.2.2) becomes
\[
\frac{dx}{dt} = (\delta x - y)(x^2 + y^2)^n + \sum_{k=0}^{n} X_k(x, y),
\]
\[
\frac{dy}{dt} = (x + \delta y)(x^2 + y^2)^n + \sum_{k=0}^{n} Y_k(x, y).
\]
(5.2.8)

Hence, we have the following conclusion.

**Theorem 5.2.1.** By the transformation
\[
x = \frac{u}{u^2 + v^2}, \quad y = \frac{v}{u^2 + v^2}, \quad \frac{dt}{d\tau} = (u^2 + v^2)^n,
\]
system (5.2.8) becomes the following polynomial system
\[
\frac{du}{d\tau} = - \delta u - v + \sum_{k=0}^{n} (u^2 + v^2)^{n-k} [(u^2 - v^2)X_k(u, v) + 2uvY_k(u, v)],
\]
\[
\frac{dv}{d\tau} = u - \delta v + \sum_{k=0}^{n} (u^2 + v^2)^{n-k} [(u^2 - v^2)Y_k(u, v) - 2uvX_k(u, v)],
\]
(5.2.10)
for which the origin is an elementary singular point.
We can use the following transformation
\[ x = \frac{u}{(u^2 + v^2)^{m_1}}, \quad y = \frac{v}{(u^2 + v^2)^{m_1}}, \quad \frac{dt}{d\tau} = (u^2 + v^2)^{m_2}, \tag{5.2.11} \]
such that infinity of system (5.2.2) changes to the origin which is an elementary singular point.

**Theorem 5.2.2.** By the transformation
\[ x = \frac{u}{(u^2 + v^2)^{n+1}}, \quad y = \frac{v}{(u^2 + v^2)^{n+1}}, \quad \frac{dt}{d\tau} = (u^2 + v^2)^{n(2n+1)}, \tag{5.2.12} \]
system (5.2.2) becomes the following polynomial system:
\[
\begin{align*}
\frac{du}{d\tau} &= -\delta u^{2n+1} - v + \sum_{k=1}^{2n+1} P_{2nk+k+1}(u, v), \\
\frac{dv}{d\tau} &= u - \delta v^{2n+1} + \sum_{k=1}^{2n+1} Q_{2nk+k+1}(u, v),
\end{align*}
\tag{5.2.13}
\]
for which the origin is an elementary singular point, where
\[
P_{2nk+k+1}(u, v) = \left[ \left( v^2 - \frac{1}{2n+1} u^2 \right) X_{2n+1-k}(u, v) - \frac{2n+2}{2n+1} uv Y_{2n+1-k}(u, v) \right] (u^2 + v^2)^{(k-1)(n+1)} \tag{5.2.14}
\]
and
\[
Q_{2nk+k+1}(u, v) = \left[ \left( u^2 - \frac{1}{2n+1} v^2 \right) Y_{2n+1-k}(u, v) - \frac{2n+2}{2n+1} uv X_{2n+1-k}(u, v) \right] (u^2 + v^2)^{(k-1)(n+1)} \tag{5.2.15}
\]
are homogeneous polynomials of degree \((2n+1)k+1\) of \(u\) and \(v\), \(k = 1, 2, \cdots, 2n+1\).

This theorem tells us that the studies of the center-focus problem and bifurcation of limit cycles at infinity of system (5.2.2) can be changed to the studies of the corresponding problems at the elementary singular point \(O(0, 0)\) of system (5.2.13). Because system (5.2.13) is a class of particular systems of (2.1.1). Therefore, we can apply all known theory for the center-focus problem of system (2.1.1) to system (5.2.13).

Of course, system (5.2.13) have the following particular properties.

1. The subscripts (i.e., the degree of homogeneous polynomials) of \(P_{2nk+k+1}\), \(Q_{2nk+k+1}\) form an arithmetic sequence having common difference \(2n+1\), \(k = 1, 2, \cdots, 2n+1\).
(2) \( P_{2nk+k+1} \) and \( Q_{2nk+k+1} \) have the common factor \((u^2 + v^2)^{(k-1)(n+1)}\).

(3) System (5.2.13) has a pair of conjugated complex straight line solutions \( u \pm iv = 0 \).

We can use these special properties to study the theory of center-focus at infinity for system (5.2.2).

5.3 Method of Formal Series and Singular Point Value of Infinity

By the polar coordinate transformation

\[
  u = \rho \cos \theta, \quad v = \rho \sin \theta
\]

system (5.2.13) becomes

\[
  \frac{d\rho}{d\theta} = -\frac{\rho}{2n+1} \cdot \delta + \sum_{k=1}^{2n+1} \varphi_{2n+2-k}(\theta)\rho^{k(2n+1)}.
\]

Substituting (5.3.1) and (5.1.5) into (5.2.12), we have

\[
  r = \rho^{2n+1}.
\]

Obviously, equation (5.3.2) can also be obtained from equation (5.2.4) by using transformation (5.3.3).

Let \( \rho = \tilde{\rho}(\theta, \rho_0) \) be the solution of (5.3.2) satisfying the initial condition \( \rho|_{\theta=0} = \rho_0 \). By using the particular properties of (5.2.13) mentioned in the above section, we obtain

**Proposition 5.3.1.** \( \tilde{\rho}(\theta, \rho_0)\rho_0^{-1} \) is a power series of \( \rho_0^{2n+1} \), i.e., \( \tilde{\rho}(\theta, \rho_0) \) has the following form:

\[
  \tilde{\rho}(\theta, \rho_0) = \sum_{m=1}^{\infty} \sigma_{(m-1)(2n+1)+1}(\theta)\rho_0^{(m-1)(2n+1)+1}.
\]

**Proof.** Let \( r = \tilde{r}(\theta, h) = \sum_{m=1}^{\infty} \nu_m(\theta)h^m \) be the solution of (5.2.4) satisfying the initial condition \( r|_{\theta=0} = h \). From (5.3.3), we obtain

\[
  \rho^{2n+1}(\theta, \rho_0) = \tilde{r}(\theta, \rho_0^{2n+1}).
\]
5.3 Method of Formal Series and Singular Point Value of Infinity

Let \( h_0 = \rho_0^{2n+1} \). (5.3.5) follows that
\[
\frac{\bar{\rho}(\theta, \rho_0)}{\rho_0} = \left( \frac{\bar{r}(\theta, h_0)}{h_0} \right)^{\frac{1}{2n+1}} = \left[ \sum_{m=1}^{\infty} \nu_m(\theta) h_0^{m-1} \right]^{\frac{1}{2n+1}}. \tag{5.3.6}
\]

Since the right hand of (5.3.6) can be expanded as a power series of \( h_0 \), it follows the conclusion of this proposition.

Clearly,
\[
\nu_1(2\pi) - 1 = e^{-2\pi \delta} - 1 = -2\pi \delta + o(\delta),
\]
\[
\sigma_1(2\pi) - 1 = e^{\frac{-2\pi \delta}{2n+1}} - 1 = \frac{-2\pi \delta}{2n+1} + o(\delta). \tag{5.3.7}
\]

**Theorem 5.3.1.** If \( \delta = 0 \), for any positive integer \( k \), we have
\[
\sigma_{2k(2n+1)+1}(2\pi) \sim \frac{1}{2n+1} \nu_{2k+1}(2\pi),
\]
\[
\sigma_{(2k-1)(2n+1)+1}(2\pi) \sim 0, \tag{5.3.8}
\]
and when \( m \) is not an integer multiple of \( 2n+1 \), \( \sigma_{m+1}(2\pi) = 0 \), where \( \nu_{2k+1}(2\pi) \) is the \( k \)-th focal value at infinity of system (5.2.2) and \( \sigma_{2k(2n+1)+1}(2\pi) \) is the \( k(2n+1) \)-th focal value at the origin of system (5.2.13).

**Proof.** First, when \( \delta = 0 \), we see from (5.3.7) that \( \nu_1(2\pi) = \sigma_1(2\pi) = 1 \). Thus, from (5.3.4), we have
\[
\rho^{2n+1}(2\pi, \rho_0) - \rho_0^{2n+1} = \sum_{j=0}^{2n} \rho_0^{2n-j} \bar{\rho}^j(2\pi, \rho_0)[\bar{\rho}(2\pi, \rho_0) - \rho_0]
\]
\[
= (2n+1) \rho_0^{2n} G(\rho_0)[\bar{\rho}(2\pi, \rho_0) - \rho_0]
\]
\[
= (2n+1) G(\rho_0) \sum_{m=2}^{\infty} \sigma_{(m-1)(2n+1)+1}(2\pi) \rho_0^{m(2n+1)}, \tag{5.3.9}
\]
where \( G(\rho_0) \) is an unit formal power series in \( \rho_0 \) (see Definition 1.2.3).

On the other hand, (5.3.5) follows that
\[
\rho^{2n+1}(2\pi, \rho_0) - \rho_0^{2n+1} = \bar{r}(2\pi, \rho_0^{2n+1}) - \rho_0^{2n+1}
\]
\[
= \sum_{m=2}^{\infty} \nu_m(2\pi) \rho_0^{m(2n+1)}. \tag{5.3.10}
\]

By (5.3.9) and (5.3.10), we have
\[
\sum_{m=2}^{\infty} \sigma_{(m-1)(2n+1)+1}(2\pi) \rho_0^{m(2n+1)}
\]
\[
= \frac{1}{(2n+1) G(\rho_0)} \sum_{m=2}^{\infty} \nu_m(2\pi) \rho_0^{m(2n+1)}. \tag{5.3.11}
\]
Comparing the coefficients of the same power of $\rho_0$ on the two sides of (5.3.11), it gives rise to the conclusion of this theorem.

By the transformation

$$z = x + iy, \quad w = x - iy, \quad T = it,$$

system (5.2.2) becomes

$$\frac{dz}{dT} = (1 - i\delta)z^{n+1}w^n + \sum_{k=0}^{2n} Z_k(z, w),$$

$$\frac{dw}{dT} = -(1 + i\delta)w^{n+1}z^n - \sum_{k=0}^{2n} W_k(z, w),$$

where

$$Z_k(z, w) = \sum_{\alpha+\beta=k} a_{\alpha\beta}z^\alpha w^\beta$$

$$W_k(z, w) = \sum_{\alpha+\beta=k} b_{\alpha\beta}w^\alpha z^\beta$$

We say that system (5.2.2) is the associated system of (5.3.13) and vice versa.

Let

$$\xi = u + iv, \quad \eta = u - iv, \quad T = i\tau.$$  

Then, from (5.3.12), (5.3.15) and (5.2.12), we have

$$z = \frac{\xi}{(\xi \eta)^{n+1}}, \quad w = \frac{\eta}{(\xi \eta)^{n+1}}, \quad \frac{dT}{dT} = (\xi \eta)^{n(2n+1)}.$$  

By transformation (5.3.16), system (5.3.13) can be reduced to

$$\frac{d\xi}{dT} = \left(1 + \frac{i\delta}{2n+1}\right) \xi + \xi \sum_{k=1}^{2n+1} \Phi_{k(2n+1)}(\xi, \eta),$$

$$\frac{d\eta}{dT} = -\left(1 - \frac{i\delta}{2n+1}\right) \eta - \eta \sum_{k=1}^{2n+1} \Psi_{k(2n+1)}(\xi, \eta),$$

where

$$\Phi_{k(2n+1)}(\xi, \eta) = \left[ \frac{n}{2n+1} \eta Z_{2n+1-k}(\xi, \eta) + \frac{n+1}{2n+1} \xi W_{2n+1-k}(\xi, \eta) \right] (\xi \eta)^{(k-1)(n+1)}.$$
\[ \Psi_{k(2n+1)}(\xi, \eta) = \left[ \frac{n}{2n+1} \xi W_{2n+1-k}(\xi, \eta) + \frac{n+1}{2n+1} \eta Z_{2n+1-k}(\xi, \eta) \right] (\xi \eta)^{(k-1)(n+1)} \] (5.3.18)

are homogeneous polynomials of degree \(k(2n+1)\) in \(\xi, \eta\).

Obviously, system (5.3.17) can also be obtained from system (5.2.13) by using transformation (5.3.15), thus, system (5.2.13) is the associated system of (5.3.17) and vice versa.

We next consider the case of \(\delta = 0\). When \(\delta = 0\), system (5.2.2), (5.2.13), (5.3.13) and (5.3.17) take the following forms, respectively,

\[
\begin{align*}
\frac{dx}{dt} &= -y(x^2 + y^2)^n + \sum_{k=0}^{2n} X_k(x, y) = X(x, y), \\
\frac{dy}{dt} &= x(x^2 + y^2)^n + \sum_{k=0}^{2n} Y_k(x, y) = Y(x, y); \\
\frac{du}{d\tau} &= -v + \sum_{k=1}^{2n+1} P_{2nk+k+1}(u, v) = P(u, v), \\
\frac{dv}{d\tau} &= u + \sum_{k=1}^{2n+1} Q_{2nk+k+1}(u, v) = Q(u, v); \\
\frac{dz}{dT} &= z^{n+1}w^n + \sum_{k=0}^{2n} Z_k(z, w) = Z(z, w), \\
\frac{dw}{dT} &= -w^{n+1}z^n - \sum_{k=0}^{2n} W_k(z, w) = -W(z, w); \\
\frac{d\xi}{dT} &= xi + \xi \sum_{k=1}^{2n+1} \Phi_{k(2n+1)}(\xi, \eta) = \Phi(\xi, \eta), \\
\frac{d\eta}{dT} &= -\eta - \eta \sum_{k=1}^{2n+1} \Psi_{k(2n+1)}(\xi, \eta) = -\Psi(\xi, \eta).
\end{align*}
\] (5.3.19) (5.3.20) (5.3.21) (5.3.22)

The right hand of system (5.3.22) have the following particular properties:

(1) The subscripts (the degree of homogeneous polynomials) of \(\Phi_{k(2n+1)}\), \(\Psi_{k(2n+1)}\) form an arithmetic sequence with common difference \(2n + 1\), \(k = 1, 2, \ldots, 2n + 1\).

(2) \(\Phi_{k(2n+1)}\) and \(\Psi_{k(2n+1)}\) have the common factor \((\xi \eta)^{(k-1)(n+1)}\).

(3) System (5.3.22) has a pair of straight line solutions \(\xi = 0\) and \(\eta = 0\).

From these properties of the right hand of system (5.3.22), we have
Theorem 5.3.2. For system (5.3.22), one can derive uniquely and successively the terms of the following formal series

\[ F(\xi, \eta) = (\xi \eta)^{2n+1} \left[ 1 + \sum_{m=1}^{\infty} f_m(2n+1)(\xi, \eta) \right], \]  

(5.3.23)

such that

\[ \frac{dF}{dT} \bigg|_{(5.3.22)} = \sum_{m=1}^{\infty} \mu_m (\xi \eta)^{(m+1)(2n+1)}, \]  

(5.3.24)

where

\[ f_m(2n+1)(\xi, \eta) = \sum_{\alpha+\beta=m(2n+1)} c_{\alpha\beta} \xi^\alpha \eta^\beta \]  

(5.3.25)

are homogeneous polynomials of degree \( m(2n+1) \) in \( \xi, \eta \) (\( m = 1, 2, \ldots \)) and we take

\[ c_{00} = 1, \quad c_{k(2n+1),k(2n+1)} = 0, \quad k = 1, 2, \ldots. \]  

(5.3.26)

Definition 5.3.1. For any positive integer \( m \), \( \mu_m \) given by (5.3.24) is called the \( m \)-th singular point value at infinity of system (5.3.21).

If there exists a positive integer \( k \), such that \( \mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0, \mu_k \neq 0 \), then infinity of system (5.3.21) is called a weak critical singular point of order \( k \).

If for all positive integer \( k \), \( \mu_k = 0 \), then infinity of system (5.3.21) is called a complex center.

Theorem 5.3.3. In the (5.3.25), for all pairs \( (\alpha, \beta) \), when \( \alpha \neq \beta \), and \( \alpha + \beta \geq 1 \), \( c_{\alpha\beta} \) is given by

\[ c_{\alpha\beta} = \frac{1}{(2n+1)(\beta - \alpha)} \times \sum_{k+j=1}^{2n+1} \left\{ [n\alpha - (n+1)\beta + (n-k)(2n+1)]a_{k,j-1} 
- [n\beta - (n+1)\alpha + (n-j)(2n+1)]b_{j,k-1} \right\} \times c_{\alpha+nk+(n+1)j-(n+1)(2n+1),\beta+nj+(n+1)k-(n+1)(2n+1)}. \]  

(5.3.27)

For any positive integer \( m \), \( \mu_m \) is given by

\[ \mu_m = \sum_{k+j=1}^{2n+1} [(n-k-m)a_{k,j-1} - (n-j-m)b_{j,k-1}] \times c_{nk+(n+1)j+(m-n-1)(2n+1),nj+(n+1)k+(m-n-1)(2n+1)}. \]  

(5.3.28)

where for all pairs \( (\alpha, \beta) \), when \( \alpha < 0 \) or \( \beta < 0 \), we take \( a_{\alpha\beta} = b_{\alpha\beta} = c_{\alpha\beta} = 0 \).
5.3 Method of Formal Series and Singular Point Value of Infinity

Proof. From (5.3.23), we have

\[
\frac{dF}{dT}\bigg|_{(5.3.22)} = (\xi \eta)^{2n+1} \left\{ \sum_{m=1}^{\infty} \left( \frac{\partial f_m(2n+1)}{\partial \xi} \xi - \frac{\partial f_m(2n+1)}{\partial \eta} \eta \right) \right. \\
+ \sum_{m=1}^{\infty} \sum_{s=1}^{2n+1} \left[ \xi \frac{\partial f_{m-s}(2n+1)}{\partial \xi} + (2n + 1)f_{m-s}(2n+1) \right] \Phi_s(2n+1) \\
- \sum_{m=1}^{\infty} \sum_{s=1}^{2n+1} \left[ \eta \frac{\partial f_{m-s}(2n+1)}{\partial \eta} + (2n + 1)f_{m-s}(2n+1) \right] \Psi_s(2n+1) \right\}. 
\]

By (5.3.25) and (5.3.29), we obtain

\[
\frac{dF}{dT}\bigg|_{(5.3.22)} = (\xi \eta)^{2n+1} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m(2n+1)} \sum_{s=1}^{2n+1} \left[ (\alpha + 2n + 1)\Phi_s(2n+1) \\
- (\beta + 2n + 1)\Psi_s(2n+1) \right] c_{\alpha\beta} \xi^{\alpha} \eta^{\beta}. 
\]

From (5.3.18) and (5.3.30), we get

\[
(\xi \eta)^{-(2n+1)} \frac{dF}{dT}\bigg|_{(5.3.22)} = \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m(2n+1)} \sum_{s=1}^{2n+1} \left[ (\alpha - \beta) \Phi_{s(2n+1)} \\
+ \frac{1}{2n+1} \sum_{\alpha+\beta=(m-s)(2n+1)} \left[ (n\alpha - n\beta - \beta - 2n - 1)\eta Z_{2n+1-s} \\
- (n\beta - n\alpha - \alpha - 2n - 1)\xi W_{2n+1-s} \right] c_{\alpha\beta} \xi^{\alpha+(s-1)(n+1)} \eta^{\beta+(s-1)(n+1)} \right]. 
\]

(5.3.14) becomes

\[
Z_{2n+1-s}(\xi, \eta) = \sum_{k+j=2n+2-s} a_{k,j-1} \xi^{k} \eta^{j-1}, \\
W_{2n+1-s}(\xi, \eta) = \sum_{k+j=2n+2-s} b_{j,k-1} \xi^{k-1} \eta^{j}. 
\]
Thus, (5.3.31) and (5.3.32) follow that

\[
(\xi\eta)^{-(2n+1)} \frac{dF}{dT}\bigg|_{(5.3.22)} = \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m(2n+1)} (\alpha - \beta) c_{\alpha\beta} \xi^{\alpha} \eta^{\beta} 
\]

\[
+ \frac{1}{2n+1} \sum_{m=1}^{\infty} \sum_{s=1}^{2n+1} \sum_{\alpha+\beta=(m-s)(2n+1)}^{\infty} \sum_{k+j=2n+2-s} \left[ (n\alpha - n\beta - \beta - 2n - 1) a_{k,j-1} - (n\beta - n\alpha - \alpha - 2n - 1) b_{j,k-1} \right] c_{\alpha\beta} \xi^{\alpha+(s-1)(n+1)+k} \eta^{\beta+(s-1)(n+1)+j}. 
\]  (5.3.33)

Write that

\[
\alpha_1 = \alpha + (s-1)(n+1) + k, \\
\beta_1 = \beta + (s-1)(n+1) + j. 
\]  (5.3.34)

Hence, when \( k + j = 2n + 2 - s \), \( \alpha + \beta = (m-s)(2n+1) \), we have

\[
\alpha_1 + \beta_1 = m(2n+1), \\
\alpha = \alpha_1 + nk + (n+1)j - (n+1)(2n+1), \\
\beta = \beta_1 + nj + (n+1)k - (n+1)(2n+1), \\
n\alpha - n\beta - \beta - 2n - 1 = n\alpha_1 - (n+1)\beta_1 + (n-k)(2n+1), \\
n\beta - n\alpha - \alpha - 2n - 1 = n\beta_1 - (n+1)\alpha_1 + (n-j)(2n+1). 
\]  (5.3.35)

Substituting (5.3.34), (5.3.35) into (5.3.33), and using the symbols \( \alpha, \beta \) instead of \( \alpha_1, \beta_1 \), we obtain

\[
\frac{dF}{dT}\bigg|_{(5.3.22)} = (\xi\eta)^{2n+1} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m(2n+1)} \left[ (\alpha - \beta) c_{\alpha\beta} + H_{\alpha\beta} \right] \xi^{\alpha} \eta^{\beta}, 
\]  (5.3.36)

where

\[
H_{\alpha\beta} = \frac{1}{2n+1} \sum_{k+j=1}^{2n+1} \left\{ [n\alpha - (n+1)\beta + (n-k)(2n+1)] a_{k,j-1} - [n\beta - (n+1)\alpha + (n-j)(2n+1)] b_{j,k-1} \right\} \\
\times c_{\alpha+nk+(n+1)j-(n+1)(2n+1), \beta+nj+(n+1)k-(n+1)(2n+1)}. 
\]  (5.3.37)

From (5.3.24) and (5.3.36), it gives rise to the conclusion of this theorem. \( \square \)

For any positive integer \( m \), let \( \mu'_m \) be the \( m \)-th singular point values at the origin of system (5.3.22), we have
Theorem 5.3.4. For any positive integer \( k \), we have

\[
\mu'_{k(2n+1)} \sim \frac{\mu_k}{2n+1}
\]  

(5.3.38)

and when \( m \) is not an integer multiple of \( 2n + 1 \), we have \( \mu'_m = 0 \).

Proof. For the function \( F(\xi, \eta) \) given by (5.3.23), let

\[
\hat{F}(\xi, \eta) = F_{\frac{2\pi}{2n+1}}(\xi, \eta) = \xi \eta \left[ 1 + \sum_{m=1}^{\infty} f_m(2n+1)(\xi, \eta) \right]^{\frac{1}{2n+1}}.
\]  

(5.3.39)

From (5.3.24) and (5.3.39), we have

\[
\frac{d\hat{F}}{dT} \bigg|_{(5.3.22)} = \frac{\sum_{m=1}^{\infty} \mu_m(\xi \eta)^m(2n+1)+1}{(2n+1) \left[ 1 + \sum_{m=1}^{\infty} f_m(2n+1)(\xi, \eta) \right]^{2n/(2n+1)}}.
\]  

(5.3.40)

(5.3.40) follows the conclusion of the theorem.

From Theorem 5.3.1, Theorem 5.3.4 and Theorem 1.4.4, we have

Theorem 5.3.5. For any positive integer \( k \),

\[
\sigma_{2k(2n+1)+1}(2\pi) \sim \frac{i\pi}{2n+1} \mu_k,
\]

\[
\nu_{2k+1}(2\pi) \sim i\pi \mu_k,
\]  

(5.3.41)

where \( \sigma_{2k(2n+1)+1}(2\pi) \) is the \( k(2n+1) \)-th focal value at the origin of system (5.3.20), \( \nu_{2k+1}(2\pi) \) is the \( k \)-th focal value at infinity of system (5.3.19) and \( \mu_k \) is the \( k \)-th singular point value at infinity of system (5.3.21).

From Theorem 2.3.4 and the particular properties of the right hand of system (5.3.22), we have

Theorem 5.3.6. For system (5.3.22), one can derive successively the terms of the following formal series

\[
M(\xi, \eta) = 1 + \sum_{m=1}^{\infty} g_m(2n+1)(\xi, \eta),
\]  

(5.3.42)

such that

\[
\frac{\partial(M\Phi)}{\partial\xi} - \frac{\partial(M\Psi)}{\partial\eta} = \sum_{m=1}^{\infty} \frac{2mn + m + 1}{2n+1} \lambda_m(\xi \eta)^m(2n+1),
\]  

(5.3.43)
where, for any positive integer $m$,

$$g_m(2n+1)(\xi, \eta) = \sum_{\alpha + \beta = m(2n+1)} d_{\alpha \beta} \xi^\alpha \eta^\beta \quad (5.3.44)$$

is a homogeneous polynomial of degree $m(2n+1)$ in $\xi, \eta$, and

$$\lambda_m \sim (2n+1)\mu'_m(2n+1) \sim \mu_m. \quad (5.3.45)$$

Similar to Theorem 5.3.3, we have

**Theorem 5.3.7.** In the right hand of (5.3.42), letting $d_{00} = 1$ and taking $d_{k(2n+1),k(2n+1)}$ as arbitrary numbers, then for all $(\alpha, \beta)$, when $\alpha \neq \beta$, and $\alpha + \beta \geq 1$, $d_{\alpha \beta}$ is given by

$$d_{\alpha \beta} = \frac{1}{(2n+1)(\beta - \alpha)} \sum_{k+j=1}^{2n+1} \left\{ [n\alpha - (n+1)\beta - 1]a_{k,j-1} 
- [n\beta - (n+1)\alpha - 1]b_{j,k-1} \right\}
\times d_{\alpha + nk + (n+1)j - (n+1)(2n+1), \beta + nj + (n+1)k - (n+1)(2n+1)} \quad (5.3.46)$$

and for any positive integer $m$, $\lambda_m$ is determined by

$$\lambda_m = \sum_{k+j=1}^{2n+1} (b_{j,k-1} - a_{k,j-1})
\times d_{nk + (n+1)j + (m-n-1)(2n+1), nj + (n+1)k + (m-n-1)(2n+1)}, \quad (5.3.47)$$

where for all $(\alpha, \beta)$, when $\alpha < 0$ or $\beta < 0$, we take $a_{\alpha \beta} = b_{\alpha \beta} = d_{\alpha \beta} = 0$.

Theorem 5.3.3 and Theorem 5.3.7 give the recursive formulas to compute singular point values at infinity of system (5.3.21).

**Theorem 5.3.8.** For system (5.3.21), one can derive successively the terms of the following formal series

$$F(z, w) = \frac{1}{zw} \left[ 1 + \sum_{m=1}^{\infty} \frac{f_m(2n+1)(z, w)}{(zw)^{m(n+1)}} \right], \quad (5.3.48)$$

such that

$$\frac{dF}{dT}_{(5.3.21)} = (zw)^n \sum_{m=1}^{\infty} \frac{\mu_m}{(zw)^{m+1}}, \quad (5.3.49)$$

where $\mu_m$ is the $m$-th singular point value at infinity of system (5.3.21), $m = 1, 2, \cdots$. 
5.3 Method of Formal Series and Singular Point Value of Infinity

Proof. The inverse transformation of (5.3.16) is

\[
\xi = z(zw)^{-\frac{(n+1)}{2n+1}}, \quad \eta = w(zw)^{-\frac{(n+1)}{2n+1}}, \quad \frac{d\mathcal{T}}{dT} = (zw)^n. \tag{5.3.50}
\]

By (5.3.23) and (5.3.48), we have

\[
\mathcal{F}(z, w) = F\left(z(zw)^{-\frac{(n+1)}{2n+1}}, w(zw)^{-\frac{(n+1)}{2n+1}}\right). \tag{5.3.51}
\]

From (5.3.50), (5.3.51) and (5.3.24), it gives rise to the conclusion of this theorem. \(\Box\)

**Theorem 5.3.9.** For system (5.3.21), one can derive successively the terms of the following formal series

\[
\mathcal{M}(z, w) = (zw)^{-n-1-\frac{1}{2n+1}} \left[ 1 + \sum_{m=1}^{\infty} g_m (2n+1) (zw)^{m(n+1)} \right], \tag{5.3.52}
\]

such that

\[
\frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} = (zw)^{-1-\frac{1}{2n+1}} \sum_{m=1}^{\infty} (2mn + m + 1) \lambda_m (zw)^m, \tag{5.3.53}
\]

where \(\lambda_m \sim \mu_m, m = 1, 2, \ldots\).

Proof. First, by (5.3.42), we have

\[
\mathcal{M}(z, w) = (zw)^{-n-1-\frac{1}{2n+1}} M \left[ z(zw)^{-\frac{(n+1)}{2n+1}}, w(zw)^{-\frac{(n+1)}{2n+1}} \right]. \tag{5.3.54}
\]

We consider the system

\[
\frac{dz}{d\mathcal{T}} = \frac{MZ}{(zw)^n} = Z(z, w),
\]
\[
\frac{dw}{d\mathcal{T}} = -\frac{MW}{(zw)^n} = -W(z, w). \tag{5.3.55}
\]

By the transformation

\[
\xi = z(zw)^{-\frac{(n+1)}{2n+1}}, \quad \eta = w(zw)^{-\frac{(n+1)}{2n+1}}, \tag{5.3.56}
\]

system (5.3.55) becomes

\[
\frac{d\xi}{d\mathcal{T}} = M(\xi, \eta)\Phi(\xi, \eta), \quad \frac{d\eta}{d\mathcal{T}} = -M(\xi, \eta)\Psi(\xi, \eta). \tag{5.3.57}
\]

The Jacobin determinant of transformation (5.3.56) is given by

\[
J = \left| \frac{\partial\xi}{\partial z} \frac{\partial\eta}{\partial w} - \frac{\partial\xi}{\partial w} \frac{\partial\eta}{\partial z} \right| = \frac{-1}{2n+1} (zw)^{-1-\frac{1}{2n+1}}. \tag{5.3.58}
\]
Then, from (5.3.54), (5.3.55) and (5.3.58), we have
\[ \frac{-1}{2n+1} MZ = JZ, \quad \frac{-1}{2n+1} MW = JW. \tag{5.3.59} \]
Thus,
\[ \frac{-1}{2n+1} \left[ \frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} \right] = \frac{\partial(JZ)}{\partial z} - \frac{\partial(JW)}{\partial w}. \tag{5.3.60} \]
By applying Proposition (1.1.3) to systems (5.3.55) and (5.3.57), from (5.3.57) we get
\[ \frac{\partial(JZ)}{\partial z} - \frac{\partial(JW)}{\partial w} = J \left[ \frac{\partial(M\Phi)}{\partial \xi} - \frac{\partial(M\Psi)}{\partial \eta} \right] \]
\[ = \frac{-1}{2n+1} (zw)^{-1} \left[ \frac{\partial(M\Phi)}{\partial \xi} - \frac{\partial(M\Psi)}{\partial \eta} \right]. \tag{5.3.61} \]
From (5.3.60) and (5.3.61), it follows that
\[ \frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} = (zw)^{-1} \left[ \frac{\partial(M\Phi)}{\partial \xi} - \frac{\partial(M\Psi)}{\partial \eta} \right]. \tag{5.3.62} \]
(5.3.62), (5.3.43) and (5.3.56) give rise to the conclusion of this theorem. □

We now consider the following formal series
\[ H(z, w) = 1 + \sum_{m=1}^{\infty} \frac{h_{m(2n+1)}(z, w)}{(zw)^{m(n+1)}}, \tag{5.3.63} \]
where
\[ h_{m(2n+1)}(\xi, \eta) = \sum_{\alpha+\beta=m(2n+1)} e_{\alpha\beta} z^\alpha w^\beta \tag{5.3.64} \]
are homogeneous polynomials of degree \( m(2n+1) \) in \( z, w \) (\( m = 1, 2, \cdots \)), and \( h_0 = e_{00} = 1 \).

Reference [Liu Y.R., 2001] gave the following two theorems.

**Theorem 5.3.10.** For all \( s \neq 0, \gamma \neq 0 \), one can derive successively the terms of the formal series
\[ \tilde{F}(z, w) = (zw)^s H^\frac{1}{\gamma}(z, w), \tag{5.3.65} \]
such that
\[ \frac{d\tilde{F}}{dT}\bigg|_{(5.3.21)} = \frac{1}{\gamma} (zw)^{n+s} H^\frac{1}{\gamma} \sum_{m=1}^{\infty} \frac{\lambda'_m}{(zw)^m} \tag{5.3.66} \]
and for any positive integer \( m \),
\[ \lambda'_m \sim -s\gamma \mu_m. \tag{5.3.67} \]
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Theorem 5.3.11. Let \( s, \gamma \) be two constants, if for any positive integer \( m \), \( \gamma(s+n+1-m) \neq 0 \), then one can derive successively the terms of the formal series

\[
\tilde{M}(z, w) = (zw)^{s}H_{\frac{1}{\gamma}}^{1}(z, w),
\]

such that

\[
\frac{\partial(\tilde{M}Z)}{\partial z} - \frac{\partial(\tilde{M}W)}{\partial w} = \frac{1}{\gamma} (zw)^{n+s}H_{\frac{1}{\gamma}-1}^{1}(z, w) \sum_{m=1}^{\infty} \frac{\lambda''_{m}}{(zw)^{m}}
\]

and for any positive integer \( m \),

\[
\lambda''_{m} \sim -\gamma(s+n+1-m)\mu_{m}.
\]

By Theorem 5.3.10 and Theorem 5.3.11, the authors of [Chen H.B. etc, 2005b] gave the recursive formulas to compute singular point values of infinity of system (5.3.21) as follows.

Theorem 5.3.12. For the formal series \( \tilde{F} \) given by Theorem 5.3.10, \( e_{k(2n+1),k(2n+1)} \) can be taken arbitrarily, \( k = 1, 2, \cdots \). If \( \alpha \neq \beta \) and \( \alpha + \beta \geq 1 \), \( e_{\alpha\beta} \) is given by

\[
e_{\alpha\beta} = \frac{1}{(2n+1)(\beta - \alpha)}
\times \sum_{2n+1}^{k+j=1} \left\{[n\alpha - (n+1)\beta + (\gamma s + n + 1 - k)(2n+1)]a_{k,j-1}
- [n\beta - (n+1)\alpha + (\gamma s + n + 1 - j)(2n+1)]b_{j,k-1}\right\}
\times e_{\alpha+nk+(n+1)j-(n+1)(2n+1),\beta+nj+(n+1)k-(n+1)(2n+1)}.
\]

(5.3.71)

For any positive integer \( m \), \( \lambda'_{m} \) is given by

\[
\lambda'_{m} = \sum_{2n+1}^{k+j=1} [(\gamma s + n + 1 - k - m)a_{k,j-1}
- (\gamma s + n + 1 - j - m)b_{j,k-1}]
\times e_{nk+(n+1)j+(m-n-1)(2n+1),nj+(n+1)k+(m-n-1)(2n+1)}.
\]

(5.3.72)

In above two recursive formulas, for all \((\alpha, \beta)\), if \( \alpha < 0 \) or \( \beta < 0 \), we take \( a_{\alpha\beta} = b_{\alpha\beta} = e_{\alpha\beta} = 0 \).

Theorem 5.3.13. For the formal series \( \tilde{M} \) given by Theorem 5.3.11, \( e_{k(2n+1),k(2n+1)} \) can be taken arbitrarily, \( k = 1, 2, \cdots \). If \( \alpha \neq \beta \), and \( \alpha + \beta \geq 1 \), \( e_{\alpha\beta} \) is given by
\[ e_{\alpha\beta} = \frac{1}{(2n+1)(\beta - \alpha)} \times \sum_{k+j=1}^{2n+1} \left\{ \left[ n\alpha - (n+1)\beta + (\gamma s + n + 1 - k)(2n + 1) \right] a_{k,j-1} \right. \\
\left. - [n\beta - (n+1)\alpha + (\gamma s + \gamma j + n + 1 - j)(2n + 1)] b_{j,k-1} \right\} \times e^{\alpha nk + (n+1)j - (n+1)(2n+1),\beta + nj + (n+1)k - (n+1)(2n+1)}, \] (5.3.73)

else \( e_{\alpha\beta} = 0 \).

For any positive integer \( m \), \( \lambda''_m \) is given by

\[ \lambda''_m = \sum_{k+j=1}^{2n+1} \left[ (\gamma s + \gamma k + n + 1 - k - m)a_{k,j-1} \right. \\
\left. - (\gamma s + \gamma j + n + 1 - j - m)b_{j,k-1} \right] \times e^{nk+(n+1)j-(m-n-1)(2n+1),nj+(n+1)(2n+1)k-(m-n-1)(2n+1)}. \] (5.3.74)

In above two recursive formulas, for \( \forall (\alpha, \beta) \), when \( \alpha < 0 \) or \( \beta < 0 \), we take \( a_{\alpha\beta} = b_{\alpha\beta} = e_{\alpha\beta} = 0 \).

From Theorem 5.3.10 and Theorem 5.3.11, we have

**Theorem 5.3.14.** Infinity of system (5.3.21) is a complex center if and only if there exists a first integral \( \tilde{F}(z, w) \) with the form (5.3.65).

**Theorem 5.3.15.** Infinity of system (5.3.21) is a complex center if and only if there exists an integral factor \( \tilde{M}(z, w) \) with the form (5.3.68).

### 5.4 The Algebraic Construction of Singular Point Values of Infinity

By means of the transformation

\[ z = \rho e^{i\phi}\hat{z}, \quad w = \rho e^{-i\phi}\hat{w}, \quad T = \rho^{-2n}\hat{T}, \] (5.4.1)

system (5.3.21) becomes

\[ \frac{d\hat{z}}{dT} = (\hat{z})^{n+1}(\hat{w})^n + \sum_{\alpha+\beta=0}^{2n} \hat{a}_{\alpha\beta}(\hat{z})^\alpha(\hat{w})^\beta, \]

\[ \frac{d\hat{w}}{dT} = -(\hat{w})^{n+1}(\hat{z})^n - \sum_{\alpha+\beta=0}^{2n} \hat{b}_{\alpha\beta}(\hat{w})^\alpha(\hat{z})^\beta, \] (5.4.2)
where \( \hat{z}, \hat{w}, \hat{T} \) are new variables and \( \rho, \phi \) are complex constants, \( \rho \neq 0 \) and for all \((\alpha, \beta)\),

\[
\begin{align*}
\hat{a}_{\alpha\beta} &= a_{\alpha\beta}\rho^{\alpha+\beta-2n-1}e^{i(\alpha-\beta-1)\phi}, \\
\hat{b}_{\alpha\beta} &= b_{\alpha\beta}\rho^{\alpha+\beta-2n-1}e^{-i(\alpha-\beta-1)\phi}.
\end{align*}
\tag{5.4.3}
\]

If

\[
\begin{align*}
z &= x + iy, \quad \hat{z} = \hat{x} + i\hat{y}, \\
w &= x - iy, \quad \hat{w} = \hat{x} - i\hat{y}, \\
T &= it, \quad \hat{T} = i\hat{t},
\end{align*}
\tag{5.4.4}
\]

then transformation (5.4.1) becomes

\[
\begin{align*}
x &= \rho(\hat{x}\cos\phi - \hat{y}\sin\phi), \\
y &= \rho(\hat{x}\sin\phi + \hat{y}\cos\phi), \\
t &= \rho^{-2n}\hat{t}.
\end{align*}
\tag{5.4.5}
\]

Compared with transformation (2.4.1), transformation (5.4.1) has a new time scale \( T = \rho^{-2n}\hat{T} \). We say that transformation (5.4.1) is a generalized rotation and similar transformation with time exponent \( n \).

**Definition 5.4.1.** For systems (5.4.1), assume that \( f = f(a_{\alpha\beta}, b_{\alpha\beta}) \) is a polynomial in \( a_{\alpha\beta}, b_{\alpha\beta} \). Denote that \( \hat{f} = f(\hat{a}_{\alpha\beta}, \hat{b}_{\alpha\beta}) \), \( f^* = f(b_{\alpha\beta}, a_{\alpha\beta}) \). If there exist \( \lambda, \sigma \), such that \( \hat{f} = \rho^\lambda e^{i\sigma\phi}f \), then \( \lambda \) and \( \sigma \) are respectively called the similar exponent and the rotation exponent with time exponent \( n \) of \( f \) under the transformation (5.4.1), which are represented by \( I_s^{(n)}(f) = \lambda, \ I_r^{(n)}(f) = \sigma \).

We see from (5.4.3) and Definition 5.4.1 that

\[
\begin{align*}
I_s^{(n)}(a_{\alpha\beta}) &= \alpha + \beta - 2n - 1, \\
I_r^{(n)}(a_{\alpha\beta}) &= \alpha - \beta - 1, \\
I_s^{(n)}(b_{\alpha\beta}) &= \alpha + \beta - 2n - 1, \\
I_r^{(n)}(b_{\alpha\beta}) &= -(\alpha - \beta - 1).
\end{align*}
\tag{5.4.6}
\]

Obviously, for the generalized rotation and similar transformation in Definition Section 2.4, the similar exponent, the rotation exponent and the generalized rotation invariant all have time exponent 0. From (2.4.4) and (5.4.3), we have

\[
\hat{a}_{\alpha\beta} = \rho^{-2n}\tilde{a}_{\alpha\beta}, \quad \hat{b}_{\alpha\beta} = \rho^{-2n}\tilde{b}_{\alpha\beta}.
\tag{5.4.7}
\]

In addition, (2.4.5) and (5.4.6) imply that

\[
\begin{align*}
I_s^{(n)}(a_{\alpha\beta}) &= I_s(a_{\alpha\beta}) - 2n, \\
I_r^{(n)}(a_{\alpha\beta}) &= I_r(a_{\alpha\beta}), \\
I_s^{(n)}(b_{\alpha\beta}) &= I_s(b_{\alpha\beta}) - 2n, \\
I_r^{(n)}(b_{\alpha\beta}) &= I_r(b_{\alpha\beta}).
\end{align*}
\tag{5.4.8}
\]
Proposition 5.4.1. Suppose that \( f_1 = f_1(a_{\alpha\beta}, b_{\alpha\beta}) \) and \( f_2 = f_2(a_{\alpha\beta}, b_{\alpha\beta}) \) are polynomials in \( a_{\alpha\beta}, b_{\alpha\beta} \). If there exist \( \lambda_1, \lambda_2, \sigma_1, \sigma_2, \) such that \( \hat{f}_1 = \rho^{\lambda_1} e^{i\sigma_1} f_1, \hat{f}_2 = \rho^{\lambda_2} e^{i\sigma_2} f_2 \), then

\[
I_s^{(n)}(f_1 f_2) = I_s^{(n)}(f_1) + I_s^{(n)}(f_2), \quad I_r^{(n)}(f_1 f_2) = I_r^{(n)}(f_1) + I_r^{(n)}(f_2). \tag{5.4.9}
\]

We see from Proposition 5.4.1 and formula (5.4.6) that

Proposition 5.4.2. For \( m_1 + m_2 \) order monomial

\[
g = \prod_{j=1}^{m_1} a_{\alpha_j, \beta_j} \prod_{k=1}^{m_2} b_{\gamma_k, \delta_k}, \tag{5.4.10}
\]

given by the coefficients of system (5.3.21), we have

\[
I_s^{(n)}(g) = \sum_{j=1}^{m_1} (\alpha_j + \beta_j - 2n - 1) + \sum_{k=1}^{m_2} (\gamma_k + \delta_k - 2n - 1),
\]

\[
I_r^{(n)}(g) = \sum_{j=1}^{m_1} (\alpha_j - \beta_j - 1) - \sum_{k=1}^{m_2} (\gamma_k - \delta_k - 1). \tag{5.4.11}
\]

Remark 5.4.1. For the coefficients \( a_{\alpha\beta}, b_{\alpha\beta} \) of system (5.3.21), we have \( 0 \leq \alpha + \beta \leq 2n \), thus, from (5.4.11), for \( m_1 + m_2 \) order monomial \( g \) of the coefficients of system (5.3.21), we have \( I_s^{(n)}(g) < 0 \).

From Proposition 5.4.2 and Theorem (t2.4.2), we obtain

Proposition 5.4.3.

\[
I_s^{(n)}(g) = I_s(g) - 2n(m_1 + m_2), \quad I_r^{(n)}(g) = I_r(g). \tag{5.4.12}
\]

Definition 5.4.2. (1) Suppose that \( f = f(a_{\alpha\beta}, b_{\alpha\beta}) \) is a polynomial in \( a_{\alpha\beta}, b_{\alpha\beta} \). If \( \hat{f} = \rho^{2k} f \), then \( f \) is called a \( k \)-order generalized rotation invariant with time exponent \( n \) under the transformation (5.4.1).

(2) A generalized rotation invariant \( f \) is called a monomial generalized rotation invariant, if \( f \) is a monomial of \( a_{\alpha\beta}, b_{\alpha\beta} \).

(3) A monomial generalized rotation invariant \( f \) is called an elementary generalized rotation invariant if it can not be expressed as a product of two monomial generalized rotation invariant.

(4) A generalized rotation invariant \( f \) is called self-symmetry, if \( f^* = f \). It is called antisymmetry, if \( f^* = -f \).

From Proposition 5.4.1 and Definition 5.4.2, we have
Proposition 5.4.4. Suppose that \( f_1 \) and \( f_2 \) are monomial generalized rotation invariants (or elementary generalized rotation invariants), then so are \( f_1^* \) and \( f_1f_2 \), moreover,

\[
I_s^{(n)}(f_1^*) = I_s^{(n)}(f_1), \quad I_s^{(n)}(f_1f_2) = I_s^{(n)}(f_1) + I_s^{(n)}(f_2).
\] (5.4.13)

We see from Proposition 5.4.2 and Definition 5.4.2 that

Proposition 5.4.5. The \( m_1+m_2 \) order monomial given by (5.4.10) is a \( N \)-order generalized rotation invariant if and only if

\[
I_s^{(n)}(g) = m_1 \sum_{j=1}^{m_1} (\alpha_j + \beta_j - 2n - 1) + m_2 \sum_{k=1}^{m_2} (\gamma_k + \delta_k - 2n - 1) = 2N,
\]

\[
I_r^{(n)}(g) = m_1 \sum_{j=1}^{m_1} (\alpha_j - \beta_j - 1) - m_2 \sum_{k=1}^{m_2} (\gamma_k - \delta_k - 1) = 0.
\] (5.4.14)

Lemma 5.4.1. For any positive integer \( m \), the \( m \)-th singular point value \( \mu_m \) at infinity of system (5.3.21) is a “\( -m \)” order generalized rotation invariant with time exponent \( n \) under the transformation (5.4.1), i.e.,

\[
\hat{\mu}_m = \rho^{-2m} \mu_m.
\] (5.4.15)

Proof. For the function \( \mathcal{F}(z, w) \) given by Theorem 5.3.8, let \( \hat{\mathcal{F}} = \rho^2 \mathcal{F}(\rho e^{i\phi} \hat{z}, \rho e^{-i\phi} \hat{w}) \), then, from (5.3.48), we have

\[
\hat{\mathcal{F}} = \frac{1}{\hat{z}\hat{w}} \left[ 1 + \sum_{m=1}^{\infty} \frac{f_m(2n+1)(\hat{z}e^{i\phi}, \hat{w}e^{-i\phi})}{\rho^m(\hat{z}\hat{w})^m(n+1)} \right].
\] (5.4.16)

And from (5.3.49), we have

\[
\left. \frac{d\hat{\mathcal{F}}}{dT} \right|_{(5.4.2)} = (\hat{z}\hat{w})^n \sum_{m=1}^{\infty} \frac{\mu_m}{\rho^{2m}(\hat{z}\hat{w})^m+1}.
\] (5.4.17)

(5.4.17) leads (5.4.15), thus, Lemma 5.4.1 holds.

Lemma 5.4.2. For any positive integer \( m \), the \( m \)-th singular point value \( \mu_m \) at infinity of system (5.3.21) is antisymmetry, i.e.,

\[
\hat{\mu}_m^* = -\mu_m.
\] (5.4.18)

Proof. By the antisymmetry transformation

\[
z = w^*, \quad w = z^*, \quad T = -T^*,
\] (5.4.19)
system (5.3.21) becomes
\[
\frac{dz^*}{dT^*} = (z^*)^{n+1}(w^*)^n + \sum_{k=0}^{2n} W_k(w^*, z^*) = W(w^*, z^*),
\]
\[
\frac{dw^*}{dT^*} = -(w^*)^{n+1}(z^*)^n - \sum_{k=0}^{2n} Z_k(w, z) = -Z(w^*, z^*). \tag{5.4.20}
\]

For the function \( F(z, w) \) given by Theorem 5.3.8, let \( F^* = F(w^*, z^*) \), then from (5.3.48), we have
\[
F^* = \frac{1}{z^*w^*} \left[ 1 + \sum_{m=1}^{\infty} \frac{f_m(2n+1)(w^*, z^*)}{(z^*w^*)^{m(n+1)}} \right]. \tag{5.4.21}
\]

From (5.3.49), we obtain
\[
\left. \frac{dF^*}{dT^*} \right|_{(5.4.20)} = (z^*w^*)^n \sum_{m=1}^{\infty} \frac{(-\mu_m^*)}{(z^*w^*)^{m+1}}. \tag{5.4.22}
\]

(5.4.22) follows (5.4.18). Thus, the conclusion of this lemma holds.

We see from Lemma 5.4.1 and 5.4.2 that

**Theorem 5.4.1 (The construction theorem of singular point values at infinity).** For any positive integer \( m \), the \( m \)-th singular point value \( \mu_m \) at infinity of system (5.3.21) can be represented as a linear combination of “\(-m\)” order monomial generalized rotation invariants with time exponent \( n \) and their antisymmetry forms, i.e.,
\[
\mu_m = \sum_{j=1}^{N} \gamma_{kj}(g_{kj} - g_{kj}^*), \quad k = 1, 2, \cdots, \tag{5.4.23}
\]
where \( N \) is a positive integer and \( \gamma_{kj} \) are rational numbers, \( g_{kj} \) and \( g_{kj}^* \) are \(-m\) order monomial generalized rotation invariants with time exponent \( n \) of system (5.3.21).

This theorem follows that

**Theorem 5.4.2 (The extended symmetric principle at infinity).** If all elementary generalized rotation invariants \( g \) of (5.3.21) satisfy symmetric condition \( g = g^* \), then all singular point values at infinity of system (5.3.21) are zero.

Under the translational transformation
\[
z' = z - z_0, \quad w' = w - w_0, \tag{5.4.24}
\]
5.5 Singular Point Values at Infinity and Integrable Conditions

The system (5.3.21) becomes

\[
\frac{dz'}{dT} = (z')^{n+1}(w')^n + \sum_{\alpha+\beta=0}^{2n} a'_{\alpha\beta}(z')^\alpha(w')^\beta,
\]

\[
\frac{dw'}{dT} = -(w')^{n+1}(z')^n - \sum_{\alpha+\beta=0}^{2n} b'_{\alpha\beta}(w')^\alpha(z')^\beta.
\] (5.4.25)

For any positive integer \(m\), the \(m\)-th singular point value at infinity of system (5.4.25) is written by \(\mu'_m\). In [Liu Y.R. etc, 2006c], the authors proved that

**Theorem 5.4.3.** In the sense of the algebraic equivalence, singular point value at infinity of system (5.3.21) have the property of translational invariance. Namely,

\[
\{\mu'_m\} \sim \{\mu_m\}. \tag{5.4.26}
\]

5.5 Singular Point Values at Infinity and Integrable Conditions for a Class of Cubic System

Consider a class of real planar cubic system

\[
\frac{dx}{dt} = X_1(x, y) + X_2(x, y) + (\delta x - y)(x^2 + y^2),
\]

\[
\frac{dy}{dt} = Y_1(x, y) + Y_2(x, y) + (x + \delta y)(x^2 + y^2). \tag{5.5.1}
\]

where \(X_k, Y_k\) are homogeneous polynomials of degree \(k\) in \(x, y\), \(k = 1, 2\).

When \(\delta = 0\), by means of transformation

\[
z = x + iy, \quad w = x - iy, \quad T = it,
\] (5.5.2)

system (5.5.1) can be reduced to

\[
\frac{dz}{dT} = a_{10}z + a_{01}w + a_{20}z^2 + a_{11}zw + a_{02}w^2 + z^2w,
\]

\[
\frac{dw}{dT} = -b_{10}w - a_{01}z - b_{20}w^2 - b_{11}wz - b_{02}w^2 - w^2z. \tag{5.5.3}
\]

If

\[
a_{10} = A_{10} + iB_{10}, \quad b_{10} = A_{10} - iB_{10},
\]

\[
a_{01} = A_{01} + iB_{01}, \quad b_{01} = A_{01} - iB_{01},
\]

\[
a_{20} = A_{20} + iB_{20}, \quad b_{20} = A_{20} - iB_{20},
\]

\[
a_{11} = A_{11} + iB_{11}, \quad b_{11} = A_{11} - iB_{11},
\]

\[
a_{02} = A_{02} + iB_{02}, \quad b_{02} = A_{02} - iB_{01}, \tag{5.5.4}
\]
system (5.5.1)$_{\delta=0}$ can be reduced to
\[
\frac{dx}{dt} = -(B_{10} + B_{01})x - (A_{10} - A_{01})y - (B_{20} + B_{11} + B_{02})x^2 \\
-2(A_{20} - A_{02})xy + (B_{02} - B_{11} + B_{02})y^2 - y(x^2 + y^2),
\]
\[
\frac{dy}{dt} = (A_{10} + A_{01})x - (B_{10} - B_{01})y + (A_{20} + A_{11} + A_{02})x^2 \\
-2(B_{20} - B_{02})xy - (A_{20} - A_{11} + A_{02})y^2 + x(x^2 + y^2). \quad (5.5.5)
\]

By using the following generalized rotation and similar transformation with time exponent 1:
\[
z = \rho e^{i\phi} \tilde{z}, \quad w = \rho e^{-i\phi} \tilde{w}, \quad T = \rho^{-2} \tilde{T} \quad (5.5.6)
\]
and (5.4.6), we obtain the similar exponent and rotation exponent of all $a_{\alpha\beta}, b_{\alpha\beta}$ as follows:
\[
I_s^{(1)}(a_{10}) = I_s^{(1)}(b_{10}) = -3, \quad I_r^{(1)}(a_{10}) = 0, \quad I_r^{(1)}(b_{10}) = 0,
\]
\[
I_s^{(1)}(a_{01}) = I_s^{(1)}(b_{01}) = -3, \quad I_r^{(1)}(b_{01}) = 2, \quad I_r^{(1)}(a_{01}) = -2,
\]
\[
I_s^{(1)}(a_{20}) = I_s^{(1)}(b_{20}) = -1, \quad I_r^{(1)}(a_{20}) = 1, \quad I_r^{(1)}(b_{20}) = -1,
\]
\[
I_s^{(1)}(a_{11}) = I_s^{(1)}(b_{11}) = -1, \quad I_r^{(1)}(b_{11}) = 1, \quad I_r^{(1)}(a_{11}) = -1,
\]
\[
I_s^{(1)}(a_{02}) = I_s^{(1)}(b_{02}) = -1, \quad I_r^{(1)}(b_{02}) = 3, \quad I_r^{(1)}(a_{02}) = -3. \quad (5.5.7)
\]

From (5.5.7), we know all elementary generalized rotation invariant of system (5.5.3). For example, since
\[
I_s^{(1)}(b_{01}^3a_{02}^2) = 3I_s^{(1)}(b_{01}) + 2I_s^{(1)}(a_{02}) = -8,
\]
\[
I_r^{(1)}(b_{01}^3a_{02}^2) = 3I_r^{(1)}(b_{01}) + 2I_r^{(1)}(a_{02}) = 0,
\]
$b_{01}^3a_{02}^2$ is “$-8/2 = -4$” order generalized rotation invariant. It can not be expressed as a product of two monomial generalized rotation invariants. Namely, the generalized rotation invariant is “elementary”. Similarly, $(b_{01}^3a_{02}^2)^* = a_{01}^3b_{02}^2$ is also a “$-4$” order elementary generalized rotation invariant.

**Theorem 5.5.1.** System (5.5.3) has exactly 32 elementary generalized rotation invariants with time exponent 1 at infinity, which are listed as follows.

By means of transformation
\[
z = \frac{\xi}{\xi^2\eta^2}, \quad w = \frac{\eta}{\xi^2\eta^2}, \quad \frac{dT}{d\tilde{T}} = \xi^3\eta^3, \quad (5.5.8)
\]
system (5.5.3) becomes a 7-th differential system with the elementary singular point as follows:

\[
\frac{d\xi}{dT} = \xi + \frac{1}{3} \left[ 2b_{02}\xi^3 + (a_{20} + 2b_{11})\xi^2\eta + (a_{11} + 2b_{20})\xi\eta^2 \right] \xi \\
+ \frac{1}{3} \left[ 2b_{01}\xi^2 + (a_{10} + 2b_{10})\xi\eta + a_{01}\eta^2 \right] \xi^3\eta^2 = \Phi(\xi, \eta),
\]

\[
\frac{d\eta}{dT} = -\eta - \frac{1}{3} \left[ 2a_{02}\eta^3 + (b_{20} + 2a_{11})\eta^2\xi + (b_{11} + 2a_{20})\eta\xi^2 \right] \eta \\
- \frac{1}{3} \left[ 2a_{01}\eta^2 + (b_{10} + 2a_{10})\eta\xi + b_{01}\xi^2 \right] \eta^3\xi^2 = -\Psi(\xi, \eta). \tag{5.5.9}
\]

From Theorem 5.3.6, we have

**Theorem 5.5.2.** For system (5.5.9), one can derive successively the terms of the following formal series

\[
M(\xi, \eta) = 1 + \sum_{m=1}^{\infty} \sum_{\alpha+\beta=3m} d_{\alpha\beta} \xi^\alpha \eta^\beta, \tag{5.5.10}
\]

such that

\[
\frac{\partial(M\Phi)}{\partial\xi} - \frac{\partial(M\Psi)}{\partial\eta} = \sum_{m=1}^{\infty} 3m + 1 \frac{3}{3} \lambda_m(\xi\eta)^{3m}. \tag{5.5.11}
\]

In addition, for any positive integer number \(m\),

\[
\lambda_m \sim 3\mu_{3m} \sim \mu_m, \tag{5.5.12}
\]

where \(\mu_{3m}\) is the 3\(m\)-th singular point value at the origin of system (5.5.9), and \(\mu_m\) is the \(m\)-th singular point value at infinity of system (5.5.3).

From Theorem 5.3.7, we know that
Theorem 5.5.3. For (5.5.10), letting \( d_{00} = 1 \) and taking \( d_{3k,3k} = 0 \), \( k = 1, 2, \ldots \), then for all \((\alpha, \beta)\) and \(\alpha \neq \beta\), \(d_{\alpha\beta}\) is given by the recursive formula

\[
d_{\alpha\beta} = \frac{1}{3(\beta - \alpha)} \sum_{k+j=1}^{3} [(\alpha - 2\beta - 1)a_{k,j-1}
- (\beta - 2\alpha - 1)b_{j,k-1}]d_{\alpha+k+2j-6,\beta+j+2k-6}.
\]

(5.5.13)

For any positive integer \(m\), \(\lambda_m\) is given by the recursive formula

\[
\lambda_m = \sum_{k+j=1}^{3} (b_{j,k-1} - a_{k,j-1})d_{k+2j+3m-6,j+2k+3m-6}.
\]

(5.5.14)

where for all \((\alpha, \beta)\), when \(\alpha < 0\) or \(\beta < 0\), we take \(a_{\alpha\beta} = b_{\alpha\beta} = d_{\alpha\beta} = 0\).

Theorem 5.5.1 and Theorem 5.5.2 give the recursive formulas to compute directly the singular point values at the origin of system (5.5.3). By using computer algebra system Mathematica, we obtain the terms of the first eight singular point values at infinity of system (5.5.3) as follows:

<table>
<thead>
<tr>
<th>(\mu_k)</th>
<th>(\mu_1)</th>
<th>(\mu_2)</th>
<th>(\mu_3)</th>
<th>(\mu_4)</th>
<th>(\mu_5)</th>
<th>(\mu_6)</th>
<th>(\mu_7)</th>
<th>(\mu_8)</th>
<th>(\mu_9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>terms</td>
<td>4</td>
<td>30</td>
<td>166</td>
<td>600</td>
<td>1764</td>
<td>4516</td>
<td>10378</td>
<td>21984</td>
<td>(\ldots)</td>
</tr>
</tbody>
</table>

We see from this table that the expressions of the singular point values are very long. We need to simplify them.

When \(A_{10} = \lambda, A_{01} = B_{01} = B_{10} = 0\), system (5.5.5) can be reduced to

\[
\begin{align*}
\frac{dx}{dt} &= -\lambda y - (B_{20} + B_{11} + B_{02})x^2 - 2(A_{20} - A_{02})xy \\
&\quad + (B_{20} - B_{11} + B_{02})y^2 - y(x^2 + y^2), \\
\frac{dy}{dt} &= \lambda x + (A_{20} + A_{11} + A_{02})x^2 - 2(B_{20} - B_{02})xy \\
&\quad - (A_{20} - A_{11} + A_{20})y^2 + x(x^2 + y^2).
\end{align*}
\]

(5.5.15)

In [Blows etc, 1993], the author discussed the center-focus problem and the bifurcation of limit cycles at origin and infinity of system (5.5.15) where the parameters of (5.5.15) are real. We next assume that the parameters of (5.5.15) are complex.

The associated system of (5.5.15) is given by

\[
\begin{align*}
\frac{dz}{dT} &= \lambda z + a_{20}z^2 + a_{11}zw + a_{o2}w^2 + z^2w, \\
\frac{dw}{dT} &= -\lambda w - b_{20}w^2 - b_{11}wz - b_{o2}w^2 - w^2z.
\end{align*}
\]

(5.5.16)

From Theorem 5.5.1, we have
**Lemma 5.5.1.** System (5.5.16) has exactly 14 elementary generalized rotation invariants which are listed as follows

\[
\lambda, \ a_{20}b_{20}, \ a_{11}b_{11}, \ a_{02}b_{02}, \ a_{20}a_{11}, \ b_{20}b_{11}, \\
a_{20}^2a_{02}, \ a_{20}^2b_{11}a_{02}, \ a_{20}b_{11}^2a_{02}, \ b_{11}^2a_{02}, \\
b_{20}^2b_{02}, \ b_{20}^2a_{11}b_{02}, \ b_{20}a_{11}^2b_{02}, \ a_{11}^3b_{02}.
\] (5.5.17)

By Theorem 5.5.1 and Theorem 5.5.2, we use computer algebra system-Mathematica to calculate the first 6 singular point values at infinity of system (5.5.16) for which there exist the numbers of terms 2, 14, 64, 180, 416, 846, respectively. Simplifying them, we obtain the following theorem.

**Theorem 5.5.4.** The first 6 singular point values at infinity of system (5.5.16) are given by

\[
\mu_1 = a_{20}a_{11} - b_{20}b_{11}, \\
\mu_2 \sim \frac{1}{3} (4I_0 - I_2), \\
\mu_3 \sim \frac{4(2I_1 - 2)(38\lambda - 7h_{02}) - (2I_2 - 3)(15h_{20} + 4h_{02} - 16\lambda)}{96}, \\
\mu_4 \sim \frac{1}{1680} (2I_2 - 3)f_4, \\
\mu_5 \sim \frac{(2I_2 - 3)(2085h_{20} - 1258h_{02} + 4776\lambda)}{13991704898994387712000} f_5, \\
\mu_6 \sim \frac{1}{1688725139658200521113600000} h_{02}^2(2I_2 - 3)f_6,
\] (5.5.18)

where

\[
f_4 = 210h_{20}h_{11} - 541h_{20}h_{02} + 18h_{11}h_{02} + 42h_{02}^2 - 168h_{02}\lambda, \\
f_5 = 253032857528472000J_3 + 2114537039332505919721h_{20}h_{11} \\
+ 6921878766377155200h_{20}h_{02} - 510999167489700493800h_{11}^2 \\
+ 34871062234758497441h_{11}h_{02} + 1639708991825843200h_{02}^2 \\
+ 6384378367684800000h_{20}\lambda + 162132858540896763524h_{11}\lambda \\
- 4181742245609548800h_{02}\lambda - 9251325634682880000\lambda^2, \\
f_6 = -43354482540693424129161616296852h_{20}h_{11} \\
+ 4882195524329926183734042496576h_{20}h_{02} \\
+ 10838886466163652549580429391013h_{11}^2 \\
- 1220573965726107568918619767504h_{11}h_{02} \\
+ 127643623556931256320000h_{02}^2.
\] (5.5.19)
In (5.5.18) and (5.5.19), \( \mu_1 \) and

\[
I_0 = a_{20}^3 a_{02} - b_{20}^3 b_{02}, \quad I_1 = a_{20}^2 b_{11} a_{02} - b_{20}^2 a_{11} b_{02}, \\
I_2 = a_{20} b_{11}^2 a_{02} - b_{20} a_{11}^2 b_{02}, \quad I_3 = b_{11}^3 a_{02} - a_{11}^3 b_{02}
\]

(5.5.20)

are anti-symmetric generalized rotation invariants, while \( \lambda \) and

\[
J_3 = b_{11}^3 a_{02} + a_{11}^3 b_{02}, \quad h_{20} = a_{20} b_{20}, \quad h_{11} = a_{11} b_{11}, \quad h_{02} = a_{02} b_{02}
\]

(5.5.21)

are self-symmetric generalized rotation invariants.

**Theorem 5.5.5.** For system (5.5.16), the first 6 singular point values are zero if and only if one of the following six conditions holds:

\[
C_1 : I_0 = I_1 = I_2 = I_3 = 0, \quad a_{20} a_{11} - b_{20} b_{11} = 0; \\
C_2 : 2a_{20} - b_{11} = 0, \quad 2b_{20} - a_{11} = 0; \\
C_3 : \lambda = 0, \quad b_{11} = -2a_{20}, \quad a_{11} = b_{20} = b_{02} = 0, \quad a_{20} a_{02} \neq 0; \\
C_3^* : \lambda = 0, \quad a_{11} = -2b_{20}, \quad b_{11} = a_{20} = a_{02} = 0, \quad b_{20} b_{02} \neq 0; \\
C_4 : \lambda = b_{20} = a_{20} = 0, \quad b_{02} = 0, \quad b_{11} a_{02} \neq 0; \\
C_4^* : \lambda = a_{20} = b_{20} = 0, \quad a_{02} = 0, \quad a_{11} b_{02} \neq 0.
\]

(5.5.22)

**Proposition 5.5.1.** If one of condition \( C_1 \) and \( C_2 \) in Theorem 5.5.5 holds, then the infinity of system (5.5.16) is an complex center.

**Proof.** If \( C_1 \) holds, from Lemma 5.5.1, we know that the coefficients of system (5.5.16) satisfy the condition of the extend symmetry principle. If \( C_2 \) holds, system (5.5.16) is a Hamiltonian system. Thus, this proposition holds.

**Proposition 5.5.2.** If one of condition \( C_3 \) and \( C_3^* \) in Theorem 5.5.5 is satisfied, then the infinity of system (5.5.16) is an complex center.

**Proof.** If \( C_3 \) holds, system (5.5.16) can be reduced to

\[
\frac{dz}{dT} = a_{20} z^2 + a_{02} w^2 + z^2 w, \quad \frac{dw}{dT} = 2a_{20} zw - zw^2.
\]

(5.5.23)

This system has the following integral:

\[
\frac{(w - 2a_{20})^3(3z^2 + 2a_{02} w)}{w} = \text{constant}.
\]

(5.5.24)

Thus, infinity of system (5.5.16) is an complex center.

Similarly, when \( C_3^* \) holds, the conclusion of Proposition 5.5.2 is true.

**Proposition 5.5.3.** If one of condition \( C_4 \) and \( C_4^* \) in Theorem 5.5.5 holds, then the infinity of system (5.5.16) is an complex center.
5.5 Singular Point Values at Infinity and Integrable Conditions

Proof. When \( C_4 \) holds, system (5.5.16) becomes

\[
\frac{dz}{dT} = w(a_{11}z + a_{02}w + z^2), \quad \frac{dw}{dT} = -zw(b_{11} + w). \tag{5.5.25}
\]

By means of transformation (5.5.8), system (5.5.25) can be changed into

\[
\frac{d\xi}{dT} = \frac{1}{3}\xi(3 + 2b_{11}\xi^2\eta + a_{11}\xi\eta^2 + a_{02}\eta^3), \quad \frac{d\eta}{dT} = \frac{1}{3}\eta(3 + b_{11}\xi^2\eta + 2a_{11}\xi\eta^2 + 2a_{02}\eta^3). \tag{5.5.26}
\]

By Theorem 1.6.7, system (5.5.26) is linearizable in a neighborhood of the origin. Thus, when \( C_4 \) holds, the conclusion of Proposition 5.5.3 holds.

Similarly, when \( C_4^* \) is satisfied, the conclusion of Proposition 5.5.3 is true. \( \square \)

Propositions 5.5.1, 5.5.2, 5.5.3 and Theorem 5.5.5 follow that

**Theorem 5.5.6.** Infinity of system (5.5.16) is an complex center, if and only if the first 6 singular point values are zero, i.e., one of the six conditions in Theorem 5.5.5 is satisfied.

We next discuss the conditions of infinity of system (5.5.16) to be a 6-order weak singular point. From Theorem 5.5.4, we have

**Theorem 5.5.7.** For system (5.5.16), infinity is a 6-order weak singular point if and only if one the following conditions holds:

\[
C_5: \quad \begin{cases} a_{11} + 2b_{20} = b_{11} + 2a_{20} = 0, \quad \lambda = \frac{1}{960}(99 \mp \sqrt{2761})h_{20}, \quad h_{02} \neq 0, \\ h_{11} = \frac{1}{60}(61 \pm \sqrt{2761})b_{02}, \quad J_3 = \frac{-12638443 \pm 238497\sqrt{2761}}{378000} b_{02}^2; \\ C_6: \quad a_{02}b_{02} \neq 0, \quad \lambda = \frac{1}{4}a_{02}b_{02}, \quad a_{11} = 0, \quad 63b_{11}^3 + 4a_{02}b_{02}^2 = 0; \\ C_6^*: \quad a_{02}b_{02} \neq 0, \quad \lambda = \frac{1}{4}a_{02}b_{02}, \quad b_{11} = 0, \quad 63a_{11}^3 + 4b_{02}a_{02}^2 = 0. \tag{5.5.27}
\]

**Theorem 5.5.8.** If \( x, y, t \) are real variables and all the coefficients of system (5.5.15) are real, then it is impossible that infinity of system (5.5.15) is a 6-th weak focus.

Proof. From the conditions given in Theorem 5.5.8, we have

\[
b_{20} = \bar{a}_{20}, \quad b_{11} = \bar{a}_{11}, \quad b_{02} = \bar{a}_{02}. \tag{5.5.28}
\]

By Theorem 5.5.7, we only need to prove that when one of conditions of \( C_5, C_6 \) and \( C_6^* \) holds, it is impossible that (5.5.28) is satisfied.
In fact, if condition $C_5$ holds, then
\[
\frac{J_3^2 - 4h_{11}^3}{h_{02}^2} = \frac{157420332562049 \pm 2995809288171\sqrt{2761}}{71442000000} > 0. \quad (5.5.29)
\]

On the other hand, from (5.5.20) and (5.5.21), we know that when (5.5.28) is satisfied, $I_3$ is a pure imaginary. Then, $I_3 = J_3^2 - 4h_{11}^3 < 0$. It implies that if condition $C_5$ is satisfied, then (5.5.28) does not hold.

In addition, it is easy to see that when one of condition $C_6$ and $C_6^*$ holds, we have $a_{11}b_{11} = 0$, but not all $a_{11}$ and $b_{11}$ are zero. Thus, if one of condition $C_6$ and $C_6^*$ is satisfied, then (5.5.28) does not hold. \qed

### 5.6 Bifurcation of Limit Cycles at Infinity

Consider the following perturbed system of (5.5.1) with two small parameters $\varepsilon, \delta$

\[
\frac{dx}{dt} = \sum_{k=0}^{2n+1} X_k(x, y, \varepsilon, \delta), \quad \frac{dy}{dt} = \sum_{k=0}^{2n+1} Y_k(x, y, \varepsilon, \delta), \quad (5.6.1)
\]

where $X_k(x, y, \varepsilon, \delta), Y_k(x, y, \varepsilon, \delta)$ are homogeneous polynomials of degree $k$ in $x, y$, and the coefficients are power series in $\varepsilon, \delta$ having nonzero convergent radius. Assume that there is an integer $d$, such that

\[
x Y_{2n+1}(x, y, 0, 0) - y X_{2n+1}(x, y, 0, 0) \geq d(x^2 + y^2)^{n+1} \quad (5.6.2)
\]

and

\[
\int_0^{2\pi} \frac{\cos \theta X_{2n+1}(\cos \theta, \sin \theta, 0, 0) + \sin \theta Y_{2n+1}(\cos \theta, \sin \theta, 0, 0)}{\cos \theta Y_{2n+1}(\cos \theta, \sin \theta, 0, 0) - \sin \theta X_{2n+1}(\cos \theta, \sin \theta, 0, 0)} \, d\theta = 0. \quad (5.6.3)
\]

By means of transformation (5.1.5), system (5.6.1) can be changed into

\[
\frac{dr}{d\theta} = \varphi_{2n+2}(\theta, \varepsilon, \delta) + \sum_{k=1}^{2n+1} \varphi_{2n+2-k}(\theta, \varepsilon, \delta) r^k, \quad \frac{d\theta}{d\theta} = \psi_{2n+2}(\theta, \varepsilon, \delta) + \sum_{k=1}^{2n+1} \psi_{2n+2-k}(\theta, \varepsilon, \delta) r^k
\]

\[
= -\varphi_{2n+2}(\theta, \varepsilon, \delta) r + o(r), \quad (5.6.4)
\]

where

\[
\varphi_k(\theta, \varepsilon, \delta) = \cos \theta X_{k-1}(\cos \theta, \sin \theta, \varepsilon, \delta) + \sin \theta Y_{k-1}(\cos \theta, \sin \theta, \varepsilon, \delta),
\]
\[
\psi_k(\theta, \varepsilon, \delta) = \cos \theta Y_{k-1}(\cos \theta, \sin \theta, \varepsilon, \delta) - \sin \theta X_{k-1}(\cos \theta, \sin \theta, \varepsilon, \delta). \quad (5.6.5)
\]
For a sufficiently small $h$, we write the solution of (5.6.4) with the initial condition $r|_{\theta=0} = h$ and the Poincaré succession function respectively as follows:

$$r = \tilde{r}(\theta, h, \varepsilon, \delta) = \sum_{k=1}^{\infty} \nu_k(\theta, \varepsilon, \delta) h^k,$$

$$\Delta(h, \varepsilon, \delta) = \tilde{r}(2\pi, h, \varepsilon, \delta) - h. \quad (5.6.6)$$

Clearly, $\nu_1(\theta)$ can be expressed as

$$\nu_1(\theta, \varepsilon, \delta) = \exp \int_{0}^{\theta} \frac{-\varphi_{2n+2}(\theta, \varepsilon, \delta)}{\psi_{2n+2}(\theta, \varepsilon, \delta)} \, d\theta. \quad (5.6.7)$$

From (5.6.3) and (5.6.7), we have

$$\nu_1(2\pi, 0, 0) = 1. \quad (5.6.8)$$

Particularly, if $X_{2n+1}, Y_{2n+1}$ are given by (5.2.1), then $\nu_1(\theta, \varepsilon, \delta) = e^{-\delta \theta}$.

Obviously, equation (5.6.4) is a particular case of equation (4.1.7).

If $\delta = \delta(\varepsilon)$ in the right hand of system (5.6.1) is a power series of $\varepsilon$ having nonzero convergent radius, and $\delta(0) = 0$, we can obtain a quasi succession function $L(h, \varepsilon)$ by computing the focal values at infinity. The method mentioned in Chapter 3 can be used to study the bifurcation of limit cycles in a neighborhood of infinity of system (5.6.1). For an example, we discuss a class of real planar cubic system

$$\frac{dx}{dt} = (\delta x - y)(x^2 + y^2) + X_2(x, y),$$

$$\frac{dy}{dt} = (x + \delta y)(x^2 + y^2) + Y_2(x, y), \quad (5.6.9)$$

where $X_2(x, y)$, $Y_2(x, y)$ are homogeneous polynomials of degree 2 in $x, y$. By means of transformation (5.5.10), system (5.6.9) can be changed into

$$\frac{dr}{d\theta} = -r \frac{\delta + [\cos \theta X_2(\cos \theta, \sin \theta) + \sin \theta Y_2(\cos \theta, \sin \theta)] r}{1 + [\cos \theta Y_2(\cos \theta, \sin \theta) - \sin \theta X_2(\cos \theta, \sin \theta)] r}. \quad (5.6.10)$$

It is interesting that under the polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, real planar quadratic system (4.4.1) can reduce to

$$\frac{dr}{d\theta} = r \frac{\delta + [\cos \theta X_2(\cos \theta, \sin \theta) + \sin \theta Y_2(\cos \theta, \sin \theta)] r}{1 + [\cos \theta Y_2(\cos \theta, \sin \theta) - \sin \theta X_2(\cos \theta, \sin \theta)] r}. \quad (5.6.11)$$

The vector fields defined by (5.5.10) and (5.6.11) are opposite oriented. Therefore, we can use known conclusions of the center-focus problem and bifurcations of limit cycles for the quadratic system.
By means of transformation \( z = x + iy, \ w = x - iy, \ T = it, \ i = \sqrt{-1} \), system (5.6.9) reduce to

\[
\frac{dz}{dT} = (1 - i\delta)z^2w + a_{20}z^2 + a_{11}zw + a_{02}w^2, \\
\frac{dw}{dT} = -(1 + i\delta)w^2z - b_{20}w^2 - b_{11}wz - b_{02}z^2.
\]  

(5.6.12)

where the relations between \( a_{\alpha\beta}, b_{\alpha\beta} \) and the coefficients of \( X_2(x, y), Y_2(x, y) \) are given by (3.4.3) and (3.4.4).

Obviously, system (5.6.12)\( \delta = 0 \) is the case of \( \lambda = 0 \) of system (5.5.16), namely it is the particular case of the system (5.5.3) with \( a_{10} = a_{01} = b_{10} = b_{01} \). From Theorem 5.5.1, we have

**Lemma 5.6.1.** When \( \delta = 0 \), system (5.6.12) has exactly 13 elementary generalized rotation invariants with time exponent 1 at infinity, which are listed as follows:

<table>
<thead>
<tr>
<th>degree</th>
<th>generalized rotation invariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>( a_{20}b_{20}, \ a_{11}b_{11}, \ a_{02}b_{02} ) (self-symmetry)</td>
</tr>
<tr>
<td></td>
<td>( a_{20}a_{11}, \ b_{20}b_{11} )</td>
</tr>
<tr>
<td>-2</td>
<td>( a_{20}^3a_{02}, \ a_{20}^2b_{11}a_{02}, \ a_{20}b_{11}^2a_{02}, \ b_{11}^3a_{02} )</td>
</tr>
<tr>
<td></td>
<td>( b_{20}^3b_{02}, \ b_{20}^2a_{11}b_{02}, \ b_{20}a_{11}^2b_{02}, \ a_{11}^3b_{02} )</td>
</tr>
</tbody>
</table>

**Remark 5.6.1.** From Corollary 2.5.1 and Lemma 5.6.1, system (5.6.12)\( \delta = 0 \) and the quadric system 2.5.4 have the same elementary generalized rotation invariants. But, the order of the same generalized rotation invariant in the two systems are difference as the time exponents are difference.

By computing the singular point values at infinity of system (5.6.12) and simplifying them, we have

**Theorem 5.6.1.** When \( \delta = 0 \), the first 4 singular point values at infinity of system (5.6.12) are listed as follows:

\[
\begin{align*}
\mu_1 &= a_{20}a_{11} - b_{20}b_{11}, \\
\mu_2 &\sim \frac{1}{3}(4I_0 - I_1), \\
\mu_3 &\sim \frac{1}{48}(2a_{02}b_{02} - 3a_{20}b_{20})(-14I_1 + 5I_2 + I_3), \\
\mu_4 &\sim \frac{1}{1800}a_{02}^2b_{02}^2(-364I_1 + 220I_2 - 19I_3),
\end{align*}
\]  

(5.6.13)

where \( I_k \) are given by (5.5.20).

From Theorem 5.6.1, we obtain
**Theorem 5.6.2.** When \( \delta = 0 \), the first 4 singular point values at infinity of system (5.6.12) are zero if and only if one of the following two conditions holds:

\[
a_{20}a_{11} - b_{20}b_{11} = I_0 = I_1 = I_2 = I_3 = 0, \tag{5.6.14}
\]

\[
b_{11} = 2a_{20}, \quad a_{11} = 2b_{20}. \tag{5.6.15}
\]

From Lemma 5.6.1, we have

**Theorem 5.6.3.** When \( \delta = 0 \), if (5.6.14) holds, the coefficients of system (5.6.12) give rise to the condition of the extend symmetric principle. While if (5.6.15) holds, system (5.6.12) is a Hamiltonian system.

Theorem 5.6.1, 5.6.2 and 5.6.3 follow that

**Theorem 5.6.4.** When \( \delta = 0 \), infinity of system (5.6.12) is a complex center if and only if \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0 \), namely, one of the two conditions in Theorem 5.6.3 holds.

We next discuss the bifurcation of limit cycles created from infinity of system (5.6.9). Assume that \( a_{\alpha\beta} = A_{\alpha\beta} + iB_{\alpha\beta}, \ b_{\alpha\beta} = A_{\alpha\beta} - iB_{\alpha\beta} \) and

\[
\delta = \frac{1}{2} \varepsilon^{10+N},
\]

\[
A_{20} = -1 - \frac{33}{40} \varepsilon^3, \quad B_{20} = -\frac{1}{4} \varepsilon^{6+N},
\]

\[
A_{11} = 2, \quad B_{11} = 0,
\]

\[
A_{02} = \frac{\sqrt{18150 - 15972\varepsilon - 625\varepsilon^{2N}}}{110}, \quad B_{02} = -\frac{5}{22} \varepsilon^N. \tag{5.6.16}
\]

From Theorem 5.6.1, we have

**Lemma 5.6.2.** When (5.6.16) holds, for infinity of system (5.6.9), we have

\[
\nu_1(2\pi, \varepsilon, \delta(\varepsilon)) - 1 = -\pi \varepsilon^{10+N} + o(\varepsilon^{10+N}),
\]

\[
\nu_3(2\pi, \varepsilon, \delta(\varepsilon)) \simeq \pi \varepsilon^{6+N}, \quad \nu_6(2\pi, \varepsilon, \delta(\varepsilon)) \simeq -\pi \varepsilon^{3+N},
\]

\[
\nu_7(2\pi, \varepsilon, \delta(\varepsilon)) \simeq \pi \varepsilon^{1+N}, \quad \nu_9(2\pi, \varepsilon, \delta(\varepsilon)) \simeq -\pi \varepsilon^N. \tag{5.6.17}
\]

By Lemma 5.6.2, if (5.6.16) is satisfied, then when \( \varepsilon = 0 \), if \( N = 0 \), infinity of system (5.6.9) is a 4-order weak focus. If \( N > 0 \), infinity of system (5.6.9) is a center, for which the vector field is symmetric with respect to \( x - \)axis. To obtain the quasi succession function at infinity, we need to prove the following conclusions.

**Lemma 5.6.3.** If (5.6.16) holds, for any positive integer \( k > 4 \), the \( k \)-th focal value at infinity of system (5.6.9) satisfies the following formula

\[
\nu_{2k+1}(2\pi, \varepsilon, \delta(\varepsilon)) = O(\varepsilon^N). \tag{5.6.18}
\]
Proof. When (5.6.16) holds, it is easy to prove that
\[ a_{20}a_{11} - b_{20}b_{11} = O(\varepsilon^N), \]
\[ I_0 = O(\varepsilon^N), \quad I_1 = O(\varepsilon^N), \]
\[ I_2 = O(\varepsilon^N), \quad I_3 = O(\varepsilon^N). \]  
(5.6.19)
Thus, Lemma 5.6.1 and the construction theorem of singular point values at infinity (Theorem 5.4.1) lead to the conclusion of this Lemma.

From Lemma 5.6.2 and 5.6.3, we have

**Lemma 5.6.4.** When (5.6.16) holds, the quasi succession function at infinity of system (5.6.9) is given by
\[ L(h, \varepsilon) = -\pi(\varepsilon^{10} - \varepsilon^6 h^2 + \varepsilon^3 h^4 - \varepsilon h^6 + h^8). \]  
(5.6.20)

From Theorem 3.3.4 and Lemma 5.6.4, we obtain

**Theorem 5.6.5.** If (5.6.16) holds, then when \(0 < \varepsilon \ll 1\), there exist at least 4 limit cycles in a sufficiently small neighborhood of infinity of system (5.6.9), which are close to the circles \((x^2 + y^2)^{-1} = \varepsilon^k, k = 1, 2, 3, 4\).

### 5.7 Isochronous Centers at Infinity of a Polynomial Systems

In this section, we extended the definition of the isochronous center to the case of infinity for a class of polynomial systems.

We consider the following real system:
\[
\begin{align*}
\frac{dx}{dt} &= \frac{1}{(x^2 + y^2)^n} \sum_{k=0}^{2n} X_k(x, y) - y, \\
\frac{dy}{dt} &= \frac{1}{(x^2 + y^2)^n} \sum_{k=0}^{2n} Y_k(x, y) + x,
\end{align*}
\]  
(5.7.1)
where \(n\) is a positive integer, and \(X_k(x, y), Y_k(x, y)\) are homogeneous polynomials of degree \(k\).

By a time rescaling \(t \to (x^2 + y^2)^n t\), system (5.7.1) becomes the system (5.2.2)\(\delta=0\).

**Definition 5.7.1.** For system (5.7.1), infinity is called an isochronous center, if it is a center and the period of all periodic solutions in a neighborhood of infinity is the same constant.

By using the transformation
\[
\begin{align*}
x &= \frac{u}{(u^2 + v^2)^{n+1}}, \quad y = \frac{v}{(u^2 + v^2)^{n+1}},
\end{align*}
\]  
(5.7.2)
system (5.7.1) can be reduced to the system (5.2.13)\(\delta=0\). We immediately have the following conclusion.
**Theorem 5.7.1.** Infinity of system (5.7.1) is a center (isochronous center) if and only if the origin of system (5.2.13)_{δ=0} (or system (5.3.17)_{δ=0}) is a center (isochronous center).

**Definition 5.7.2.** Infinity of system (5.7.1) is called a complex isochronous center, if the origin of system (5.2.13)_{δ=0} (or system (5.3.17)_{δ=0}) is a complex isochronous center.

Infinity of system (5.5.1)_{δ=0} is called a complex quasi isochronous center, if infinity of system (5.7.1) is a complex isochronous center.

By Theorem 5.7.1 and the above knowledge, to determine center conditions and isochronous center conditions at infinity of system (5.7.1), we only need to consider the computation problem of singular point values and period constants at the origin for system (5.3.17)_{δ=0}.

To explain the mentioned idea, we consider the following real rational system

\[
\frac{dx}{dt} = -y + \frac{X_3(x, y)}{(x^2 + y^2)^2}, \quad \frac{dy}{dt} = x + \frac{Y_3(x, y)}{(x^2 + y^2)^2},
\]  

(5.7.3)

where \(X_3(x, y), Y_3(x, y)\) are homogeneous polynomials of degree 3 in \(x, y\). The associated system of system (5.7.3) has the form

\[
\frac{dz}{dT} = z + \frac{1}{z^2w^2}(a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3),
\]

\[
\frac{dw}{dT} = -w - \frac{1}{z^2w^2}(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{03}z^3).
\]

(5.7.4)

By means of transformations

\[
z = \frac{\xi}{\xi^3\eta^3}, \quad w = \frac{\eta}{\xi^3\eta^3},
\]

(5.7.5)

system (5.7.4) becomes

\[
\frac{d\xi}{dT} = \xi + \frac{3}{5}b_{03}\xi^8\eta^3 + \frac{1}{5}(2a_{30} + 3b_{12})\xi^7\eta^4 + \frac{1}{5}(2a_{21} + 3b_{21})\xi^6\eta^5
\]

\[+ \frac{1}{5}(2a_{12} + 3b_{30})\xi^5\eta^6 + \frac{2}{5}a_{03}\xi^4\eta^7,
\]

\[
\frac{d\eta}{dT} = -\eta - \frac{3}{5}a_{03}\eta^8\xi^3 - \frac{1}{5}(2b_{30} + 3a_{12})\eta^7\xi^4 - \frac{1}{5}(2b_{21} + 3a_{21})\eta^6\xi^5
\]

\[+ \frac{1}{5}(2a_{12} + 3b_{30})\eta^5\xi^6 - \frac{2}{5}b_{03}\eta^4\xi^7.
\]

(5.7.6)

### 5.7.1 Conditions of Complex Center for System (5.7.6)

First, we discuss the center conditions. By using Theorem 2.3.6 to compute the singular point values at origin of system (5.7.6) and simplify them, we obtain
Theorem 5.7.2. The first 35 singular point values at the origin of system (5.7.6) are given by
\[
\begin{align*}
\mu_5 &= \frac{1}{5}(-a_{21} + b_{21}), \\
\mu_{10} &\sim \frac{1}{5}(a_{12}a_{30} - b_{12}b_{30}), \\
\mu_{15} &\sim \frac{1}{40}(-9a_{03}a_{30}^2 - a_{12}^2b_{03} + a_{03}b_{12}^2 + 9b_{03}b_{30}^2), \\
\mu_{20} &\sim \frac{1}{20}(a_{21} + b_{21})(3a_{03}a_{30}^2 - a_{03}a_{30}b_{12} + a_{12}b_{03}b_{30} - 3b_{03}b_{30}^2), \\
\mu_{25} &\sim \frac{1}{120}(16a_{30}b_{30} - 3a_{03}b_{03})(3a_{03}a_{30}^2 - a_{03}a_{30}b_{12} + a_{12}b_{03}b_{30} - 3b_{03}b_{30}^2), \\
\mu_{30} &\sim 0, \\
\mu_{35} &\sim \frac{88}{675} a_{30}^2 b_{30} (3a_{03}a_{30}^2 - a_{03}a_{30}b_{12} + a_{12}b_{03}b_{30} - 3b_{03}b_{30}^2). 
\end{align*}
\] (5.7.7)

We know from Theorem 5.7.2 that

Theorem 5.7.3. For system (5.7.6), the first 35 singular point values are zero if and only if one of the following four conditions holds:

\[C_1: \quad a_{21} = b_{21}, \quad a_{12} = 3b_{30}, \quad b_{12} = 3a_{30},\]
\[C_2: \quad \begin{cases} a_{21} = b_{21}, & |3a_{30} - b_{12}| + |3b_{30} - a_{12}| \neq 0, \quad a_{12}a_{30} = b_{12}b_{30}, \\
 a_{20}a_{03} = b_{30}b_{03}, & a_{30}b_{12}a_{03} = b_{30}a_{12}b_{03}, \quad b_{12}b_{03} = a_{12}^2b_{30}, \end{cases}\]
\[C_3: \quad a_{21} = b_{21} = a_{30} = b_{12} = a_{03} = 0, \quad a_{12} = -3b_{30},\]
\[C_5: \quad a_{21} = b_{21} = b_{30} = a_{12} = b_{03} = 0, \quad b_{12} = -3a_{30}.\] (5.7.8)

We next discuss the conditions that the origin is a complex center.

(1) When condition \(C_1\) holds, there exist a constant \(s\), such that \(a_{21} - b_{21} = s\). Thus system (5.7.6) becomes
\[
\frac{d\xi}{dT} = \xi + \frac{3}{5}b_{03}\xi^8\eta^3 + \frac{11}{5}a_{30}\xi^7\eta^4 + s\xi^6\eta^5 + \frac{9}{5}b_{30}\xi^5\eta^6 + \frac{2}{5}a_{03}\xi^4\eta^7, \\
\frac{d\eta}{dT} = -\eta - \frac{3}{5}a_{03}\eta^8\xi^3 - \frac{11}{5}b_{30}\eta^7\xi^4 - s\eta^6\xi^5 - \frac{9}{5}a_{30}\eta^5\xi^6 - \frac{2}{5}b_{03}\eta^4\xi^7. 
\] (5.7.9)

System (5.7.9) has an analytic first integral
\[
F = \frac{\xi^{15}\eta^{15}}{4 + 3b_{03}\xi^7\eta^3 + 12a_{30}\xi^6\eta^4 + 6s\xi^5\eta^5 + 12b_{30}\xi^4\eta^6 + 3a_{03}\xi^3\eta^7}. 
\] (5.7.10)

Thus, the origin of system (5.7.9) is a complex center.

(2) When condition \(C_2\) holds, we denote that \(3a_{30} - b_{12} = \beta, \quad 3b_{30} - a_{12} = \alpha\), then there exist complex \(p, q, s\), such that
\[
a_{30} = p\beta, \quad b_{30} = p\alpha, \quad a_{03} = q\alpha^2, \quad b_{03} = q\beta^2, \quad a_{21} = b_{21} = s. 
\] (5.7.11)
Thus, system (5.7.6) becomes

\[
\frac{d\xi}{dT} = \xi + \frac{3}{5} q_0^2 \xi^8 \eta^3 + \frac{1}{5} (11p - 3) \xi^7 \eta^4 + s \xi^6 \eta^5 + \frac{1}{5} (11p - 3) \xi^5 \eta^6 - \frac{2}{5} q_0^2 \xi^4 \eta^7.
\]

\[
\frac{d\eta}{dT} = -\eta - \frac{3}{5} q_0 \eta^8 \xi^3 - \frac{1}{5} (11p - 3) \eta^7 \xi^4 - \frac{2}{5} q_0 \eta^4 \xi^7.
\]

(5.7.12)

For system (5.7.12), the conditions of the extended symmetric principle are satisfied. Thus the origin of system (5.7.12) is a complex center.

(3) When one of conditions \( C_3 \) and \( C_3^* \) holds, we have

**Proposition 5.7.1.** If one of conditions \( C_3 \) and \( C_3^* \) holds, then the origin of system (5.7.6) is a complex isochronous center.

**Proof.** When conditions \( C_3 \) holds, system (5.7.6) becomes

\[
\frac{d\xi}{dT} = \xi + \frac{3}{5} b_{03} \xi^8 \eta^3 - \frac{3}{5} b_{30} \xi^5 \eta^6,
\]

\[
\frac{d\eta}{dT} = -\eta + \frac{7}{5} b_{30} \eta^7 \xi^4 - \frac{2}{5} b_{03} \eta^4 \xi^7.
\]

(5.7.13)

System (5.7.13) is linearizable by using the transformation

\[
u = \xi \left(1 - 3b_{30} \xi^4 \eta^6\right)^{1/5}, \quad \nu = \eta \left(1 - 3b_{03} \xi^7 \eta^3\right)^{1/5}. \quad (5.7.14)
\]

Thus the origin of system (5.7.14) is a complex isochronous center.

Similarly, if Condition \( C_3^* \) holds, the origin of system (5.7.6) is also a complex isochronous center.

To sum up, we have

**Theorem 5.7.4.** All singular point values of the origin of (5.7.6) are zero if and only if the first 35 singular point values of the origin are zero, i.e., one of the four conditions in Theorem 5.7.3 holds.

**Theorem 5.7.5.** For system (5.7.1), infinity is a complex center if and only if one of the four conditions in Theorem 5.7.3 holds.
5.7.2 Conditions of Complex Isochronous Center for System (5.7.6)

We next discuss the isochronous center conditions.

(1) When condition $C_1$ in Theorem 5.7.3 holds, we have

**Proposition 5.7.2.** The first 20 period constants of the origin of system (5.7.9) are given by

\[
\begin{align*}
\tau_5 &= 2s, \quad \tau_{10} \sim -\frac{1}{2}(a_{03}b_{03} + 16a_{30}b_{30}), \\
\tau_{15} &\sim 0, \quad \tau_{20} \sim -80a_{30}^2b_{30}^2. 
\end{align*}
\] (5.7.15)

From Proposition 5.7.2, we have

**Proposition 5.7.3.** The first 20 period constants of the origin of system (5.7.9) are all zeros, if and only if one of the following conditions holds:

- $C_{11}: s = 0, a_{30} = 0, a_{03} = 0$;
- $C_{11}^*: s = 0, b_{30} = 0, b_{03} = 0$;
- $C_{21}: s = 0, a_{30} = 0, b_{03} = 0$;
- $C_{21}^*: s = 0, b_{30} = 0, a_{03} = 0$. (5.7.16)

**Proposition 5.7.4.** If one of condition $C_{11}$ and $C_{11}^*$ holds, then the origin of (5.7.9) is a complex isochronous center.

**Proof.** When condition $C_{11}$ holds, system (5.7.9) becomes

\[
\begin{align*}
\frac{d\xi}{dT} &= \xi + \frac{3}{5}b_{03}\xi^8\eta^3 + \frac{9}{5}b_{30}\xi^5\eta^6, \\
\frac{d\eta}{dT} &= -\eta - \frac{2}{5}b_{03}\xi^7\eta^4 - \frac{11}{5}b_{30}\xi^4\eta^7. 
\end{align*}
\] (5.7.17)

System (5.7.17) is linearizable by using the transformation

\[
\begin{align*}
u &= \frac{\xi \sqrt{1 + 3b_{30}\xi^4\eta^6}}{(1 + 3b_{30}\xi^4\eta^6 + \frac{3}{4}b_{03}\xi^7\eta^3)^{\frac{1}{2}}} \\
v &= \frac{\eta(1 + 3b_{30}\xi^4\eta^6 + \frac{3}{4}b_{03}\xi^7\eta^3)^{\frac{1}{2}}}{\sqrt{1 + 3b_{30}\xi^4\eta^6}}.
\end{align*}
\] (5.7.18)

Thus, the origin of (5.7.17) is a complex isochronous center.

If Condition $C_{21}^*$ is satisfied, then by using the same method as the above, we know that the origin of system (5.7.9) is also a complex isochronous center. □

**Proposition 5.7.5.** If one of condition $C_{12}$ and $C_{12}^*$ holds, then the origin of (5.7.9) is a complex isochronous center.
5.7 Isochronous Centers at Infinity of a Polynomial Systems

**Proof.** When condition $C_{12}$ holds, system (5.7.9) becomes

\[
\frac{d\xi}{dT} = \xi + \frac{9}{5} b_{30} \xi^5 \eta^6 + \frac{2}{5} a_{03} \xi^4 \eta^7,
\]
\[
\frac{d\eta}{dT} = -\eta - \frac{11}{5} b_{30} \xi^4 \eta^7 - \frac{3}{5} a_{03} \xi^3 \eta^8.
\] (5.7.19)

By using the transformation

\[ u = \xi^4 \eta^6, \quad v = \xi^3 \eta^7, \] (5.7.20)

system (5.7.20) becomes

\[
\frac{du}{dT} = -2u(1 + 3b_{30}u + a_{03}v), \quad \frac{dv}{dT} = -v(4 + 10b_{30}u + 3a_{03}v). \] (5.7.21)

The origin of system (5.7.21) is an integer-ratio node, similar to the proof of Theorem 1.6.1, we can prove that the origin of (5.7.19) is a complex isochronous center.

Similarly, when condition $C_{12}^*$ holds, the origin of (5.7.9) is also a complex isochronous center. \qed

(2) When condition $C_2$ in Theorem 5.7.3 holds, we have

**Proposition 5.7.6.** The first 25 period constants of the origin of system (5.7.12) are given by

\[
\tau_5 = 2s, \quad \tau_{10} \sim -\frac{1}{2} \alpha \beta (-4p + 16p^2 + \alpha \beta q^2),
\]
\[
\tau_{15} \sim \frac{1}{4} \alpha^2 \beta^2 q(6p - 1), \quad \tau_{20} \sim -\frac{1}{48} \alpha^3 \beta^3 q^2. \] (5.7.22)

From Proposition 5.7.6, we have

**Proposition 5.7.7.** The first 20 period constants of the origin of system (5.7.12) are all zeros, if and only if one of the following conditions holds:

\[
C_{21} : \quad s = 0, \quad q = 0, \quad p = 0;
\]
\[
C_{22} : \quad s = 0, \quad q = 0, \quad p = \frac{1}{4};
\]
\[
C_{23} : \quad s = 0, \quad \alpha = 0;
\]
\[
C_{23}^* : \quad s = 0, \quad \beta = 0. \] (5.7.23)

**Proposition 5.7.8.** If condition $C_{21}$ holds, then the origin of (5.7.12) is a complex isochronous center.

**Proof.** When condition $C_{21}$ holds, system (5.7.12) becomes

\[
\frac{d\xi}{dT} = \xi - \frac{3}{5} \beta \xi^7 \eta^4 - \frac{2}{5} \alpha \xi^5 \eta^6,
\]
\[
\frac{d\eta}{dT} = -\eta + \frac{3}{5} \alpha \eta^7 \xi^4 + \frac{2}{5} \beta \eta^5 \xi^6. \] (5.7.24)
System (5.7.16) is linearizable by using the transformation

\[
\xi = \frac{\xi(1 - \alpha \xi^4 \eta^6)^{1/5}}{(1 - \beta \xi^6 \eta^4)^{1/10}}, \quad \eta = \frac{\eta(1 - \beta \xi^6 \eta^4)^{1/5}}{(1 - \alpha \xi^4 \eta^6)^{1/10}}.
\] (5.7.25)

Thus, the origin of system (5.7.24) is an complex isochronous center.

**Proposition 5.7.9.** If condition \(C_{22}\) holds, then the origin of (5.7.12) is a complex isochronous center.

**Proof.** When condition \(C_{22}\) holds, system (5.7.12) becomes

\[
\frac{d\xi}{dT} = \xi - \frac{1}{20} \beta \xi^7 \eta^4 + \frac{1}{20} \alpha \xi^5 \eta^6,
\]

\[
\frac{d\eta}{dT} = -\eta + \frac{1}{20} \alpha \eta^7 \xi^4 - \frac{1}{20} \beta \eta^5 \xi^6.
\] (5.7.26)

By using the polar coordinates \(\xi = \rho e^{i\theta}, \eta = \rho e^{-i\theta}\) and \(T = it\), we have \(\frac{d\theta}{dt} \equiv 1\). System (5.7.26) has an isochronous center at the origin.

**Proposition 5.7.10.** If one of condition \(C_{23}\) and \(C_{23}^*\) holds, then the origin of (5.7.12) is a complex isochronous center.

**Proof.** When condition \(C_{23}\) holds, system (5.7.12) becomes

\[
\frac{d\xi}{dT} = \xi + \frac{3}{5} \beta^2 q \xi^8 \eta^3 + \frac{1}{5} \beta (11p - 3) \xi^7 \eta^4,
\]

\[
\frac{d\eta}{dT} = -\eta - \frac{2}{5} \beta^2 q \xi^7 \eta^4 - \frac{1}{5} \beta (9p - 2) \xi^6 \eta^5.
\] (5.7.27)

Letting

\[
u = \xi^7 \eta^3, \quad v = \xi^6 \eta^4,
\] (5.7.28)

system (5.7.27) becomes

\[
\frac{du}{dT} = 4u + 3 \beta^2 q u^2 + \beta (10p - 3) uv, \quad \frac{dv}{dt} = 2v + 2 \beta^2 q uv + 2 \beta (3p - 1) v^2.
\] (5.7.29)

The origin of system (5.7.29) is an integer-ratio node, similar to the proof of Theorem 1.6.1, we can prove that the origin of (5.7.29) is a complex isochronous center.

Similarly, when condition \(C_{23}^*\) holds, the origin of (5.7.12) is also a complex isochronous center.

(3) Finally, when one of conditions \(C_3\) and \(C_3^*\) in Theorem 5.7.3 holds, according to Proposition 5.7.1, the origin of system (5.7.6) is a complex isochronous center.

The problem of a complex isochronous center for the infinity of system (5.7.3) and (5.7.4) are already solved completely in this section.
5.7 Isochronous Centers at Infinity of a Polynomial Systems

The problem of a complex quasi isochronous center for the infinity of system

\[
\frac{dx}{dt} = -y(x^2 + y^2)^2 + X_3(x, y), \quad \frac{dy}{dt} = x(x^2 + y^2)^2 + Y_3(x, y)
\]  

(5.7.30)

are also already solved completely.

Bibliographical Notes

The center-focus problem at infinity and the bifurcation of limit cycles created from \( \Gamma_{\infty} \) are essentially difficult problems. There are only a few results on the study for these problems before 2000 year (see [Ruienok, 1987; Blows etc, 1993; Han M.A., 1993] et al). Recent years, many papers have been published (see [Liu Y.R. etc, 2003b; Chen H.B. etc, 2002; Liu Y.R. etc, 2002a; Chen H.B. etc, 2003; Huang W.T. etc, 2004b; Zhang L. etc, 2006; Zhang Q. etc, 2006a; Liu Y.R. etc, 2006a; Liu Y.R. etc, 2006c; Zhang Q. etc, 2006b; Wang Q. L. etc, 2007; Huang W.T. etc, 2007] et al.)

The materials of this chapter are taken from [Liu Y.R., 2001] and [Liu Y.R. etc, 2004].