A NOTE ON GENERALIZED ABSOLUTE SUMMABILITY

BY

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Abstract. In this paper, a main theorem on $|A|_k$ summability method has been proved. This theorem also includes two results.

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1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums $(s_n)$, and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

\begin{equation}
A_n(s) = \sum_{v=0}^{n} a_{nv}s_v, \quad n = 0, 1, \ldots
\end{equation}

The series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$, if (see [5])

\begin{equation}
\sum_{n=1}^{\infty} n^{k-1}|\Delta A_n(s)|^k < \infty,
\end{equation}

where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$. When $A$ is a Riesz matrix, the series $\sum a_n$ is said to be summable $|R, p_n|_k, k \geq 1$, if (see [4]) (1.2) holds. By a Riesz matrix we mean one such that $a_{nv} = \frac{p_v}{P_n}$ for $0 \leq v \leq n$, and $a_{nv} = 0$ for $v > n$, where $(p_n)$ is a sequence of positive real numbers and $P_n = \sum_{v=0}^{n} p_v$ as $n \to \infty$, ($P_i = p_i = 0, i \geq 1$). Also if we take $a_{nv} = \frac{1}{n^i}$, i.e. Cesàro matrix, the series $\sum a_n$ is said to be summable $|C, 1|_k, k \geq 1$ (see [1]).
Given any sequences \((x_n), (y_n)\), it is customary to write \(y_n = O(x_n)\), if there exist \(\eta\) and \(N\), for every \(n > N, \frac{|y_n|}{x_n} \leq \eta\). For any matrix entry \(a_{nv}, \Delta_n a_{nv} = a_{nv} - a_{n,v+1}\).

2. **Known results**

Given a normal matrix \(A = (a_{nv})\), we may associate two lower semi-matrices \(\overline{A} = (\overline{a}_{nv})\) and \(\hat{A} = (\hat{a}_{nv})\) as follows:

\[
\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, n, v = 0, 1, \ldots ,
\]

\(a_{00} = \overline{a}_{00} = \hat{a}_{00}; \overline{a}_{nv} = a_{nv} - a_{n-1,v}, n = 1, 2, \ldots \)

It may be noted that \(\overline{A}\) and \(\hat{A}\) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i = \sum_{i=0}^{n} a_i \sum_{v=i}^{n} a_{nv} = \sum_{i=0}^{n} \overline{a}_{ni} a_i
\]

\[
\Delta A_n(s) = \sum_{i=0}^{n} \overline{a}_{ni} a_i - \sum_{i=0}^{n-1} \overline{a}_{n-1,i} a_i = \overline{a}_{nn} a_n + \sum_{i=0}^{n-1} (\overline{a}_{ni} - \overline{a}_{n-1,i}) a_i
\]

\[
\Delta \overline{a}_{nn} a_n + \sum_{i=0}^{n-1} \hat{a}_{ni} a_i = \sum_{i=0}^{n} \hat{a}_{ni} a_i.
\]

The aim of this paper is to prove the following main theorem. For this we need the following lemma.

**Lemma 2.1 ([2]).** *Under the conditions*

\[
|\Delta \lambda_n| \leq \beta_n,
\]

\[
\beta_n \to 0, \ n \to \infty,
\]

\[
\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty,
\]

*where \((X_n)\) is a positive non-decreasing sequence, we have that*

\[
n \beta_n X_n = O(1), \ n \to \infty,
\]

\[
\sum_{n=1}^{\infty} \beta_n X_n < \infty.
\]
3. The main result

Before we state our main result, we show \( A = (a_{nv}) \) is said to be of class \( U \) if the following hold:

- \( A \) is lower triangular
- \( a_{nv} \geq 0, \ n, v = 0, 1, \ldots \);
- \( a_{n-1,v} \geq a_{nv} \) for \( n \geq v + 1 \);
- \( \sigma_{n0} = 1, \ n = 0, 1, \ldots \).

A given by \( A_1(x) = x \) and \( A_n(x) = \frac{x^{n-1} + x^n}{2} \) for \( n > 1 \) is an example of a matrix of class \( U \).

**Theorem 3.1.** Let \( A \in U \) satisfying

\[
\begin{align*}
na_{nn} & = O(1), \\
\hat{a}_{n,v+1} & = O(v|\Delta_v \hat{a}_{nv}|)
\end{align*}
\]

and let there be sequences \((X_n), (\beta_n)\) and \((\lambda_n)\) such that the conditions taken in the statement of Lemma 2.1 be satisfied. If

\[
\sum_{n=1}^{m} a_{nn} |s_n|^k = O(X_m), \ m \to \infty,
\]

\[
|\lambda_n|X_n = O(1), \ n \to \infty,
\]

then the series \( \sum a_n \lambda_n \) is summable \(|A|_k, k \geq 1\).

**Proof.** Let \( (T_n) \) be the \( n \)-th term of the \( A \)-transform of the series \( \sum_{i=0}^{n} a_i \lambda_i \). Then, by means of (2.3), we have \( T_n = \sum_{v=0}^{n-1} \sigma_{nv} a_v \lambda_v \). Applying Abel’s transformation we have that

\[
\Delta T_n = \sum_{v=0}^{n} \Delta_v \hat{a}_{nv} \lambda_v = \sum_{v=0}^{n-1} \Delta_v \hat{a}_{nv} \lambda_v s_v + \sum_{v=0}^{n-1} \hat{a}_{n,v+1} \Delta \lambda_v s_v + a_{nn} \lambda_n s_n
\]

\[
= T_n(1) + T_n(2) + T_n(3), \ \text{say}.
\]

To complete the proof of the Theorem, it is sufficient to show that

\[
\sum_{n=1}^{\infty} n^{k-1} |T_n(r)|^k < \infty, \ \text{for} \ r = 1, 2, 3.
\]
Firstly, since
\[
\Delta_v \hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n,v+1} = \hat{a}_{nv} - \hat{a}_{n-1,v} + \hat{a}_{n-1,v+1} - \hat{a}_{nv} = \alpha_{n,v} - \alpha_{n-1,v} \leq 0.
\] (3.9)

By using (3.2) and (3.3),
\[
\sum_{n=1}^{m+1} n^{k-1} |T_n(1)|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{v=0}^{n-1} |\Delta_v \hat{a}_{nv}| |\lambda_v||s_v|^k \right)^k \leq O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{v=0}^{n-1} |\Delta_v \hat{a}_{nv}| |\lambda_v||s_v|^k \right)^k \leq O(1) \sum_{v=0}^{m} |\lambda_v||s_v|^k \sum_{n=v+1}^{m+1} |\Delta_v \hat{a}_{nv}|.
\]

By (3.1) and (3.9),
\[
\sum_{n=v+1}^{m+1} |\Delta_v \hat{a}_{nv}| = \sum_{n=v+1}^{m+1} (a_{n-1,v} - a_{nv}) = a_{vv} - a_{n+1,v} \leq a_{vv}.
\]

Thus, we have
\[
I_1 = O(1) \sum_{v=0}^{m} |\lambda_v|^{k-1} |\lambda_v||s_v|^k a_{vv} = O(1) \sum_{v=0}^{m} |\lambda_v| a_{vv} |s_v|^k
\]
\[
= O(1) \sum_{v=0}^{m-1} |\lambda_v| \sum_{i=0}^{v} a_{ii} |s_i|^k + O(1) |\lambda_m| \sum_{v=0}^{m} a_{vv} |s_v|^k
\]
\[
= O(1) \sum_{v=0}^{m-1} |\Delta \lambda_v| X_v + O(1) |\lambda_m| X_m
\]
\[
= O(1) \sum_{v=0}^{m-1} |\beta_v| X_v + O(1) |\lambda_m| X_m = O(1), \ m \to \infty
\]
by virtue of the hypotheses of Theorem 3.1 and Lemma 2.1.
Since \( n\beta_n = O(1/X_n) = O(1) \) by (2.8), we have that
\[
I_2 = \sum_{n=1}^{m+1} n^{k-1} |T_n(2)|^k \leq \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{v=0}^{n-1} \hat{a}_{n,v+1} |\Delta \lambda_v||s_v| \right)^k
\]
\[
= O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{v=1}^{n-1} v|\Delta \hat{a}_{nv}||\beta_v||s_v| \right)^k
\]
\[
= O(1) \sum_{n=1}^{m+1} n^{k-1} \left( \sum_{v=1}^{n-1} (v|\beta_v|^k|\Delta \hat{a}_{nv}|)^k \times \left( \sum_{v=1}^{n-1} |\Delta \hat{a}_{nv}| \right)^{-k-1} \right)
\]
\[
= O(1) \sum_{v=1}^{m} (v|\beta_v|^k|s_v|^k) \sum_{n=v+1}^{m+1} |\Delta \hat{a}_{nv}|
\]
\[
= O(1) \sum_{v=1}^{m} (v|\beta_v|^k|s_v|^k a_{vv}) = O(1) \sum_{v=1}^{m} v|\beta_v|^k|s_v|^k a_{vv}.
\]

Applying Abel's transformation, we get that
\[
I_2 = O(1) \sum_{v=1}^{m} \Delta(v|\beta_v|^k) \sum_{i=1}^{v} a_{ii}|s_i|^k + O(1)m|\beta_m| \sum_{v=1}^{m} a_{vv}|s_v|^k
\]
\[
= O(1) \sum_{v=1}^{m} v|\Delta \beta_v||X_v| + O(1) \sum_{v=1}^{m} \beta_{v+1}X_{v+1} + O(1)mX_m
\]
\[
= O(1), \ m \to \infty
\]
by virtue of the hypotheses of Theorem 3.1 and Lemma 2.1.Finally, we have
\[
I_3 = \sum_{n=1}^{m} n^{k-1} |T_n(3)|^k = O(1) \sum_{n=1}^{m} n^{k-1} a_{nn}^k |\lambda_n|^k|s_n|^k
\]
\[
= O(1) \sum_{n=1}^{m} a_{nn} |\lambda_n||s_n|^k = O(1), \ m \to \infty
\]
as in the proof of \( I_1 \).

Thus, we obtain (3.8). This completes the proof of the Theorem. \( \Box \)

4. Corollaries

Setting \( a_{nv} = \frac{p_{nv}}{p_n} \) and \( a_{nv} = \frac{1}{n} \) in the Theorem 3.1, the following corollaries can be stated.
Corollary 4.1. Let \((X_n)\) be a positive non-decreasing sequence and let there be sequences \((\beta_n)\) and \((\lambda_n)\) such that the conditions (2.5)-(2.7) and (3.7) are satisfied. If
\begin{align*}
(4.1) \quad np_n &= O(P_n), \quad n \to \infty, \\
(4.2) \quad P_n &= O(np_n), \quad n \to \infty, \\
(4.3) \quad \sum_{n=1}^{m} \frac{P_n}{P_n} |s_n|^k &= O(X_m), \quad m \to \infty,
\end{align*}
then the series \(\sum a_n\lambda_n\) is summable \(|R, p_n|_k, k \geq 1\).

Corollary 4.2 ([3]). Let \((X_n)\) be a positive non-decreasing sequence and let there be sequences \((\beta_n)\) and \((\lambda_n)\) such that the conditions (2.5)-(2.7) and (3.7) are satisfied. If
\begin{align*}
(4.4) \quad \sum_{n=1}^{m} \frac{1}{n} |s_n|^k &= O(X_m), \quad m \to \infty,
\end{align*}
then the series \(\sum a_n\lambda_n\) is summable \(|C, 1|_k, k \geq 1\).

REFERENCES


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