1 Preliminaries

The aim of this chapter is to introduce the notations, notions and results which will be useful in subsequent chapters. The results will be given without proof as they refer to basic notions of mathematical analysis, differential calculus and linear algebra.

1.1 $\mathbb{R}^p$ Space

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Take $p \in \mathbb{N}^*$. We denote by $\mathbb{R}$ the set of real numbers and we introduce the set

$$\mathbb{R}^p := \{(x^1, x^2, \ldots, x^p) \mid x^i \in \mathbb{R},\ \forall i \in \overline{1,p}\}.$$

This set can be organized as a p dimensional real vector space, with respect to the standard operations defined as follows: for every $x = (x^1, x^2, \ldots, x^p)$, $y = (y^1, y^2, \ldots, y^p) \in \mathbb{R}^p$ and every $a \in \mathbb{R}$

$$\begin{aligned} &x + y := (x^1 + y^1, x^2 + y^2, \ldots, x^p + y^p) \in \mathbb{R}^p, \\ &ax := (ax^1, ax^2, \ldots, ax^p) \in \mathbb{R}^p. \end{aligned}$$

Recall that the canonical base in $\mathbb{R}^p$ is the set $\{e_1, \ldots, e_p\}$, where for any $i \in \overline{1,p}$, $e_i := (0, \ldots, 1, \ldots, 0)$, and 1 is placed on the i–th coordinate. In some situations, when there is no risk of confusion, we will use the notation with subscript indices of the components. We will extend these operations also for sets: if $A, B \subset \mathbb{R}^p$ are nonempty, $\alpha \in \mathbb{R} \setminus \{0\}$ and $C \subset \mathbb{R}$ is nonempty, one defines $A + B := \{a + b \mid a \in A,\ b \in B\}$, $\alpha A := \{\alpha a \mid a \in A\}$, $CA := \{\alpha a \mid \alpha \in C,\ a \in A\}$, $A - B := A + (-1)B$.

One can consider an element x of $\mathbb{R}^p$ as a matrix of dimension $1 \times p$. The corresponding transposed matrix will be denoted by x^t. Also, one defines the usual scalar product of two vectors $x, y \in \mathbb{R}^p$ by

$$\langle x, y \rangle := \sum_{i=1}^{p} x^i y^i = xy^t.$$

Moreover, $\mathbb{R}^p$ can be seen as a normed vector space (in particular, as a metric space) endowed with the Euclidean norm $\|\cdot\| : \mathbb{R}^p \to \mathbb{R}_+$ given by

$$\|x\| := \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{p} (x^i)^2}.$$

It is easy to prove that for every $x, y \in \mathbb{R}^p$, the next relation (the parallelogram law) holds

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The angle between two vectors $x, y \in \mathbb{R}^p \setminus \{0\}$ is the value $\theta \in [0, \pi]$ given by

$$\cos\theta := \frac{\langle x, y\rangle}{\|x\|\,\|y\|}.$$

The open (closed) ball and the sphere centered at $\overline{x} \in \mathbb{R}^p$ with radius $\varepsilon > 0$ are given, respectively, by:

$$B(\overline{x}, \varepsilon) := \{x \in \mathbb{R}^p \mid \|x - \overline{x}\| < \varepsilon\},$$

$$D(\overline{x}, \varepsilon) := \{x \in \mathbb{R}^p \mid \|x - \overline{x}\| \le \varepsilon\},$$

and

$$S(\overline{x}, \varepsilon) := \{x \in \mathbb{R}^p \mid \|x - \overline{x}\| = \varepsilon\}.$$

One says that a subset $A \subset \mathbb{R}^p$ is bounded if it is contained in an open ball centered in the origin i.e., if there exists $M > 0$ such that $A \subset B(0, M)$.

A neighborhood of an element $\overline{x} \in \mathbb{R}^p$ is a subset of $\mathbb{R}^p$ which contains an open ball centered in $\overline{x}$. We denote by $\mathcal{V}(\overline{x})$ the class of all neighborhoods of $\overline{x}$. Let us summarize some facts:

– A subset of $\mathbb{R}^p$ is open if it is empty or it is neighborhood for all of its points.
– A subset of $\mathbb{R}^p$ is closed if its complement with respect to $\mathbb{R}^p$ is open.
– An element a is an interior point of the set $A \subset \mathbb{R}^p$ if A is a neighborhood of a. We denote by $\operatorname{int} A$ the interior of A (i.e., the set of all interior points of A).
– An element a is an accumulation point (or a limit point) of $A \subset \mathbb{R}^p$ if every neighborhood of a has at least one element in common with the set A which is different from a. We denote by A' the set of all limit points of A. If $a \in A \setminus A'$, one says that a is an isolated point of A.
– An element a is an adherent point (or a closure point) of $A \subset \mathbb{R}^p$ if every neighborhood of a has at least one element in common with the set A. We will use the notations $\operatorname{cl} A$ and $\overline{A}$ to denote the closure of A (i.e., the set of all the adherent points of A).
– A subset of $\mathbb{R}^p$ is compact if it is bounded and closed.

We denote by $\operatorname{bd} A$ the set $\operatorname{cl} A \setminus \operatorname{int} A = \operatorname{cl} A \cap \operatorname{cl}(\mathbb{R}^p \setminus A)$ and we call it the boundary of A.

Proposition 1.1.1. *(i) A subset of $\mathbb{R}^p$ is open if and only if it coincides with its interior.*
(ii) A subset of $\mathbb{R}^p$ is closed if and only if it coincides with its closure.

Definition 1.1.2. *One says that a function $f : \mathbb{N} \to \mathbb{R}^p$ is a sequence of elements from $\mathbb{R}^p$.*

The value of the function f in $n \in \mathbb{N}$, $f(n)$, is denoted by x_n (or $y_n, z_n, \ldots$), and the sequence defined by f is denoted by (x_n) (respectively, by $(y_n), (z_n), \ldots$).

Definition 1.1.3. *A sequence is bounded if the set of its terms is bounded.*

Definition 1.1.4. *One says that (y_k) is a subsequence of (x_n) if for every $k \in \mathbb{N}$, one has $y_k = x_{n_k}$, where by (n_k) one denotes a strictly increasing sequence of natural numbers (i.e., $n_k < n_{k+1}$ for every $k \in \mathbb{N}$).*

Definition 1.1.5. *One says that a sequence $(x_n) \subset \mathbb{R}^p$ is convergent (or converges) if there exists $x \in \mathbb{R}^p$ such that*

$$\forall\, V \in \mathcal{V}(x),\ \exists n_V \in \mathbb{N},\ \forall n \geq n_V : \ x_n \in V.$$

The element x is called the limit of (x_n).

If it exists, the limit of a sequence is unique.

We will use the notations $x_n \to x$, $\lim\limits_{n\to\infty} x_n = x$ or, simplified, $\lim x_n = x$ to formalize the previous definition.

Proposition 1.1.6. *A sequence (x_n) is convergent to $x \in \mathbb{R}^p$ if and only if*

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon : \|x_n - x\| < \varepsilon.$$

Proposition 1.1.7. *The sequence $(x_n) \subset \mathbb{R}^p$ converges to $x \in \mathbb{R}^p$ if and only if the coordinate sequences (x_n^i) converge (in $\mathbb{R}$) to x^i for every $i \in \overline{1,p}$.*

Proposition 1.1.8. *A sequence is convergent to $x \in \mathbb{R}^p$ if and only if all of its subsequences are convergent to x.*

Proposition 1.1.9. *Every convergent sequence is bounded.*

Proposition 1.1.10 (Characterization of the closure points using sequences)**.** *Consider $A \subset \mathbb{R}^p$. A point $x \in \mathbb{R}^p$ is a closure point of A if and only if there exists a sequence $(x_n) \subset A$ such that $x_n \to x$.*

Proposition 1.1.11. *The set $A \subset \mathbb{R}^p$ is closed if and only if every convergent sequence from A has its limit in A.*

Proposition 1.1.12. *The set $A \subset \mathbb{R}^p$ is compact if and only if every sequence from A has a subsequence which converges to a point of A.*

Theorem 1.1.13 (Cesàro Lemma)**.** *Every bounded sequence contains a convergent subsequence.*

Definition 1.1.14. *One says that $(x_n) \subset \mathbb{R}^p$ is a Cauchy sequence or a fundamental sequence if*

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n, m \geq n_\varepsilon : \|x_n - x_m\| < \varepsilon.$$

The above definition can be reformulated as follows: (x_n) is a Cauchy sequence if:

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N},\ \forall n \geq n_\varepsilon,\ \forall p \in \mathbb{N} : \|x_{n+p} - x_n\| < \varepsilon.$$

Theorem 1.1.15 (Cauchy). *The space $\mathbb{R}^p$ is complete, i.e., a sequence from $\mathbb{R}^p$ is convergent if and only if it is a Cauchy sequence.*

The next results are specific to the case of real sequences.

Definition 1.1.16. *One says that a sequence (x_n) of real numbers is increasing (strictly increasing, decreasing, strictly decreasing) if for every $n \in \mathbb{N}$, $x_{n+1} \geq x_n$ ($x_{n+1} > x_n$, $x_{n+1} \leq x_n$, $x_{n+1} < x_n$). If (x_n) is either increasing or decreasing, then it is called monotone.*

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be the set of extended real numbers. A neighborhood of $+\infty$ is a subset of $\overline{\mathbb{R}}$ which contains an interval of the form $(x, +\infty]$, where $x \in \mathbb{R}$. The neighborhoods of $-\infty$ are defined in a similar manner.

Definition 1.1.17. *(i) One says that the sequence $(x_n) \subset \mathbb{R}$ has the limit equal to $+\infty$ if*

$$\forall V \in \mathcal{V}(+\infty),\ \exists n_V \in \mathbb{N},\ \forall n \geq n_V : x_n \in V.$$

(ii) One says that the sequence $(x_n) \subset \mathbb{R}$ has the limit equal to $-\infty$ if

$$\forall V \in \mathcal{V}(-\infty),\ \exists n_V \in \mathbb{N},\ \forall n \geq n_V : x_n \in V.$$

Proposition 1.1.18. *(i) A sequence $(x_n) \subset \mathbb{R}$ has the limit equal to $+\infty$ if and only if*

$$\forall A > 0,\ \exists n_A \in \mathbb{N},\ \forall n \geq n_A : x_n > A.$$

(ii) A sequence $(x_n) \subset \mathbb{R}$ has the limit equal to $-\infty$ if and only if

$$\forall A > 0,\ \exists n_A \in \mathbb{N},\ \forall n \geq n_A : x_n < -A.$$

Proposition 1.1.19. *Let $(x_n), (y_n), (z_n)$ be sequences of real numbers, $x, y \in \mathbb{R}$ and $n_0 \in \mathbb{N}$. Then:*

(i) (Passing to the limit in inequalities) *if $x_n \to x$, $y_n \to y$ and $x_n \leq y_n$ for every $n \geq n_0$, then $x \leq y$;*

(ii) (The boundedness criterion) *if $|x_n - x| \leq y_n$ for every $n \geq n_0$, and $y_n \to 0$, then $x_n \to x$;*

(iii) if $x_n \geq y_n$ for every $n \geq n_0$, and $y_n \to +\infty$, then $x_n \to +\infty$;

(iv) if $x_n \geq y_n$ for every $n \geq n_0$, and $x_n \to -\infty$, then $y_n \to -\infty$;

(v) if (x_n) is bounded and $y_n \to 0$, then $x_n y_n \to 0$;

(vi) if $x_n \leq y_n \leq z_n$ for every $n \geq n_0$, and $x_n \to x$, $z_n \to x$, then $y_n \to x$;

(vii) $x_n \to 0 \Leftrightarrow |x_n| \to 0 \Leftrightarrow x_n^2 \to 0$.

We now present some fundamental results in the theory of real sequences.

Theorem 1.1.20. *Every monotone real sequence has its limit in $\overline{\mathbb{R}}$. Moreover, if the sequence is bounded, then it is convergent, as follows: if it is increasing, then its limit is the supremum of the set of its terms, and if it is decreasing, the limit is the infimum of the set of its terms. If it is unbounded, then its limit is either $+\infty$ if the sequence is increasing, or $-\infty$ if the sequence is decreasing.*

Theorem 1.1.21 (Weierstrass theorem for sequences)**.** *If (x_n) is a bounded and monotone sequence of real numbers, then (x_n) is convergent.*

Definition 1.1.22. *Let $(x_n)_{n\geq 0}$ be a sequence of real numbers. An element $x \in \overline{\mathbb{R}}$ is called a limit point of (x_n) if there exists a subsequence (x_{n_k}) of (x_n) such that $x = \lim\limits_{k\to\infty} x_{n_k}$.*

We finalize this section with two useful convergence criteria.

Proposition 1.1.23. *Let (x_n) be a sequence of strictly positive real numbers such that there exists $\lim \dfrac{x_{n+1}}{x_n} = x$. If $x < 1$, then $x_n \to 0$, and if $x > 1$, then $x_n \to +\infty$.*

Proposition 1.1.24 (Stolz-Cesàro Criterion)**.** *Let (x_n) and (y_n) be real sequences such that (y_n) is strictly increasing and its limit is equal to $+\infty$. If there exists $\lim \dfrac{x_{n+1} - x_n}{y_{n+1} - y_n} = x \in \overline{\mathbb{R}}$, then $\lim \dfrac{x_n}{y_n}$ exists and is equal to x.*

1.2 Limits of Functions and Continuity

In this section, we expand on some issues related to the concepts of limit and continuity for functions. Let $p, q \in \mathbb{N}^\star$.

Definition 1.2.1. *Let $f : A \to \mathbb{R}^q$, $A \subset \mathbb{R}^p$ and $a \in A'$. One says that the element $l \in \mathbb{R}^q$ is the limit of the function f at a, if for every $V \in \mathcal{V}(l)$, there exists $U \in \mathcal{V}(a)$ such that if $x \in U \cap A$, $x \neq a$, then $f(x) \in V$. We will denote this situation by $\lim\limits_{x\to a} f(x) = l$.*

Theorem 1.2.2. *Let $f : A \to \mathbb{R}^q$, $A \subset \mathbb{R}^p$ and $a \in A'$. The next assertions are equivalent:*

(i) $\lim\limits_{x\to a} f(x) = l$;

(ii) for every $B(l, \varepsilon) \subset \mathbb{R}^q$, there exists $B(a, \delta) \subset \mathbb{R}^p$ such that if $x \in B(a, \delta) \cap A$, $x \neq a$, then $f(x) \in B(l, \varepsilon)$;

(iii) for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $\|x - a\| < \delta$, $x \in A$, $x \neq a$, then $\|f(x) - l\| < \varepsilon$;

(iv) for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $|x_i - a_i| < \delta$ for every $i \in \overline{1, p}$, where $x = (x_1, x_2, .., x_p) \in A$, $a = (a_1, a_2, .., a_p)$, $x \neq a$, then $\|f(x) - l\| < \varepsilon$;

(v) for every sequence $(x_n) \subset A \setminus \{a\}$, $x_n \to a$ implies that $f(x_n) \to l$.

Theorem 1.2.3. *Let $f : A \to \mathbb{R}^q$, $A \subset \mathbb{R}^p$, $l \in \mathbb{R}^q$ and $a \in A'$. If the function f has the limit l at a, then this limit is unique.*

Remark 1.2.4. *If there exist two sequences (x'_n), $(x''_n) \subset A \setminus \{a\}$, $x'_n \to a$, $x''_n \to a$ such that $f(x'_n) \to l'$, $f(x''_n) \to l''$ and $l' \neq l''$, then the limit of the function f at $a \in A'$ does not exist.*

Theorem 1.2.5. *Let $f : A \to \mathbb{R}^q$, $A \subset \mathbb{R}^p$, $f = (f_1, f_2, \ldots, f_q)$ and $a \in A'$. Then f has the limit $l = (l_1, l_2, \ldots, l_q) \in \mathbb{R}^q$ at a if and only if there exists $\lim\limits_{x \to a} f_i(x) = l_i$, for every $i \in \overline{1, q}$.*

Definition 1.2.6. *Let $a \in \mathbb{R}$, $A \subset \mathbb{R}$ and denote $A_s = A \cap (-\infty, a]$, $A_d = A \cap [a, \infty)$. One says that the element a is a left (right) accumulation point for A, if it is an accumulation point for A_s (A_d, respectively). We will denote the set of left (right) accumulation points of A by A'_s (A'_d, respectively).*

Definition 1.2.7. *Let $f : A \to \mathbb{R}^q$, $A \subset \mathbb{R}$ and a be a left (right) accumulation point of A. One says that the element $l \in \mathbb{R}^q$ is the left-hand (right-hand) limit of the function f in a if for every neighborhood $V \in \mathcal{V}(l)$ there exists $U \in \mathcal{V}(a)$, such that if $x \in U \cap A_s$ ($x \in U \cap A_d$, respectively), $x \neq a$, then $f(x) \in V$. In this case we will write $\lim\limits_{x \to a, x<a} f(x) = l$, or $\lim\limits_{x \to a-} f(x) = l$, or $\lim\limits_{x \uparrow a} f(x) = l$ ($\lim\limits_{x \to a, x>a} f(x) = l$, or $\lim\limits_{x \to a+} f(x) = l$, or $\lim\limits_{x \downarrow a} f(x) = l$, respectively).*

Theorem 1.2.8. *Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}^q$, and $a \in I$. Then there exists $\lim\limits_{x \to a} f(x) = l$ if and only if the left-hand and the right-hand limits of f at a exist and they are equal. In this case, all three limits are equal:*

$$\lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = l.$$

A well-known result says that the monotone real functions admit lateral limits at every accumulation point of their domains.

Theorem 1.2.9 (The boundedness criterion)**.** *Let $f : A \to \mathbb{R}^q$, $g : A \to \mathbb{R}$, $A \subset \mathbb{R}^p$ and $a \in A'$. If there exist $l \in \mathbb{R}^q$ and $U \in \mathcal{V}(a)$ such that $\|f(x) - l\| \leq |g(x)|$ for every $x \in U \setminus \{a\}$, and $\lim\limits_{x \to a} g(x) = 0$, then there exists $\lim\limits_{x \to a} f(x) = l$.*

Theorem 1.2.10. *Let $f, g : A \subset \mathbb{R}^p \to \mathbb{R}^q$, and $a \in A'$. If $\lim\limits_{x \to a} f(x) = 0$ and there exists $U \in \mathcal{V}(a)$ such that g is bounded on U, then there exists the limit $\lim\limits_{x \to a} f(x) g(x) = 0$.*

Theorem 1.2.11. *Let $f : A \subset \mathbb{R}^p \to \mathbb{R}^q$, and $a \in A'$. If there exists $\lim\limits_{x \to a} f(x) = l$, $l > 0$ ($l < 0$), then there exists $U \in \mathcal{V}(a)$ such that for every $x \in U \cap A$, $x \neq a$, one has $f(x) > 0$ (respectively, $f(x) < 0$).*

Theorem 1.2.12. *Let $f : A \subset \mathbb{R}^p \to \mathbb{R}^q$, and $a \in A'$. If there exists $\lim\limits_{x \to a} f(x) = l$, then there exists $U \in \mathcal{V}(a)$ such that f is bounded on U (i.e., there exists $M > 0$ such that for every $x \in U$, one has $\|f(x)\| \leq M$).*

Definition 1.2.13. *Let $f : A \subset \mathbb{R}^p \to \mathbb{R}$ and $a \in A'$. One says that the function f has the limit equal to $+\infty$ (respectively, $-\infty$) at a, if for every $V \in \mathcal{V}(+\infty)$ (respectively, $V \in \mathcal{V}(-\infty)$), there exists $U \in \mathcal{V}(a)$ such that for every $x \in U \cap A$, $x \neq a$, one has $f(x) \in V$. In this case, we will write $\lim\limits_{x \to a} f(x) = +\infty$ (respectively, $\lim\limits_{x \to a} f(x) = -\infty$).*

Theorem 1.2.14. *Let $f : A \subset \mathbb{R}^p \to \mathbb{R}$ and $a \in A'$. Then there exists $\lim\limits_{x \to a} f(x) = +\infty$ (respectively, $\lim\limits_{x \to a} f(x) = -\infty$) if and only if for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $\|x - a\| < \delta$, $x \in A$, $x \neq a$, one has $f(x) > \varepsilon$ (respectively, $f(x) < -\varepsilon$).*

Definition 1.2.15. *Let $f : A \subset \mathbb{R} \to \mathbb{R}^q$, such that $+\infty$ (respectively, $-\infty$) is an accumulation point of A. One says that the element $l \in \mathbb{R}^q$ is the limit of f at $+\infty$ (respectively, $-\infty$), if for every $V \in \mathcal{V}(l)$, there exists $U \in \mathcal{V}(+\infty)$ (respectively, $U \in \mathcal{V}(-\infty)$) such that for every $x \in U \cap A$, one has $f(x) \in V$. In this case, we will write $\lim\limits_{x \to +\infty} f(x) = l$ (respectively, $\lim\limits_{x \to -\infty} f(x) = l$).*

Theorem 1.2.16. *Let $f : A \subset \mathbb{R} \to \mathbb{R}^q$, such that $+\infty$ (respectively, $-\infty$) is an accumulation point of A. Then there exists $\lim\limits_{x \to +\infty} f(x) = l$ (respectively, $\lim\limits_{x \to -\infty} f(x) = l$) if and only if for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $x > \delta$ (respectively, $x < -\delta$), $x \in A$, one has $\|f(x) - l\| < \varepsilon$.*

Definition 1.2.17. *Let $f : A \subset \mathbb{R}^p \to \mathbb{R}^q$, and $a \in A$. One says that the function f is continuous at a if for every $V \in \mathcal{V}(f(a))$, there exists $U \in \mathcal{V}(a)$ such that for every $x \in U \cap A$, one has $f(x) \in V$.*

If the function f is not continuous at a, one says that f is discontinuous at a, or that a is a discontinuity point of the function f.

Theorem 1.2.18. *Let $f : A \subset \mathbb{R}^p \to \mathbb{R}^q$, and $a \in A' \cap A$. The function f is continuous at a if and only if $\lim\limits_{x \to a} f(x) = f(a)$. If a is an isolated point of A, then f is continuous at a.*

Theorem 1.2.19. *Let $f : A \subset \mathbb{R}^p \to \mathbb{R}^q$, and $a \in A$. The next assertions are equivalent:*

(i) f is continuous at a;

(ii) ($\varepsilon - \delta$ characterization) *for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $\|x - a\| < \delta$, $x \in A$, then $\left\|f(x) - f(a)\right\| < \varepsilon$;*

(iii) (sequential characterization) *for every $(x_n) \subset A$, $x_n \to a$, one has $f(x_n) \to f(a)$.*

Theorem 1.2.20. *The image of a compact set through a continuous function is a compact set.*

Theorem 1.2.21 (Weierstrass Theorem). *Let K be a compact subset of $\mathbb{R}^p$. If $f : K \to \mathbb{R}$ is a continuous function, then f is bounded and it attains its extreme values on the set K (i.e., there exist $a, b \in K$, such that $\sup_{x \in K} f(x) = f(a)$ and $\inf_{x \in K} f(x) = f(b)$).*

Definition 1.2.22. *Let $f : D \subset \mathbb{R}^p \to \mathbb{R}^q$. One says that the function f is uniformly continuous on the set D if for every $\varepsilon > 0$, there exists $\delta > 0$, such that for every $x', x'' \in D$ with $\|x' - x''\| < \delta$, one has $\|f(x') - f(x'')\| < \varepsilon$.*

Remark 1.2.23. *Every function which is uniformly continuous on D is continuous on D, i.e., it is continuous at every point of D.*

Theorem 1.2.24 (Cantor Theorem). *Every function which is continuous on a compact set $K \subset \mathbb{R}^p$ and takes values in $\mathbb{R}^q$ is uniformly continuous on K.*

Definition 1.2.25. *Let $L \geq 0$ be a real number. One says that a function $f : A \subset \mathbb{R}^p \to \mathbb{R}^q$ is Lipschitz on A with modulus L, or L–Lipschitz on A, if $\|f(x) - f(y)\| \leq L \|x - y\|$, for every $x, y \in A$.*

Proposition 1.2.26. *Every Lipschitz function on $A \subset \mathbb{R}^p$ is uniformly continuous on A.*

Theorem 1.2.27. *Let $I \subset \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is injective and continuous, then f is strictly monotone on I.*

Definition 1.2.28. *Let $I \subset \mathbb{R}$ be an interval. One says that the function $f : I \to \mathbb{R}$ has the Darboux property if for every $a, b \in I$, $a < b$ and every $\lambda \in (f(a), f(b))$ or $\lambda \in (f(b), f(a))$, there exists $c_\lambda \in (a, b)$ such that $f(c_\lambda) = \lambda$.*

Theorem 1.2.29. *Let $I \subset \mathbb{R}$ be an interval. If the function $f : I \to \mathbb{R}$ has the Darboux property and there exist $a, b \in I$, $a < b$, such that $f(a)f(b) < 0$, then the equation $f(x) = 0$ has at least one solution in (a, b).*

Theorem 1.2.30. *Let $I \subset \mathbb{R}$ be an interval. The function $f : I \to \mathbb{R}$ has the Darboux property if and only if for every interval $J \subset I$, $f(J)$ is an interval.*

Theorem 1.2.31. *Let $I \subset \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is continuous, then f has the Darboux property.*

Recall that every linear operator $T : \mathbb{R}^p \to \mathbb{R}^q$ is continuous. For such a map, one uses the constant

$$\begin{aligned}\|T\| &:= \inf\{M > 0 \mid \|Tx\| \le M\|x\|\,,\ \forall x \in \mathbb{R}^p\}\\ &= \sup\left\{\|Tx\| \mid x \in D(0,1)\right\}.\end{aligned}$$

The mapping $T \mapsto \|T\|$ satisfies the axioms of a norm, therefore it is called the norm of the operator T. Consequently, the set of linear operators from $\mathbb{R}^p$ to $\mathbb{R}^q$ is a real normed vector space, with respect to the usual algebraic operations and to the norm previously defined. This space is denoted by $L(\mathbb{R}^p, \mathbb{R}^q)$ and can be isomorphically identified with $\mathbb{R}^{pq}$. Every operator $T \in L(\mathbb{R}^p, \mathbb{R}^q)$ can be naturally associated with a $q \times p$ matrix, denoted by $A_T = (a_{ji})_{j\in\overline{1,q},i\in\overline{1,p}}$, as follows: if $(e_i)_{i\in\overline{1,p}}$ and $(e'_i)_{i\in\overline{1,q}}$ are the canonical bases of the spaces $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, then $(a_{ji})_{j\in\overline{1,q},i\in\overline{1,p}}$ are the coordinates of the expressions of the images of the elements $(e_i)_{i\in\overline{1,p}}$ through T with respect to the basis $(e'_i)_{i\in\overline{1,q}}$, i.e.,

$$T(e_i) = \sum_{j=1}^{q} a_{ji}e'_j,\ \forall i \in \overline{1,p}.$$

Consequently, $T \mapsto A_T$ is an isomorphism of linear spaces between $L(\mathbb{R}^p, \mathbb{R}^q)$ and the space of real $q \times p$ matrices. Also, for every $x \in \mathbb{R}^p$:

$$T(x) = (A_T x^t)^t.$$

Moreover, for every $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$, one has that

$$\left\langle (A_T x^t)^t, y\right\rangle = \left\langle x, (A_T^t y^t)^t\right\rangle.$$

If A is a $q\times p$ matrix, then the linear operator associated with A is surjective if and only if the map associated with A^t is injective.

Recall also that if $T : \mathbb{R}^p \to \mathbb{R}^q$ is a linear operator, then its kernel,

$$\operatorname{Ker}(T) := \{x \in \mathbb{R}^p \mid T(x) = 0\},$$

is a linear subspace of $\mathbb{R}^p$, and its image,

$$\operatorname{Im}(T) := \{T(x) \mid x \in \mathbb{R}^p\},$$

is a linear subspace of $\mathbb{R}^q$. Moreover,

$$p = \dim(\operatorname{Ker}(T)) + \dim(\operatorname{Im}(T)),$$

where by dim we denote the algebraic dimension.

From the theory of linear algebra, one knows that if A is a symmetric square matrix of order p, then its eigenvalues are real and, moreover, there exists an orthogonal matrix B (i.e., $BB^t = B^tB = I$) such that B^tAB is the diagonal matrix having the eigenvalues on its main diagonal. Recall that, as usual, I denotes the identity matrix.

One says that a matrix A as above is positive semidefinite if $\langle (Ax^t)^t, x\rangle \geq 0$ for every $x \in \mathbb{R}^p$, and positive definite if $\langle (Ax^t)^t, x\rangle > 0$ for every $x \in \mathbb{R}^p \setminus \{0\}$. Actually, A is positive definite if and only if it is positive semidefinite and invertible.

We end this section by mentioning the celebrated result of Hahn-Banach. Recall that a function $f : \mathbb{R}^p \to \mathbb{R}$ is called sublinear if it is positive homogeneous (i.e., $f(\alpha x) = \alpha f(x)$ for all $\alpha \geq 0$ and $x \in \mathbb{R}^p$) and subadditive (i.e., $f(x+y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}^p$).

Theorem 1.2.32 (Hahn-Banach)**.** *Let X be a linear subspace of $\mathbb{R}^p$, $\chi : \mathbb{R}^p \to \mathbb{R}$ be a sublinear function, and $\varphi_0 : X \to \mathbb{R}$ be a linear function. If $\varphi_0(x) \leq \chi(x)$ for every $x \in X$, then there exists a linear function $\varphi : \mathbb{R}^p \to \mathbb{R}$ such that $\varphi\big|_X = \varphi_0$ and $\varphi(x) \leq \chi(x)$ for every $x \in \mathbb{R}^p$.*

1.3 Differentiability

Definition 1.3.1. *Let $f : D \subset \mathbb{R}^p \to \mathbb{R}^q$ and $a \in \operatorname{int} D$. One says that f is Fréchet differentiable (or, simply, differentiable) at a if there exists a linear operator denoted by $\nabla f(a) : \mathbb{R}^p \to \mathbb{R}^q$ such that*

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - \nabla f(a)(h)}{\|h\|} = \lim_{x \to a} \frac{f(x) - f(a) - \nabla f(a)(x-a)}{\|x-a\|} = 0.$$

The map $\nabla f(a)$ is called the Fréchet differential of the function f at a.

The previous relation is equivalent to the following conditions:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(a,\delta) : \left\|f(x) - f(a) - \nabla f(a)(x-a)\right\| \leq \varepsilon \|x-a\|;$$

$$\exists \alpha : D - \{a\} \to \mathbb{R}^q : \lim_{h\to 0} \alpha(h) = \alpha(0) = 0,$$
$$f(a+h) = f(a) + \nabla f(a)(h) + \|h\|\,\alpha(h), \ \forall h \in D - \{a\}.$$

One says that $f : D \subset \mathbb{R}^p \to \mathbb{R}^q$ is of class C^1 on the open set D if f is Fréchet differentiable on D and ∇f is continuous on D. Obviously, f can be written as $f = (f_1, f_2, \ldots, f_q)$, where $f_i : \mathbb{R}^p \to \mathbb{R}$, $i \in \overline{1,q}$ and, in general, the map $\nabla f(a) \in L(\mathbb{R}^p, \mathbb{R}^q)$ will be identified to the $q \times p$ matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1}(a) & \frac{\partial f_1}{\partial x^2}(a) & \cdots & \frac{\partial f_1}{\partial x^p}(a) \\ \frac{\partial f_2}{\partial x^1}(a) & \frac{\partial f_2}{\partial x^2}(a) & \cdots & \frac{\partial f_2}{\partial x^p}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_q}{\partial x^1}(a) & \frac{\partial f_q}{\partial x^2}(a) & \cdots & \frac{\partial f_q}{\partial x^p}(a) \end{pmatrix},$$

called the Jacobian matrix of f at the point a, where $\dfrac{\partial f_i}{\partial x^j}(a)$ is the partial derivative of the function f_i with respect to the variable x^j at a.

We will subsequently refer several times to the Jacobian matrix instead of the differential. Based on a general result, if $f : D \subset \mathbb{R}^p \to \mathbb{R}^p$ and $a \in \operatorname{int} D$, $\nabla f(a)$ is an isomorphism of $\mathbb{R}^p$ if and only if the Jacobian matrix of f at a is invertible.

The next calculus rules hold.

- Let $f : \mathbb{R}^p \to \mathbb{R}^q$ be an affine function, i.e., it takes the form $f(x) := g(x) + u$ for every $x \in \mathbb{R}^p$, where $g : \mathbb{R}^p \to \mathbb{R}^q$ is linear, and $u \in \mathbb{R}^q$. Then for every $x \in \mathbb{R}^p$, $\nabla f(x) = g$.
- Let $f : \mathbb{R}^p \to \mathbb{R}$ be of the form $f(x) = \frac{1}{2}\left\langle (Ax^t)^t, x\right\rangle + \langle b, x\rangle$, where A is a symmetric square matrix of order p, and $b \in \mathbb{R}^p$. Then for every $x \in \mathbb{R}^p$, $\nabla f(x) = (Ax^t)^t + b$.
- Let $D \subset \mathbb{R}^p$, $E \subset \mathbb{R}^q$, $\overline{x} \in \operatorname{int} D$, $\overline{y} \in \operatorname{int} E$ and $f, g : D \to \mathbb{R}^q$, $\varphi : D \to \mathbb{R}$, $h : E \to \mathbb{R}^k$.
  - If f, g are differentiable at $\overline{x}$, and $\alpha, \beta \in \mathbb{R}$, then the function $\alpha f + \beta g$ is differentiable at $\overline{x}$ and
  $$\nabla(\alpha f + \beta g)(\overline{x}) = \alpha \nabla f(\overline{x}) + \beta \nabla g(\overline{x}).$$
  - If f, φ are differentiable at $\overline{x}$, then φf is differentiable at $\overline{x}$ at
  $$\nabla(\varphi f)(\overline{x}) = \varphi(\overline{x}) \nabla f(\overline{x}) + f(\overline{x}) \nabla \varphi(\overline{x}),$$
  where $\left(f(\overline{x})\nabla\varphi(\overline{x})\right)(x) := \nabla\varphi(\overline{x})(x) \cdot f(\overline{x})$.
  - (Chain rule) If $f(D) \subset E$, $\overline{y} = f(\overline{x})$, f is differentiable at $\overline{x}$ and h is differentiable at $\overline{y}$, then $h \circ f$ is differentiable at $\overline{x}$ and
  $$\nabla(h \circ f)(\overline{x}) = \nabla h(\overline{y}) \circ \nabla f(\overline{x}).$$

A case which deserves special attention is $p = 1$. In this case one says that f is derivable at a if there exists
$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \in \mathbb{R}^q. \tag{1.3.1}$$
One denotes this limit by $f'(a)$ and it is called the derivative of f at a.

Proposition 1.3.2. *Let $f : D \subset \mathbb{R} \to \mathbb{R}^q$ and $a \in \operatorname{int} D$. The next assertions are equivalent:*

(i) f is derivable at a;

(ii) f is Fréchet differentiable at a.

In every one of these cases, $\nabla f(a)(x) = xf'(a)$ for every $x \in \mathbb{R}$.

Let $r \in \mathbb{N}^*$. If $f : D \subset \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^r$, and $(a, b) \in \operatorname{int} D$ is fixed, one defines $D_1 := \{x \in \mathbb{R}^p \mid (x, b) \in D\}$ and $f_1 : D_1 \to \mathbb{R}^r$, $f_1(x) := f(x, b)$. One says that f is Fréchet differentiable with respect to x at a if f_1 is Fréchet differentiable at a, and in this case the differential is denoted by $\nabla_x f(a,b)$. If f is differentiable at (a,b), then f is differentiable with respect to x and y at a and b, respectively, and

$$\nabla_x f(a,b) = \nabla f(a,b)(\cdot,0), \quad \nabla_y f(a,b) = \nabla f(a,b)(0,\cdot).$$

In the general case, one says that $f : D \subset \mathbb{R}^p \to \mathbb{R}^q$ is twice Fréchet differentiable at $a \in \operatorname{int} D$ if f is Fréchet differentiable on a neighborhood $V \subset D$ of a and $\nabla f : V \to L(\mathbb{R}^p, \mathbb{R}^q)$ is Fréchet differentiable at a, i.e., there exists a functional denoted by $\nabla^2 f(a)$, from the space $L^2(\mathbb{R}^p, \mathbb{R}^q) := L(\mathbb{R}^p, L(\mathbb{R}^p, \mathbb{R}^q))$, and $\alpha : D - \{a\} \to L(\mathbb{R}^p, \mathbb{R}^q)$, such that $\lim_{h\to 0} \alpha(h) = \alpha(0) = 0$ and for every $h \in D - \{a\}$, one has

$$\nabla f(a+h) = \nabla f(a) + \nabla^2 f(a)(h,\cdot) + \|h\|\,\alpha(h).$$

Recall that the space $L^2(\mathbb{R}^p, \mathbb{R}^q)$ mentioned above can be identified with the space of bilinear maps from $\mathbb{R}^p \times \mathbb{R}^p$ to $\mathbb{R}^q$.

One says that f is of class C^2 on the open set D if it is twice Fréchet differentiable on D and $\nabla^2 f : D \to L^2(\mathbb{R}^p, \mathbb{R}^q)$ is continuous.

Theorem 1.3.3. *Let $f : D \subset \mathbb{R}^p \to \mathbb{R}^q$ and $a \in \operatorname{int} D$. If f is twice Fréchet differentiable at a, then $\nabla^2 f(a)$ is a symmetric bilinear map.*

In the case when $q = 1$, the map $\nabla^2 f(a)$ is defined by the symmetric square matrix $H(a) = \left(\dfrac{\partial^2 f}{\partial x^i \partial x^j}(a)\right)_{i,j\in\overline{1,p}}$, which is called the Hessian matrix of f at a. Moreover, $\left\langle \left(H(a)u^t\right)^t, v\right\rangle = \nabla^2 f(a)(u,v)$ for every $u, v \in \mathbb{R}^p$, i.e.,

$$\nabla^2 f(a)(u,v) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^i \partial x^j}(a) u_i v_j.$$

If $a, b \in \mathbb{R}^p$, one defines the closed and the open line segments between a and b as follows:

$$[a,b] := \{\alpha a + (1-\alpha)b \mid \alpha \in [0,1]\},$$
$$(a,b) := \{\alpha a + (1-\alpha)b \mid \alpha \in (0,1)\}.$$

Theorem 1.3.4 (Lagrange and Taylor Theorems)**.** *Let $U \subset \mathbb{R}^p$ be an open set, $f : U \to \mathbb{R}$ and $a, b \in U$ with $[a,b] \subset U$. If f is of class C^1 on U, then there exists $c \in (a,b)$ such that*

$$f(b) = f(a) + \nabla f(c)(b-a).$$

If f is of class C^2 on U, then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + \nabla f(a)(b-a) + \frac{1}{2}\nabla^2 f(c)(b-a, b-a).$$

Theorem 1.3.5 (Implicit Function Theorem). *Let $D \subset \mathbb{R}^p \times \mathbb{R}^q$ be an open set, $h : D \to \mathbb{R}^q$ be a function and $\overline{x} \in \mathbb{R}^p$, $\overline{y} \in \mathbb{R}^q$ be such that:*

(i) $h(\overline{x}, \overline{y}) = 0$;

(ii) the function h is of class C^1 on D;

(iii) $\nabla_y h(\overline{x}, \overline{y})$ is invertible.

Then there exist two neighborhoods U and V of $\overline{x}$ and $\overline{y}$, respectively, and a unique continuous function $\varphi : U \to V$ such that:

(a) $h(x, \varphi(x)) = 0$ for every $x \in U$;

(b) if $(x, y) \in U \times V$ and $h(x, y) = 0$, then $y = \varphi(x)$;

(c) φ is differentiable on U and

$$\nabla \varphi(x) = -[\nabla_y h(x, \varphi(x))]^{-1} \nabla_x h(x, \varphi(x)), \ \forall x \in U.$$

Some fundamental results from the theory of differentiability of the real functions are briefly given at the end of this section.

In the case $p = q = 1$ one can apply Proposition 1.3.2. It is also sensible to speak of the existence of the derivative at points of the domain which are accumulation points of it: consider the limit from the relation (1.3.1) at accumulation points. Moreover, as in the case of the lateral limits, one can speak about the left and right-hand derivatives, by considering the lateral limits in the expression from relation (1.3.1). When they exist, we will call these limits the left, and the right-hand derivatives of the function f at a and we will denote them by $f'_-(a)$ and $f'_+(a)$, respectively.

Definition 1.3.6. *Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. One says that $a \in A$ is a local minimum (maximum) point for f if there exists a neighborhood V of a such that $f(a) \le f(x)$ (respectively, $f(a) \ge f(x)$), for every $x \in A \cap V$. One says that a point is a local extremum if it is a local minimum or a local maximum.*

Theorem 1.3.7 (Fermat Theorem). *Let $I \subset \mathbb{R}$ be an interval and $a \in \operatorname{int} I$. If $f : I \to \mathbb{R}$ is derivable at a, and a is a local extremum point for f, then $f'(a) = 0$.*

Theorem 1.3.8 (Rolle Theorem). *Let $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$, derivable on (a, b), and satisfies $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 1.3.9 (Lagrange Theorem). *Let $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$, derivable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Proposition 1.3.10. *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be derivable on I.*

(i) If $f'(x) = 0$ for every $x \in I$, then f is constant on I.

(ii) If $f'(x) > 0$ (respectively, if $f'(x) \geq 0$) for every $x \in I$, then f is strictly increasing (respectively, it is increasing) on I.

(iii) If $f'(x) < 0$ (respectively, if $f'(x) \leq 0$), for every $x \in I$, then f is strictly decreasing (respectively, it is decreasing) on I.

Theorem 1.3.11 (Rolle Sequence). *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a derivable function. If $x_1, x_2 \in I$, $x_1 < x_2$ are consecutive roots of the derivative f' (i.e., $f'(x_1) = 0$, $f'(x_2) = 0$ and $f'(x) \neq 0$ for any $x \in (x_1, x_2)$) then:*

(i) if $f(x_1)f(x_2) < 0$, the equation $f(x) = 0$ has exactly one root in the interval (x_1, x_2);

(ii) if $f(x_1)f(x_2) > 0$, the equation $f(x) = 0$ has no roots in the interval (x_1, x_2);

(iii) if $f(x_1) = 0$ or $f(x_2) = 0$, then x_1 or x_2 is a multiple root of the equation $f(x) = 0$ and this equation has no other root in the interval (x_1, x_2).

Theorem 1.3.12 (Cauchy Rule). *Let $I \subset \mathbb{R}$ be an interval and $f, g : I \to \mathbb{R}$, $a \in I$, which satisfy:*

(i) $f(a) = g(a) = 0$;

(ii) f, g are derivable at a;

(iii) $g'(a) \neq 0$.

Then there exists $V \in \mathcal{V}(a)$ such that $g(x) \neq 0$, for any $x \in V \setminus \{a\}$ and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Theorem 1.3.13 (L'Hôpital Rule). *Let $f, g : (a, b) \to \mathbb{R}$, where $-\infty \leq a < b \leq \infty$. If:*

(i) f, g are derivable on (a, b) with $g' \neq 0$ on (a, b);

(ii) there exists $\lim\limits_{\substack{x \to a \\ x > a}} \dfrac{f'(x)}{g'(x)} = L \in \overline{\mathbb{R}}$;

(iii) $\lim\limits_{\substack{x \to a \\ x > a}} f(x) = \lim\limits_{\substack{x \to a \\ x > a}} g(x) = 0$ or

(iii)' $\lim\limits_{\substack{x \to a \\ x > a}} g(x) = \infty$,

then there exists $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = L$.

Theorem 1.3.14. *Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ be a n–times derivable function at $a \in I$, ($n \in \mathbb{N}$, $n \geq 2$), such that*

$$f'(a) = 0,\ f''(a) = 0, \ldots, f^{(n-1)}(a) = 0,\ f^{(n)}(a) \neq 0.$$

(i) If n is even, then a is an extremum point, more precisely: a local maximum if $f^{(n)}(a) < 0$, and a local minimum if $f^{(n)}(a) > 0$.

(ii) If n is odd, then a is not an extremum point.

1.4 The Riemann Integral

At the end of this chapter we discuss the main aspects concerning the Riemann integral. Let $a, b \in \mathbb{R}$, $a < b$.

Definition 1.4.1. *(i) A partition of the interval $[a, b]$ is a finite set of real numbers $x_0, x_1, \ldots, x_n$ ($n \in \mathbb{N}^\star$), denoted by Δ, such that*

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

(ii) The norm of the partition Δ is the number

$$\|\Delta\| := \max\{x_i - x_{i-1} \mid i \in \overline{1, n}\}.$$

(iii) A tagged partition of the interval $[a, b]$ is a partition Δ, together with a finite set of real numbers $\Xi := \{\xi_i \mid i \in \overline{1, n}\}$, such that $\xi_i \in [x_{i-1}, x_i]$ for any $i \in \overline{1, n}$. The set Ξ is called the intermediate points system associated to Δ.

(iv) Let $f : [a, b] \to \mathbb{R}$ be a function. The Riemann sum associated to a tagged partition of the interval $[a, b]$ is

$$S(f, \Delta, \Xi) := \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

Definition 1.4.2. *Let $f : [a, b] \to \mathbb{R}$ be a function. One says that f is Riemann integrable on $[a, b]$ if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition Δ of the interval $[a, b]$ with the property $\|\Delta\| < \delta$, and for any intermediate points system Ξ associated to Δ, the next inequality holds:*

$$\left|S(f, \Delta, \Xi) - I\right| < \varepsilon.$$

The real number I from the previous definition, which is unique, is called the Riemann integral of f an $[a, b]$ and is denoted by

$$\int_a^b f(x)dx.$$

Theorem 1.4.3. *Any function which is Riemann integrable on $[a, b]$ is bounded on $[a, b]$.*

Definition 1.4.4. *Let $f : [a, b] \to \mathbb{R}$ be a function. One says that a function $F : [a, b] \to \mathbb{R}$ is an antiderivative (or, equivalently, a primitive integral, or an indefinite integral) of f on $[a, b]$ if F is derivable on $[a, b]$ and $F'(x) = f(x)$ for any $x \in [a, b]$.*

If an antiderivative exists for a given function, then infinitely many antiderivatives exist for that function and the difference of any two such antiderivatives is a constant.

The next result is sometimes called the fundamental theorem of calculus.

Theorem 1.4.5 (Leibniz-Newton)**.** *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ and it admits an antiderivative F on $[a, b]$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Continuous functions satisfy both hypotheses of the preceding theorem.

Theorem 1.4.6. *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$ and it admits antiderivatives on $[a, b]$.*

Theorem 1.4.7. *If $f : [a, b] \to \mathbb{R}$ is bounded and has a finite set of discontinuity points, then f is Riemann integrable on $[a, b]$. Every function which is monotone on $[a, b]$ is Riemann integrable on $[a, b]$.*

We present now the main properties of the Riemann integral.

Theorem 1.4.8. *(i) If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable on $[a, b]$, and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is Riemann integrable on $[a, b]$ and*

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

(ii) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and $m \leq f(x) \leq M$ for every $x \in [a, b]$ $(m, M \in \mathbb{R})$, then

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a).$$

In particular, if $f(x) \geq 0$ for every $x \in [a, b]$, then

$$\int_a^b f(x)dx \geq 0,$$

and if $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable and $f(x) \leq g(x)$ for every $x \in [a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

(iii) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then $|f|$ is Riemann integrable on $[a, b]$.

(iv) If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable $[a, b]$, then $f \cdot g$ is Riemann integrable on $[a, b]$.

Theorem 1.4.9. *(i) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then f is Riemann integrable on every subinterval of $[a, b]$.*

(ii) If $c \in (a, b)$ and f is Riemann integrable on $[a, c]$ and on $[c, b]$, then f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Theorem 1.4.10. *Let $f : [a, b] \to \mathbb{R}$ be a function, and $f^* : [a, b] \to \mathbb{R}$ be another function which coincides with f on $[a, b]$, except on a finite set of points. If f^* is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$ and*

$$\int_a^b f(x)dx = \int_a^b f^*(x)dx.$$

Theorem 1.4.11 (integration by parts)**.** *If $f, g : [a, b] \to \mathbb{R}$ are C^1functions, then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Theorem 1.4.12 (change of variable)**.** *Let $\varphi : [a, b] \to [c, d]$ be a C^1 function, and let $f : [c, d] \to \mathbb{R}$ be a continuous function. Then*

$$\int_a^b f(\varphi(t)) \cdot \varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx. \tag{1.4.1}$$

We end this section by the next multidimensional variant of Taylor Theorem. In what follows, the equality is understood on components (i.e., for every function $f_i : \mathbb{R}^p \to \mathbb{R}$, $i = \overline{1, p}$, where $f = (f_1, \ldots, f_p)$).

Theorem 1.4.13. *Suppose $f : \mathbb{R}^p \to \mathbb{R}^p$ is continuously differentiable on some convex open set D and that $x, x + y \in D$. Then there is $t \in (0, 1)$ such that*

$$f(x + y) = f(x) + \int_0^1 \nabla f(x + ty)(y)\,dt.$$