

A Non-Probabilistic Proof of the Assouad Embedding Theorem with Bounds on the Dimension

Abstract

We give a non-probabilistic proof of a theorem of Naor and Neiman that asserts that if (E, d) is a doubling metric space, there is an integer $N > 0$, depending only on the metric doubling constant, such that for each exponent $\alpha \in (1/2, 1)$, one can find a bilipschitz mapping $F = (E, d^\alpha) \rightarrow \mathbb{R}^N$.

Keywords

Assouad Embedding • doubling metric spaces • snowflake distance

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1. Introduction

The purpose of this paper is to give a simpler proof of a theorem proved by Naor and Neiman [3] that asserts that if (E, d) is a doubling metric space, there is an integer $N > 0$, depending only on the metric doubling constant, such that for each exponent $\alpha \in (1/2, 1)$, one can find a bilipschitz mapping $F : (E, d^\alpha) \rightarrow \mathbb{R}^N$.

Here \mathbb{R}^N is equipped with its Euclidean metric, the snowflake distance d^α is simply defined by $d^\alpha(x, y) = d(x, y)^\alpha$ for $x, y \in E$, and metrically doubling means that there is an integer $C_0 \geq 1$ such that for every $r > 0$, every (closed) ball of radius $2r$ in E can be covered with no more than C_0 balls of radius r . We call C_0 a metric doubling constant for (E, d) .

Remark 1.

Notice that in a doubling metric space with doubling constant C_0 , every ball of radius $2^m r$ can be covered with C_0^m balls of radius r . More generally, for $\lambda > 0$, every ball of radius λr can be covered by $C_0 \lambda^{N_0}$ balls of radius r , where $N_0 = \log_2 C_0$. To see this replace λ by the next power of 2.

A consequence of Naor and Neiman's main result in [3] is the following.

Theorem 2.

For each $C_0 \geq 1$, there is an integer N and, for $1/2 < \alpha < 1$, a constant $C = C(C_0, \alpha)$ such that if (E, d) is a metric space that admits the metric doubling constant C_0 , we can find an injection $F : E \rightarrow \mathbb{R}^N$ such that

$$C^{-1} d(x, y)^\alpha \leq |F(x) - F(y)| \leq C d(x, y)^\alpha$$

for $x, y \in E$.

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If we did not ask N to be independent of α , this would just be the usual Assouad embedding theorem from [1]. The theorem of Naor and Neiman is more general than Theorem 2, but its proof is also more complicated. In particular, the existence of the bilipschitz embedding F is demonstrated in [3] by constructing a random embedding into \mathbb{R}^N and showing (via a version of the Lovász Local Lemma) that with positive probability the map is bilipschitz.

In the present paper we do not try to get optimal values of N and C ; see [3] for better control of these constants, and for more information on the context. There is nothing special about the constant $1/2$ in our statement; we just do not want to consider the case when α is close to 0, for which the dimension independence fails. Indeed, even when $E = \mathbb{R}$, with the usual distance, one needs many dimensions to construct an α -snowflake (or equivalently, a mapping F as above) with α small.

The proof will be a rather simple modification of the standard proof of Assouad’s theorem (see, for example [2]), but it took a surprising amount of energy to the authors, plus the knowledge of the fact that the result is true, to make it work. Rather than using a probabilistic proof, we use an adaptive argument, and work at small relative scales to use the fact that there is a lot of space in \mathbb{R}^N ; the difficult part (for the authors) was to realize that using a very sparse collection of scales would not kill the argument, and instead helps control the residual terms. The constant $C(C_0, \alpha)$ gets very large (when α gets close to 1), but this is expected.

2. Acknowledgements

The issue of giving a simpler proof of Theorem 2 came out in the AIM workshop on Mapping Theory in Metric Spaces in Palo Alto (January 2012); the authors wish to thank Mario Bonk for asking the question, other participants of the task force, and in particular U. Lang, C. Smart, J. Tyson, for very interesting discussions and their patience with at least one wrong proof, and L Capogna, J. Tyson, and S. Wenger who organized the event.

3. Proof of Theorem 2

We need some notation before we start the proof. Here the doubling metric space (E, d) is fixed, and $B(x, r)$ will denote the closed ball with center x and radius r .

We note that, in fact, if one considers values of α bounded away from both 0 and 1, one can use the standard Assouad proof (as in the proof of Lemma 12.7 in [2]) to show that the resulting snowflake spaces can all be embedded in a fixed dimension Euclidean space. Hence we need only consider α very close to 1.

We shall use a small parameter $\tau > 0$, with $\tau \leq 1 - \alpha$, and work at the scales

$$r_k = \tau^{2k}, k \in \mathbb{Z}. \tag{1}$$

We first prove Theorem 2 in the case that E has finite diameter; this allows us to choose an initial scale $k = k_0$, where k_0 is such that $r_{k_0} \geq \text{diam}(E)$. But our construction, and the main constants, will not depend on this choice. We shall dispense with this extra assumption and treat the general case near (21).

For each $k \geq k_0$, select a maximal collection $\{x_j\}$, $j \in J_k$, of points of E , with $d(x_i, x_j) \geq r_k$ for $i \neq j$. Thus, by maximality

$$E \subset \bigcup_{j \in J_k} B(x_j, r_k). \tag{2}$$

For convenience, we write $B_j = B(x_j, r_k)$ and $\lambda B_j = B(x_j, \lambda r_k)$ for $\lambda > 0$.

Letting $N(x)$ denote the number of indices $j \in J_k$ such that $d(x_j, x) \leq 10r_k$, we now check that

$$N(x) \leq C_0^5 \text{ for } x \in E. \tag{3}$$

Indeed, we can cover $B(x, 10r_k)$ with fewer than C_0^5 balls D_l of radius $r_k/3$. Each D_l contains at most one x_j (because $d(x_i, x_j) \geq r_k$ for $i \neq j$). Because all the x_j that lie in $B(x, 10r_k)$ are contained in some D_l , (3) follows.

From (3) and the assumption (for the present case) that E is bounded, we conclude J_k is finite. Note, however, that even if E were unbounded, we could use the preceding discussion to construct the collection $\{x_j\}$ without using the axiom of choice.

Set $\Xi = \{1, 2, \dots, C_0^5\}$ (a set of colors). Now enumerate J_k , and for $j \in J_k$ let $\xi(j)$ be the first color not taken by an earlier close neighbor; specifically, choose $\xi(j)$ to be the first color not used by an earlier $i \in J_k$ such that $d(x_i, x_j) \leq 10r_k$. By construction we have

$$\xi(i) \neq \xi(j) \quad \text{for } i, j \in J_k \text{ with } i \neq j \text{ and } d(x_i, x_j) \leq 10r_k.$$

Finally, for each $\xi \in \Xi$, define the set $J_k(\xi) := \{j \in J_k; \xi(j) = \xi\}$. Thus

$$d(x_i, x_j) > 10r_k \quad \text{for } i, j \in J_k(\xi) \text{ such that } i \neq j. \quad (4)$$

For each $j \in J_k$, set $\varphi_j(x) = \max\{0, 1 - r_k^{-1} \text{dist}(x, B_j)\}$. The formula is not important; we just want to make sure that

$$\begin{aligned} 0 \leq \varphi_j(x) &\leq 1 \quad \text{everywhere,} \\ \varphi_j(x) &= 1 \quad \text{for } x \in B_j, \\ \varphi_j(x) &= 0 \quad \text{for } x \in E \setminus 2B_j, \end{aligned} \quad (5)$$

and

$$\varphi_j \text{ is Lipschitz, with } \|\varphi_j\|_{lip} \leq r_k^{-1}. \quad (6)$$

We continue with the non-surprising part of the construction. For each $\xi \in \Xi$, we will construct two mappings: $F^\xi : E \rightarrow \mathbb{R}^M$ and a slightly modified version $\tilde{F}^\xi : E \rightarrow \mathbb{R}^M$, where M is a very large integer depending only on the metric doubling constant. Our final mapping $F : E \rightarrow \mathbb{R}^N$, will be the tensor product of these $2C_0^5$ mappings. Thus, the dimension N is $2C_0^5M$, which can probably be improved.

We decide that F^ξ will be of the form:

$$F^\xi(x) = \sum_{k \geq k_0} r_k^\alpha f_k^\xi(x),$$

where

$$f_k^\xi(x) = \sum_{j \in J_k(\xi)} v_j \varphi_j(x), \quad (7)$$

with vectors $v_j \in \mathbb{R}^M$ that will be carefully chosen later. The extra room in \mathbb{R}^M will be used to give lots of different choices of v_j . \tilde{F} will take on the same form as F , but with a different choice of the vectors $\{v_j\}$. However, for both functions we will choose the v_j inductively, and so that

$$v_j \in B(0, \tau^2) \subset \mathbb{R}^M,$$

with the same very small $\tau > 0$ as in the definition of $r_k = \tau^{2k}$ above; τ will be chosen near the end. With this choice, we immediately see that

$$\|f_k^\xi\|_\infty \leq \tau^2$$

because the $\varphi_j, j \in J_k(\xi)$, have disjoint supports by (4) and (5); hence the series in (7) converges. Moreover, if we set

$$F_k^\xi(x) = \sum_{k_0 \leq \ell \leq k} r_\ell^\alpha f_\ell^\xi(x), \quad (8)$$

we get that

$$\|F^\xi - F_k^\xi\|_\infty \leq \sum_{\ell > k} r_\ell^\alpha \tau^2 = r_{k+1}^\alpha \tau^2 \sum_{\ell \geq 0} \tau^{2\ell\alpha} \leq 2\tau^2 r_{k+1}^\alpha \quad (9)$$

(because $r_l = \tau^{2l}$ and $\tau^{2\alpha} < 1/2$ when τ is small). Also, the Lipschitz norm of f_k^ξ is

$$\|f_k^\xi\|_{lip} \leq \tau^2 r_k^{-1}$$

by (6) and because the φ_j are supported in disjoint balls; we sum brutally and get that

$$\|F_k^\xi\|_{lip} \leq \sum_{\ell \leq k} r_\ell^\alpha \|f_\ell^\xi\|_{lip} \leq \tau^2 \sum_{\ell \leq k} r_\ell^{\alpha-1} = \tau^2 r_k^{\alpha-1} \sum_{\ell \leq k} \tau^{2(\ell-k)(\alpha-1)} = \tau^2 r_k^{\alpha-1} (1 - \tau^{2(1-\alpha)})^{-1}$$

by (8). We take $\tau \leq 1 - \alpha$ (many other choices would do, the main point is to have a control in (10) below by a power of τ , which could even be negative); then

$$\ln(\tau^{2(1-\alpha)}) = 2(1-\alpha) \ln(\tau) = -2(1-\alpha) \ln\left(\frac{1}{\tau}\right) \leq -2\tau \ln\left(\frac{1}{\tau}\right);$$

we exponentiate and get that $\tau^{2(1-\alpha)} \leq e^{-2\tau \ln(\frac{1}{\tau})} \leq 1 - \tau \ln(\frac{1}{\tau})$ if τ is small enough, hence $1 - \tau^{2(1-\alpha)} \geq \tau \ln(\frac{1}{\tau})$, and finally

$$\|F_k^\xi\|_{lip} \leq \frac{\tau}{\ln(\frac{1}{\tau})} r_k^{\alpha-1} \leq r_k^{\alpha-1} \quad (10)$$

(again if τ is small enough; for example, $\tau < 1/2$ works).

We now describe how to choose the vectors v_j , $j \in J_k$, so that the differences $|F_k^\xi(x) - F_k^\xi(y)|$ will be as large as possible (toward proving that $(F_k^\xi)^{-1}$ is Lipschitz).

Fix $k \geq k_0$, suppose that the F_{k-1}^ξ were already constructed, and fix $\xi \in \Xi$. Put any order $<$ on the finite set $J_k(\xi)$. We shall construct F_k^ξ with the order $<$. Specifically, we will choose the v_j in Lemma 3 using the order $<$. For \tilde{F}_k^ξ , we will use the reverse order.

Recall that we defined

$$F_k^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha f_k^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha \sum_{i \in J_k(\xi)} v_i \varphi_i(y) \quad (11)$$

in (8) and (7); for each $j \in J_k(\xi)$, we shall also consider the partial sum $G_{k,j}^\xi$ defined by

$$G_{k,j}^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha \sum_{i \in J_k(\xi); i < j} v_i \varphi_i(y), \quad (12)$$

which we therefore assume to be known when we choose v_j .

Lemma 3.

For each $j \in J_k(\xi)$, we can choose $v_j \in B(0, \tau^2)$ so that

$$|F_k^\xi(x) - G_{k,j}^\xi(y)| \geq \tau^3 r_k^\alpha \quad \text{for } x \in B_j \text{ and } y \in B(x_j, 10\tau^{-2}r_k) \setminus 2B_j. \quad (13)$$

Observe that for $x \in B_j$, $\varphi_j(x) = 1$ and $\varphi_i(x) = 0$ for the other indices $i \neq j \in J_k(\xi)$ (because φ_i is supported in $2B_i$ by (5), and $2B_i$ never meets B_j by (4)). Thus

$$F_k^\xi(x) = F_{k-1}^\xi(x) + r_k^\alpha f_k^\xi(x) = F_{k-1}^\xi(x) + r_k^\alpha v_j \quad (14)$$

by (8) and (7). By (10), $\|F_k^\xi\|_{lip} \leq r_k^{\alpha-1}$, but the proof of (10) also yields

$$\|G_{k,j}^\xi\|_{lip} \leq r_k^{\alpha-1}$$

(we just add fewer terms). We shall use this to replace B_j and $B(x_j, 10\tau^{-2}r_k) \setminus 2B_j$ with discrete sets. Set $\eta = \tau^3 r_k$, and pick an η -dense set X in B_j , and an η -dense set Y in $B(x_j, 10\tau^{-2}r_k) \setminus 2B_j$. We shall soon prove that we can choose v_j so that

$$|F_k^\xi(x') - G_{k,j}^\xi(y')| \geq 3\tau^3 r_k^\alpha \quad \text{for } x' \in X \text{ and } y' \in Y, \quad (15)$$

and let us first check that the lemma will follow.

Proof of Lemma 3. Notice that for $x \in B_j$, we can find $x' \in X$ such that

$$|F_k^\xi(x') - F_k^\xi(x)| \leq \|F_k^\xi\|_{lip} \eta \leq r_k^{\alpha-1} \cdot \tau^3 r_k = \tau^3 r_k^\alpha,$$

and similarly, for $y \in B(x_j, 10\tau^{-2}r_k) \setminus 2B_j$ we can find $y' \in Y$ such that

$$|G_{k,j}^\xi(y) - G_{k,j}^\xi(y')| \leq \|G_{k,j}^\xi\|_{lip} \eta \leq \tau^3 r_k^\alpha.$$

Then (13) for x and y follows from (15), as needed.

So we want to arrange (15). First we bound $|X|$, the number of elements in X . By Remark 1, we can cover $B_j = B(x_j, r_k)$ by $C_0(2\tau^{-3})^{N_0}$ balls of radius $\eta/2$; we just keep those that meet B_j , pick an element of B_j in each such ball, and get an η -dense net X , with $|X| \leq C_0(2\tau^{-3})^{N_0}$. Similarly, we can find Y so that

$$|Y| \leq C_0 \left(2 \frac{10\tau^{-2}r_k}{\eta} \right)^{N_0} = C_0 \left(\frac{20\tau^{-2}r_k}{\tau^3 r_k} \right)^{N_0} = C_0(20\tau^{-5})^{N_0}.$$

The total number of pairs (x', y') for which we have to check (15) is thus

$$|X||Y| \leq C_0^2(40\tau^{-8})^{N_0}.$$

Now pick a maximal finite set V in $B(0, \tau^2) \subset \mathbb{R}^M$, whose points lie at mutual distances at least $7\tau^3$ from each other. For each pair (x', y') as above, the different choices of $v_j \in V$ yield the same value of $G_{k,j}^\xi(y')$ (because $G_{k,j}^\xi$ does not depend on v_j , by (12)), and values of $F_k^\xi(x')$ that differ by at least $7\tau^3 r_k^\alpha$, by (14). Thus (15) for this pair (x', y') cannot fail for more than one choice of $v_j \in V$, and it is now enough to show that V has more than $|X||Y|$ elements. Taking C_M to be the doubling constant of \mathbb{R}^M , it follows from Remark 1 that $|V| \geq C_M(1/7\tau)^{\log_2 C_M}$, which is indeed larger than $|X||Y|$ if $C_M > C_0^8$ and τ is small enough (depending on M). This completes our verification of (15); as noted earlier, Lemma 3 follows. \square

For each color ξ , choose the vectors v_j , and hence the mappings F_k^ξ , as in Lemma 3. Also define a second version \tilde{F}_k^ξ of F_k^ξ using the opposite order on $J_k(\xi)$. By (9), $F^\xi : E \rightarrow \mathbb{R}^M$ is the limit $F^\xi = \lim_{k \rightarrow \infty} F_k^\xi$. We are ready to check that F (the tensor product of the maps $\{F^\xi, \tilde{F}^\xi : \xi \in \Xi\}$) is bilipschitz from the snowflaked space (E, d^α) .

Lemma 4.

We have that

$$\frac{\tau^5}{8} d(x, y)^\alpha \leq |F(x) - F(y)| \leq 5N\tau^{-2(1-\alpha)} d(x, y)^\alpha \text{ for } x, y \in E.$$

Proof. Let $x, y \in E$ be given; we may assume that $x \neq y$. Let k be such that

$$4r_k \leq d(x, y) \leq 4r_{k-1} = 4\tau^{-2}r_k, \tag{16}$$

where the last part comes from (1). Then $r_k \leq 4r_k \leq d(x, y) \leq \text{diam}(E) \leq r_{k_0}$ by our definition of k_0 , and so $k \geq k_0$. By (10) and (16),

$$|F_k^\xi(x) - F_k^\xi(y)| \leq \|F_k^\xi\|_{lip} d(x, y) \leq r_k^{\alpha-1} d(x, y) \leq \left(\frac{d(x, y)}{4\tau^{-2}} \right)^{\alpha-1} d(x, y) = (4\tau^{-2})^{1-\alpha} d(x, y)^\alpha. \tag{17}$$

But (9) also says that for sufficiently small τ ,

$$\left| |F^\xi(x) - F^\xi(y)| - |F_k^\xi(x) - F_k^\xi(y)| \right| \leq 2\|F^\xi - F_k^\xi\|_\infty \leq 4\tau^2 r_{k+1}^\alpha = 4\tau^2 \tau^{2\alpha} r_k^\alpha \leq 2\tau^2 d(x, y)^\alpha. \tag{18}$$

Inequalities (17) and (18) (and similar estimates for the \tilde{F}_k^ξ) give the upper bound in the statement of Lemma 4.

For the lower bound, consider the same fixed $x, y \in E$. Notice that by (2) we can find $j \in J_k$ such that $x \in B_j$. Let $\xi \in \Sigma$ be the color such that $j \in J_k(\xi)$. We will consider separately the case where $y \in 2B_i$ for some $i \in J_k(\xi)$ and where $y \notin 2B_i$ for all $i \in J_k(\xi)$.

We'll need to know that

$$y \in B(x_j, 10\tau^{-2}r_k) \setminus 2B_j. \quad (19)$$

That $y \in B(x_j, 10\tau^{-2}r_k)$ follows from (16), because $d(x, x_j) \leq r_k$ since $x \in B_j$. Moreover, if $y \in 2B_j$, then $d(x, y) \leq d(x, x_j) + d(x_j, y) \leq 3r_k$, which would contradict (16). So (19) holds.

If $y \in 2B_i$ for some $i \in J_k(\xi)$, then $i \neq j$, by (19). Let us assume that $i < j$; otherwise, we would use \tilde{F}_k^ξ instead of F_k^ξ in the following calculations. Recall that all the $\varphi_l(y)$, $l \neq i$, are equal to 0, by (5) and (4). Then (11) and (12) yield

$$F_k^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha v_i \varphi_i(y) = G_{k,i}^\xi(y) \quad (20)$$

(with $F_{k-1}^\xi(y) = 0$ if $k = k_0$). By (19), we can apply (13), which says that

$$|F_k^\xi(x) - F_k^\xi(y)| = |F_k^\xi(x) - G_{k,i}^\xi(y)| \geq \tau^3 r_k^\alpha.$$

We then combine this with (18) and get that

$$|F^\xi(x) - F^\xi(y)| \geq |F_k^\xi(x) - F_k^\xi(y)| - 4\tau^2 \tau^{2\alpha} r_k^\alpha \geq \tau^3 r_k^\alpha - 4\tau^2 r_{k+1}^\alpha = \tau^3 r_k^\alpha (1 - 4\tau^{-1} \tau^{2\alpha}) \geq \frac{\tau^3 r_k^\alpha}{2}$$

by (1) and because we can take $\alpha > 2/3$ and τ small (recall that when $1/2 \leq \alpha \leq 2/3$, we could simply use the proof of [1] or [2]). Now

$$|F(x) - F(y)| \geq |F^\xi(x) - F^\xi(y)| \geq \frac{\tau^3 r_k^\alpha}{2} \geq \frac{\tau^3}{2} \left(\frac{d(x, y)}{4\tau^{-2}} \right)^\alpha \geq \frac{\tau^5}{8} d(x, y)^\alpha$$

by (16). This proves Lemma 4 when $y \in 2B_i$ for some $i \in J_k(\xi)$. If not, all the $\varphi_i(x)$ vanish by (5), and $F_k^\xi(y) = G_{k,j}^\xi(y) = F_{k-1}^\xi(y)$ for all j , by (11) and (12). That is, (20) still holds (with j again chosen so that $x \in B_j$), and we can continue just as in the previous case. Lemma 4 follows. \square

This completes the proof of Theorem 2 for bounded E .

Now suppose E is an unbounded metric space with doubling constant C_0 . Fix an origin x_0 , and apply the construction above to the sets $E_m = E \cap B(x_0, 2^m)$.

The set E_m is itself doubling, with doubling constant C_0^2 . To see this note that if $x \in E_m$ and $r > 0$, we can cover $E_m \cap B(x, 2r)$ with C_0^2 balls of radius $r/2$, which (when they meet E_m) we can replace with balls of radius r whose centers are in E_m .

We get from the proof above a mapping F_m such that

$$C^{-1} d(x, y)^\alpha \leq |F_m(x) - F_m(y)| \leq C d(x, y)^\alpha \quad (21)$$

for $x, y \in E_m$, where C depends on C_0 and α but not on m . We may assume that $F_m(x_0) = 0$, after possibly adding a constant, which would not destroy (21).

Now define for each $k \in \mathbb{Z}$, a maximal collection $\{x_j\} \subset E$, $j \in J_k$, with $d(x_i, x_j) \geq r_k$ for $i \neq j$. Although E is unbounded, each J_k is still at most countable.

Notice also that for each x_j , the sequence $\{F_m(x_j)\}$ is bounded (by (21) and because $F_m(x_0) = 0$); hence we can extract a subsequence $\{m_j\}$, so that the sequence $F_{m_j}(x_j)$ converges for each x_j . By (21) again, the convergence is uniform on each bounded subset of E , so (21) passes to the limit, and this limit F satisfies the conclusion of Theorem 2. This completes our proof.

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