AN EXTENSION OF PENROSE’S INEQUALITY ON GENERALIZED INVERSES TO THE SCHATTEN $p$-CLASSES

BY

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Abstract. Let $B(H)$ be the algebra of all bounded linear operators on a complex separable infinite dimensional Hilbert space $H$.

In this paper we minimize the Schatten $C_p$-norm of suitable affine mappings from $B(H)$ to $C_p$, using convex and differential analysis (Gâteaux derivative) as well as input from operator theory. The mappings considered generalize Penrose’s inequality which asserts that if $A^+$ and $B^+$ denote the Moore-Penrose inverses of the matrices $A$ and $B$, respectively, then

$$\|AXB-C\|_2 \geq \|AA^+CB^+B-C\|_2,$$

with $A^+CB^+$ being the unique minimizer of minimal $\|\cdot\|_2$ norm. The main results obtained characterize the best $C_p$-approximant of the operator $AXB$.

Mathematics Subject Classification 2010: 47B47, 47B10, 47A05.

Key words: Schatten $p$-classes, Gâteaux derivative, generalized inverses, Moore-Penrose inverses.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on a complex separable and infinite dimensional Hilbert space $H$ and let $T \in B(H)$ be compact, and let $s_1(T) \geq s_2(T) \geq \ldots \geq 0$ denote the singular values of $T$, i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator $T$ is said to belong to the Schatten $p$-classes $C_p$ if

$$\|T\|_p = \left[\sum_{i=1}^{\infty} s_i(T)^p\right]^\frac{1}{p} < \infty, \quad 1 \leq p < \infty.$$
Hence \( C_1 \) is the trace class, \( C_2 \) is the Hilbert-Schmidt class, and \( C_\infty \) corresponds to the class of compact operators with

\[
\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|
\]

denoting the usual operator norm. For the general theory of the Schatten \( p \)-classes the reader is referred to [8]. In this paper we minimize the Schatten \( C_p \)-norm of suitable affine mappings from \( B(H) \) to \( C_p \), using convex and differential analysis (Gâteaux derivative) as well as input from operator theory. This techniques are already used by the author to minimize \( C_1 \)-norm, for the best \( C_1 \) approximant and for the best \( L_1(X,\mu) \) approximant see ([3],[4], [5]). The mappings considered generalize Penrose’s inequality [6, Corollary 1] which asserts that if \( A^+ \) and \( B^+ \) denote the Moore-Penrose inverses of the matrices \( A \) and \( B \), respectively, then

\[
\|AXB - C\|_2 \geq \|AA^+CB^+B - C\|_2,
\]

with \( A^+CB^+ \) being the unique minimizer of minimal \( \|\cdot\|_2 \) norm. The main results obtained characterize the best \( C_p \)-approximant of the operator \( AXB \).

2. Main results

We begin by some definitions and properties of generalized inverses which will be used for the sequel.

**Definition 2.1** ([9], pp. 251). An operator \( A^- \) is said to be a generalized inverse of the operator \( A \in B(H) \) if \( AA^-A = A \). An operator \( A \in B(H) \) has a generalized inverse if its range, \( \text{ran} A \), is closed.

**Proposition 2.1** ([6], Theorem 1). For an operator \( A \in B(H) \) with closed range its Moore-Penrose inverse, denoted \( A^+ \), satisfies

\( (i) \) \( AA^+A = A \),
\( (ii) \) \( A^+AA^+ = A^+ \),
\( (iii) \) \( (AA^+)^* = AA^+ \),
\( (iv) \) \( (A^+A)^* = A^+A \),

and, further, \( A^+ \) is uniquely determined by these properties.
If an operator \( A^- \) satisfies properties (i) and (ii) of Proposition 2.1 (so that \( AA^- A = A \) and \( (AA^-)^* = AA^- \) it will be called a (i), (iii) inverse of \( A \); if \( B^- \) satisfies (i) and (iv) of Proposition 2.1 it will be called a (i), (iv) inverse of \( B \).

Let \( \phi : B(H) \to B(H) \) be a linear map and let \( C \in C_p(1 < p < \infty) \).

Let \( \psi : U \to C_p \) defined by

\[
\psi(X) = \phi(X) - c.
\]

Define the function \( F : U \to \mathbb{R}^+ \) by

\[
F(X) = \|\psi(X)\|_{C_p}.
\]

Now we are ready to prove our first result in \( C_p \)-classes \((1 < p < \infty)\). It gives a necessary and sufficient optimality condition for minimizing \( F \).

Let \( B \) be a Banach space, \( \phi \) a linear map \( B \to B \), and \( \psi(x) = \phi(x) - c \) for some element \( c \in B \). Use the notation

\[
D_x(y) = \lim_{t \to 0^+} \frac{1}{t} (\|x + ty\| - \|x\|).
\]

Elementary that \( D_x \) is sub-additive and \( D_x(y) \leq \|y\| \), also \( D_x(x) = \|x\| \) and \( D_x(-x) = -\|x\| \). For more details the reader is referred to [1]. The following theorem is a well known result in convex analysis.

**Theorem 2.1.** The map \( F_\psi = \|\psi(x)\| \) has a global minimum at \( x \in B \) if and only if

\[
(2.1) \quad D_{\psi(x)}(\phi(y)) \geq 0, \quad \forall y \in B.
\]

It is well known that this holds for all \( a \in B = C_p(H) \), since \([8] C_p(1 < p < \infty) \) is always uniformly convex. This fails when either \( p = 1 \) or \( p = \infty \).

**Theorem 2.2** ([8]). Let \( X, Y \in C_p \). Then, there holds

\[
D_X(Y) = pRe\{tr(|X|^{p-1}U^*)Y\},
\]

where \( X = U|X| \) is the polar decomposition of \( X \).

Now we are ready to characterize the global minimum of \( F_\psi \) on \( C_p(1 < p < \infty) \), when \( \phi \) is a linear map satisfying the following useful condition:

\[
(2.2) \quad tr(X\phi(Y)) = tr(\phi^*(X)Y), \quad \forall X, Y \in C_p,
\]
where $\phi^*$ is an appropriate conjugate of the linear map $\phi$. We state some example of $\phi$ and $\phi^*$ satisfying the above condition (2.2).

The elementary operator $E_{A,B} : I \mapsto I$ defined by

$$E_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i,$$

where $A_i, B_i \in B(H)$, $(1 \leq i \leq n)$ and $I$ is a separable ideal of compact operators in $B(H)$ associated with some unitary invariant norm. It is easy to show that the conjugate operators $E^*_{A,B} : I^* \mapsto I^*$ of $E_{A,B}$ has the form

$$E^*_{A,B}(X) = \sum_{i=1}^{n} B_i X A_i,$$

and that the operators $E_{A,B}$ and $E^*_{A,B}$ satisfy the condition (2.2). Now we are in position to prove the following theorem.

**Theorem 2.3.** Let $V \in C_p$, and let $\psi(V)$ have the polar decomposition $\psi(V) = U|\psi(V)|$. Then $F_\psi$ has a global minimum on $C_p$ at $V$ if and only if $|\psi(V)|^{p-1}U^* \in \ker \phi^*$.

**Proof.** Assume that $F_\psi$ has a global minimum on $C_p$. Then

(2.3) \[ D_{\psi(V)}(\phi(Y)) \geq 0, \]

for all $Y \in C_p$. That is,

$$p\text{Re}\{\text{tr}(|\psi(V)|^{p-1}U^*\phi(Y))\} \geq 0, \forall Y \in C_p.$$  

This implies that

(2.4) \[ \text{Re}\{\text{tr}(|\psi(V)|^{p-1}U^*\phi(Y))\} \geq 0, \forall Y \in C_p. \]

Let $f \otimes g$ be the rank one operator defined by $x \mapsto \langle x, f \rangle g$, where $f, g$ are arbitrary vectors in the Hilbert space $H$. Take $Y = f \otimes g$, since the map $\phi$ satisfies (2.2) one has

$$\text{tr}(|\psi(V)|^{p-1}U^*\phi(Y)) = \text{tr}(\phi^*(|\psi(V)|^{p-1}U^*)Y).$$

Then (2.4) is equivalent to

(2.5) \[ \text{Re}\{\text{tr}(\phi^*(|\psi(V)|^{p-1}U^*)Y)\} \geq 0, \]
for all \( Y \in C_p \), or equivalently
\[
\langle \phi^* (|\psi(V)|^{p-1}) g, f \rangle \geq 0, \forall f, g \in H.
\]
Since \( f, g \) are arbitrary we get
\[
\phi^* (|\psi(V)|^{p-1}U^*) = 0,
\]
or
\[
|\psi(V)|^{p-1}U^* = 0.
\]
Conversely, if \( \phi(|\psi(V)|^{p-1}U^*) = 0 \), it is easy seen (using the same arguments above) that
\[
\text{Re} \{ \text{tr} (|\psi(V)|^{p-1}U^*) \phi(Y) \} \geq 0, \forall Y \in C_p.
\]
By this we get (2.3). \( \square \)

By using Theorem 2.3 with \( \phi(X) = AXB \) we obtain the following corollary.

**Corollary 2.1.** Let \( S = U|S| \in C_p \) be the polar decomposition of \( S \) and let \( X \in B(H) \) be such that \( AXB - C \in C_p \). Then the following assertions are equivalent:

1. \( \|AXB - C\|_{C_p} \geq \|ASB - C\|_{C_p} \), \( \forall X \in C_p \).
2. \( B|ASB - C|^{p-1}U^*A = 0 \).

Note that Corollary 2.1 remains hold for more general classes of operators than the operator \( AXB \) like the elementary operator \( E_{A,B} \).

Now by using Corollary 2.1 and the following lemmas we obtain as a consequence an extension of Penrose’s inequality to the von Neumann-Schatten classes \( C_p \).

**Lemma 2.1** ([2], Assertion). Let \( A, B \in B(H) \) have closed range and let \( X \in B(H) \) be such that \( AXB - C \in C_p \). Then for \( p \geq 2 \), \( B|AXB - C|^{p-1}U^*A = 0 \) if and only if \( B|AXB - C|U^*A = 0 \).

Note that *Maher* [2] proved the previous lemma for \( p \geq 2 \) and showed that it is not valid for \( p < 2 \).

**Lemma 2.2** ([7], Theorem 2). Let \( A, B \in B(H) \) having closed range and \( A \) have a (i), (iii) inverse \( A^- \) and \( B \) have (i), (iv) inverse \( B^- \). Then the operator equation \( AXB = C \) has a solution if and only if \( AA^-CB^-B = C \) in which case the general solution is
\[
X = X_1 + L - A^-ALBB^-,
\]
where \( X_1 \) is a particular solution of \( AXB = C \) and \( L \) is arbitrary in \( B(H) \).
Theorem 2.4. Let \( A, B \in B(H) \) having closed range and \( A \) have a (i), (iii) inverse \( A^- \) and \( B \) have (i), (iv) inverse \( B^- \) and let \( X \in B(H) \) be such that \( AXB - C \in C_p \). Then for \( p \geq 2 \)

\[
\|AXB - C\|_{C_p} \geq \|ASB - C\|_{C_p},
\]

if and only if, \( S \) satisfies \( ASB = AA^-CB^-B \).

**Proof.** It follows from Lemma 2.1 that \( B|ASB - C|^{p-1}U^*A = 0 \) if and only if \( B|ASB - C|U^*A = 0 \). Hence \( B(ASB - C)^*A = 0 \), that is, \( A^*(ASB - C)B^* = 0 \). Therefore

\[
(2.7) \quad A^*ASBB^* = A^*CB^*.
\]

Multiply (2.7) on the left and on the right by \( (B^-)^* \). Since \( A^- \) is a (i), (iii) inverse of \( A \) and \( B^- \) is a (i), (iv) inverse of \( B \), \( (A^-)^*A^* = (AA^-)^* = AA^- \) and \( B^*(B^-)^* = (B^-B)^* = B^-B \). Then \( (A^-)^*A^*A = A \) and \( BB^*(B^-)^* = B \). Hence it follows from (2.6) that \( S \) satisfies

\[
(2.8) \quad ASB = AA^-CB^-B.
\]

Conversely, let \( S \) satisfies \( ASB = AA^-CB^-B \). It results from (2.6) that (2.8) has the following solution

\[
(2.9) \quad S = A^-CB^- + L - A^-ALBB^- \quad \text{for arbitrary } L \in B(H).
\]

Then \( S \) satisfies (2.8). Since \( A^- \) and \( B^- \) are (i), (iii) and (i), (iv) inverses, respectively,

\[
A^*ASBB^* = A^*(AA^-)^*C(B^-B)^*B^* = A^*CB^*.
\]

Therefore \( A^*(ASB - C)B^* = 0 \) and \( B(ASB - C)^*A = B|ASB - C|U^*A = 0 \). It follows from Lemma 2.1 that \( B|ASB - C|^{p-1}U^*A = 0 \). By applying Corollary 2.1 we get \( \|AXB - C\|_{C_p} \geq \|AA^-CB^-B - C\| \). \( \square \)

Let \( M \) be a subspace of \( B \). Recall that if to each \( A \in B \) there exists a \( B \in M \) for which

\[
\|A - B\| \leq \|A - C\|
\]

for all \( C \in M \). Such \( B \) (if they exist) are called best approximants to \( A \) from \( M \). Then the previous theorem can be reformulated as follows:
Theorem 2.5. Let $A, B \in B(H)$ having closed range and $A$ have a (i), (iii) inverse $A^{-}$ and $B$ have (i), (iv) inverse $B^{-}$ and let $X \in B(H)$ be such that $AXB - C \in C_p$. Then for $p \geq 2$, the operator $ASB$ is the best unique approximant of the operator $AXB$ if and only if $S$ satisfies $ASB = AA^{-}CB^{-}B$.

In the following theorem we show that Maher’s result [2, Theorem 4.1] is a consequence of Theorem 2.3 and its Corollary 2.1

Theorem 2.6. Let $A, B \in B(H)$ having closed range and $A$ have a (i), (iii) inverse $A^{-}$ and $B$ have (i), (iv) inverse $B^{-}$ and let $X \in B(H)$ be such that $AXB - C \in C_p$. Then for $p \geq 2$

$$\|AXB - C\|_{C_p} \geq \|AA^{-}CB^{-}B - C\|.$$

Proof. Since the map $\|AXB - C\|_{C_p}$ has a global minimizer by Theorem 2.4 at $X = S = A^{-}CB^{-} + L - A^{-}ALBB^{-}$, thus $ASB = AA^{-}CB^{-}B$. Hence $\|AXB - C\|_{C_p} \geq \|AA^{-}CB^{-}B - C\|$.

Again Theorem 2.5 can be reformulated as follows:

Theorem 2.7. Let $A, B \in B(H)$ having closed range and $A$ have a (i), (iii) inverse $A^{-}$ and $B$ have (i), (iv) inverse $B^{-}$ and let $X \in B(H)$ be such that $AXB - C \in C_p$. Then, for $p \geq 2$, $AA^{-}CB^{-}B$ is the best $C_p$ approximant of the operator $AXB$.

Acknowledgements. The author would like to thank the referee for his careful reading of the paper. His valuable suggestions, and pertinent comments resulted in numerous improvements throughout.

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Revised: 12.X.2010
Accepted: 19.X.2010