FUNCTIONS WITH UNUSUAL DIFFERENTIABILITY PROPERTIES

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Abstract. The monotone increasing Cantor function is used as a "machine" for building examples of strange functions with interesting properties both from the Fractal Measure Theory and from Singular Functions viewpoints.

We give examples showing functions \( f \) where: (i) \( f \) is increasing with zero derivative a.e. (i.e. singular) and it has associated a Lebesgue-Stieltjes measure \( df \) of prescribed Hausdorff dimension \( \alpha \in [0, 1] \), (ii) \( f \) is of monotonic type on no interval (MTNI) and singular; (iii) \( f \) is a continuous nowhere differentiable function, and (iv) \( f \) is absolutely continuous and monotone on no interval (MNI).

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1. Introduction and preliminaries

Classic examples of strange functions have been mentioned in the literature since the end of the 19th century. They have inspired many other such functions and stimulated the development of new applications. We recall here some known examples of continuous nowhere monotone real-valued functions (Hill, Garg), of continuous nowhere differentiable functions (Weierstrass, Takagi, Knopp), and of singular functions (Cantor, Minkowski, Salem, de Rham). Related references with respect to these functions can be found, for example, in [13, 16].

In this paper we are dealing with the construction of peculiar or strange functions, with the aid of the Cantor function, which reveals itself to be a
"machine" building strange functions. Among the corresponding individual constructions, we wish to point out certain other topics closely connected with characterizations by means of functional equations or fractal dimensions of some sets related with these functions.

In [3] we study a parameterized family of continuous functions, which contains as its members Takagi’s function. They are functions without monotonicity on any subinterval whose derivates vanish on a set of measure one. It remains still open if these functions are or not of bounded variation. Computer calculi induce to think that they are no functions of bounded variation. By this reason, we look for examples of bounded variation functions that are monotonic on no-interval that vanish a.e.

First, we introduce notions and definitions. As usual, \( \mathcal{C}(I) \) denotes the class of all the continuous and real functions defined in the unit interval \( I := [0,1] \). Following Kairies [13], the term “peculiar” function refers to certain elements in \( \mathcal{C}(I) \) with unusual differentiability properties. The term “strange” function is taken from Kharazishvili [16].

In a series of papers (see, for example, [7]), several notions that measure different degrees of pathology in the class of continuous nowhere monotone functions have been considered. We say that \( f \) is of monotonic type on an interval \( J \) if \( f_m(x) = f(x) + mx \) is monotone on \( J \) for some \( m \in \mathbb{R} \). Let us denote by MNI those functions in \( \mathcal{C}(I) \) that are monotone on no interval (or nowhere monotone), and by MTNI those functions in \( \mathcal{C}(I) \) that are of monotonic type on no interval.

In [7], the relationships among these notions are given. In particular, MTNI implies MNI. We add the following observation: it follows from the Lebesgue Theorem on almost everywhere (a.e.) differentiability of monotone functions that a nowhere differentiable real-valued function defined on an interval \( \langle a, b \rangle \) is simultaneously nowhere monotone on \( \langle a, b \rangle \).

A non-constant \( f : I \to I \) is called singular if it is continuous and increasing with \( f'(x) = 0 \) a.e. In [13], singular functions have been discussed from a general viewpoint, and the existence of strictly singular functions can be seen in [16, Chap.2]. Singular functions come up in a wide variety of context and applications, see for example, [3] and the references therein.

We recall that there is an important connection between measures on \( \mathbb{R} \) and monotone functions. Let \( F : \mathbb{R} \to \mathbb{R} \) be a left continuous and increasing function. Let us define \( dF \) on the semi-algebra of intervals \([a,b],[a < b]. \) Set \( dF[a,b] := F(b) - F(a) \). The unique extension of \( dF \) to a complete \( \sigma \)-measure defined on a \( \sigma \)-algebra which contains the Borel sets of \( \mathbb{R} \) and is
finite on compact subsets of \( \mathbb{R} \) is called the Lebesgue-Stieltjes measure on \( \mathbb{R} \) associated with \( F \). \( dF[a, b] \) measures the mass of each interval \([a, b] \). In general, “mass” is a word that is used informally as measure. The idea is that the measure of a subset \( A \) is the amount of mass in \( A \). We use standard terms such as self-similarity, Hausdorff measure \( \mathcal{H}^s \), Hausdorff dimension \( \text{dim}_H \), box-counting dimension \( \text{dim}_B \), and so on, for which definitions and properties may be found in [11].

The key for the construction of strange functions in this paper is the Cantor function \( C \). Let us recall that the Cantor ternary set \( c \) consists of those numbers in \( I \) whose 3-base representation only has 0s and 2s. Geometrically, it can be built by iteration of the process of interval-tricotomization and removing the open central intervals at each step; therefore, there are \( 2^n \) intervals of length \( 1/3^n \), each in the \( n \)-th iteration.

The Cantor function \( C \) is defined as follows: for each element in \( c \), if \( x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}, x_n \in \{0, 2\} \), then \( C(x) := \sum_{n=1}^{\infty} \frac{x_n/2}{2^n} \) for each element in \( I \setminus c \), which can be written as \( x = \sum_{n=1}^{k} \frac{x_n}{3^n} + \frac{1}{3^{k+1}} + \frac{y}{3^{k+1}} \), with \( x_n \in \{0, 2\} \) and \( y \in I \), then \( C(x) := \sum_{n=1}^{k-1} \frac{x_n}{2^n} + \frac{1}{2^{k+1}} \). This function was simultaneously given by Cantor and Schewer in 1884 (see [8, 25]). A geometric construction of \( C \) can be found in [26].

In addition, there exist other methods, based upon the classic Fubini Theorem concerning the convergence of term-by-term differentiable functions series, which allow us to build examples of strictly increasing singular functions (see [16, p.66]).

This paper is structured as in this way: following similar ideas and starting from initial functions, in Section 2 we give a constructive method to present strictly increasing singular functions with the additional property that the corresponding associated Lebesgue-Stieltjes measures have a Hausdorff dimension \( \beta \), for prescribed \( \beta \in I \).

In Section 3 we furthermore build an MTNI function of bounded variation with derivatives that vanish on a set of measure one.

In Section 4, with the aid of a 4-base representation a continuous function without derivatives at any point is generated. We show that it satisfies the Hölder condition with index \( 1/2 \). Moreover, we prove the fact that this function has not approximate derivatives at any point.

The last section illustrates an example of an MNI function that is absolutely continuous.
2. Strictly increasing singular functions

Strictly increasing singular functions turn up in the mathematical literature in a wide number of topics. We can find them in the study of relations between two representation systems for real numbers, as for example Minkowski’s function, see [1, 6, 17, 19, 28]. They also appear as asymptotic limits, as this is the case for Schöenberg’s function studied by Erdös (see [9]). Other examples turn up in Trigonometric Series Theory, particularly for Riesz products (see [31]). Perhaps, jointly Cantor function, Riesz-Nagy’s functions (see [20, 22, 27]) are the most known among them. Among other fields, applications of these functions appear studying plastics [5] or in fuzzy logic [2]. These are particular cases of functions in the family introduced by [29] from a probabilistic viewpoint.

Here we propose two constructions of strictly singular functions.

For the function $C_\alpha$, with $\alpha \in ]0, 1[$, we remove the ratio length at each step, we proceed in a similarly way to the geometric construction of $C := C_{1/3}$, but with $\alpha$ instead of $1/3$ to determine the intervals where $C_\alpha$ will be constant (see [23, p.168, Example 8.20]). The generalized Cantor sets $c_\alpha$ are generated in the same way as $c$ was. For the generalized function $C_\alpha$ we have that the restriction of $dC_\alpha$ to $c_\alpha$ coincides with the Hausdorff measure $\mathcal{H}^s$ with $s = \frac{-\ln 2}{\ln 2 - \ln (1 - \alpha)}$, and with the zero measure on the complement of $c_\alpha$.

This function has the following properties:

i. It is monotone increasing with infinite intervals where it is constant and the sum of the lengths is 1.

ii. The associated Lebesgue-Stieltjes measure $dC_\alpha$ has Hausdorff dimension $\frac{\ln 2}{\ln 2 - \ln (1 - \alpha)}$.

iii. Clearly, if $\alpha \in ]0, y - x[$, then $C_\alpha(x) < C_\alpha(y)$.

We present a classic Fubini result on term-by-term derivatives for functions series expansion (see [16] or [22]).

**Lemma 1** (Fubini). Let $\{F_n : n \in \mathbb{Z}^+\}$ be a sequence of non-negative increasing functions given on a segment $[a, b]$. Let us suppose that, for each $x \in [a, b]$, we have $F(x) = \sum_{n=1}^{\infty} F_n(x) < +\infty$. Then, $F'(x) = \sum_{n=1}^{\infty} F'_n(x)$ is satisfied for almost every $x \in [a, b]$.

The first construction provides an application of this lemma.
Theorem 2. The function \( F := \sum_{n=1}^{\infty} \frac{1}{2^n} C_{n+1} \) is a strictly increasing singular function whose associated Lebesgue-Stieltjes measure \( dF \) concentrates its mass on a set of measure 0, but the Hausdorff dimension of \( dF \) is 1.

For the second construction we first define a set of zero Hausdorff dimension. We proceed as follows: in the first step we remove the centred open interval of length 1/3 (as in the Cantor ternary set construction). However, in the second step, we remove the two centred intervals (each of length \((3^3 - 2)/3^3\): there are four intervals, each of length 1/3^3). The third step consists in preserving eight closed subintervals, each one having length 1/3^n. Therefore, in each step we preserve extreme closed subintervals of lengths equal to the 1/3^n part of the last interval we considered. At the n-th step, we have 2^n closed intervals, each of length 1/3^{(n^2+n)/2}.

In the limit, this process yields a set we denote by \( c_* \).

We recall that, in the case of the Cantor ternary set, upper bounds for Hausdorff dimensions are obtained by finding effective covers by small sets (see [11, Chap.4]). We follow this idea to calculate the Hausdorff of \( c_* \).

Lemma 3. The Hausdorff dimension of \( c_* \) is zero.

Proof. The above observation gives the relation:

\[
\dim_H c_* \leq \lim_{k \to \infty} \frac{\ln n_k}{-\ln \delta_k},
\]

where the \( n_k \) are the cardinals of the \( \delta_k \)-coverings for \( c_* \). Our case corresponds to covers with \( n_k = 2^k \) sets whose diameters are equal to \( \delta_k = 3^{-\frac{n^2+n}{2}} \). Consequently, we conclude that \( \dim_H c_* \leq \lim_{n \to \infty} \frac{\ln 2}{\frac{n \ln 2}{\ln 3}} = 0. \)

We use the notations below.

Notation 4. Let \( f : I \to \mathbb{R} \) be a real and bounded function. For \( a, b \in I, a < b \), we write

\[
ab f(x) := \begin{cases} 
0, & \text{if } x < a \\
 f \left( \frac{x-a}{b-a} \right), & \text{if } a \leq x \leq b \\
1, & \text{if } x > b
\end{cases}
\]

and \( \mathcal{F}(x) := \sum_{a, b \in \mathbb{Q} \cap I} \frac{ab f(x)}{2^{n_{a,b}}} \), where \( n_{a,b} \) gives an enumeration for \((\mathbb{Q} \cap I) \times (\mathbb{Q} \cap I)\).
Above considerations give the following result.

**Theorem 5.** Let $C_\ast : \mathbb{I} \to \mathbb{I}$ be the function built in the same way as $C_\alpha$. Then, i) $C_\ast$ is a singular function that concentrates its mass on the set $c_\ast$, and ii) the function $\overline{C}_\ast$ is a strictly increasing singular function whose associated Lebesgue-Stieltjes measure $d\overline{C}_\ast$ has zero Hausdorff dimension.

**Remark 6.** Because the measure $dC_\alpha$ has Hausdorff dimension 

$$\frac{\ln 2}{\ln 2 - \ln (1 - \alpha)},$$

$dC_\alpha$ has the same.

Combining this fact with the previous results, for each $\beta \in \mathbb{I}$, we have constructed strictly increasing singular functions whose associated Lebesgue-Stieltjes measure has Hausdorff dimension $\beta$. This process starts from singular functions that are constant on subintervals that sum 1 for their total length.

3. A function of monotonic type on no interval (MTNI) and of bounded variation with null derivative a.e.

Let us recall that the Cantor function $C$ was defined as a constant on each segment removed at each step for the building of $c$. We use two copies of $C$ to introduce an auxiliary function.

Let $D_0 : \mathbb{I} \to \mathbb{I}$ be given by the formula

$$D_0(x) = \begin{cases} C(2x), & \text{if } 0 \leq x \leq 1/2 \\ C(2 - 2x), & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Note the set $M := \{x : x = y/2 \text{ or } x = (1 + y)/2 \text{ with } y \in c\}$ of points $x$ where there is no interval containing $x$ with $D_0$ being a constant.

We introduce the notation we will use.

**Notation 7.** Let us consider a function $f : \mathbb{I} \to \mathbb{R}$, and numbers $a, b \in \mathbb{I}$. We set

$$f^{ab}(x) := \begin{cases} f \left( \frac{x-a}{b-a} \right), & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}.$$ 

There exists an infinite sequence of intervals $\{[a_{0,n}, b_{0,n}] : n \in \mathbb{Z}^+\}$, where $D_0$ is a constant function. Moreover, the total length of these intervals is 1.
Let us now consider this new function:

(1) \[ D_1(x) := D_0(x) + \sum_{n=1}^{\infty} \frac{(b_{0,n} - a_{0,n})}{2} D_0^{a_{0,n} b_{0,n}}(x). \]

This function has analogous properties to those of \( D_0 \). Let us now denote as \( \{ [a_{1,n}, b_{1,n}] : n \in \mathbb{Z}^+ \} \) the family of intervals where \( D_1 \) reduces a constant on each one of them. Proceeding as in (1), we write

(2) \[ D_2(x) := D_1(x) + \sum_{n=1}^{\infty} \frac{(b_{1,n} - a_{1,n})}{2^2} D_0^{a_{1,n} b_{1,n}}(x); \]

and we can do it recurrently: for each positive integer \( k \), \( D_k \) is constant on the intervals of the family \( \{ [a_{k,n}, b_{k,n}] : n \in \mathbb{Z}^+ \} \), and it is possible to state

\[ D_{k+1}(x) := D_k(x) + \sum_{n=1}^{\infty} \frac{(b_{k,n} - a_{k,n})}{2^{k+1}} D_0^{a_{k,n} b_{k,n}}(x). \]

This sequence of functions converges uniformly.

**Definition 8.** Let \( D : \mathbb{I} \to \mathbb{I} \) be given by

(3) \[ D(x) := \lim_{k \to \infty} D_k(x) = D_0(x) + \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(b_{k,n} - a_{k,n})}{2^{k+1}} D_0^{a_{k,n} b_{k,n}}(x). \]

\( D \) is continuous because of the uniform convergence of the sequence of continuous functions \( D_k \). To study differentiability, we use a result due to Fubini (see for instance [14, p.148]). If \( f \) is a bounded variation function, then \( T f \) will denote its total variation. The symbol \( O \) is the big \( O \) of Landau, and \( f \preceq g \) means that \( f = O(g) \) and \( g = O(f) \).

**Lemma 9** (Fubini’s theorem for bounded variation functions). Let \( \sum_{n=1}^{\infty} f_n \) be a series expansion of bounded variation functions. Furthermore, let the series of total variation functions \( \sum_{n=1}^{\infty} T f_n \) be convergent. Then \( \sum_{n=1}^{\infty} f_n \) converges to a function \( f \) of bounded variation, and \( f' = \sum_{n=1}^{\infty} f'_n \) a.e.

**Theorem 10.** \( D \) is a function of bounded variation and \( D'(x) = 0 \) a.e.
Proof. With the aid of the above result, it is enough to prove the existence of the number
\[ \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(b_{k,n} - a_{k,n})}{2^{k+1}} TD_{0}^{a_{k,n},b_{k,n}} \]
to obtain the one we are searching for.

If \( f \) is a bounded variation function with \( f(0) = f(1) = 0 \), then it satisfies \( Tf = Tf^{ab} \), for all \( a \) and \( b \). Direct computations
\[
\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(b_{k,n} - a_{k,n})}{2^{k+1}} TD_{0}^{a_{k,n},b_{k,n}} = O \left( \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(b_{k,n} - a_{k,n})}{2^{k+1}} \right) = O \left( \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \right) = O(1),
\]
indicate that \( D \) is of bounded variation.

Because \( D^{ab}_{0} \) has null derivatives on a set of measure 1, we can establish the existence of a set of measure 1 where all the \( D_{0}^{a_{k,n},b_{n,k}} \) have null derivatives on it. But this is precisely the required condition for proving the existence of a set of measure 1 where \( D \) has null derivatives. \( \square \)

The following results describe the set of local maxima for the function \( D \). For each \([a_{k,n}, b_{k,n}]\) we set \( c_{k,n} = a_{k,n} + b_{k,n} \).

**Theorem 11.** \( D \) reaches its maxima on a set of rational numbers that is dense in \( \mathbb{I} \).

**Proof.** The way \( D_{1} \) is defined guarantees that \( D_{1}(x) < D_{0}(1/2) = D(1/2) \) for all \( x \neq 1/2 \). Similarly, on intervals \([a_{0,n}, b_{0,n}]\), one obtains \( D_{2}(x) < D_{1}(c_{0,n}) = D(c_{0,n}) \) for \( x \in [a_{0,n}, b_{0,n}] \setminus \{c_{0,n}\} \). In general, \( D_{k+1}(x) < D_{k}(c_{k-1,n}) = D(c_{k-1,n}) \), for \( x \in [a_{k-1,n}, b_{k-1,n}] \setminus \{c_{0,n}\} \).

The above shows that \( D \) has an absolute maximum at 1/2. But \([a_{k,n}, b_{k,n}]\) contains a replica of \( D \) (whose scale depends on each interval). Therefore, we deduce that \( D \) has a maximum at \( c_{k,n} \). Moreover, all these points are rational.

The rest can be divided in two cases:

a) There is an interval \([a_{k,n}, b_{k,n}]\) such that for \( y \in M \), \( y = \frac{x - a_{k,n}}{b_{k,n} - a_{k,n}} \). Hence, \( D \) does not reach a maximum at \( x \).

b) There exist infinite intervals \([a_{k,n}, b_{k,n}]\) containing \( x \). If \( x \in ]\alpha_{1}, \alpha_{2}[ \subset \mathbb{I} \), then we find \( x \in [a_{k,n}, b_{k,n}] \subset ]\alpha_{1}, \alpha_{2}[ \). In this subinterval, \( D(x) < \).
Theorem 12. $D$ reaches its minima on a set of rational numbers that is dense in $I$.

Proof. The reasoning follows similarly to Theorem 11. Minima are reached at one of the two extreme points of each interval in the form $[a_{k,n}, b_{k,n}]$. Such is the case for $a_{k,n}$ when $[a_{k,n}, b_{k,n}]$ is included in an open interval where $D_k$ is monotone decreasing.

On the other hand, it would also be the case for $b_{k,n}$ if $D_k$ is monotone increasing on that interval. □

Proposition 13. $D$ is a MTNI function.

Proof. The Cantor ternary function $C$ satisfies that $\lim_{n \to \infty} \frac{1-C\left(\frac{1}{2} - \frac{1}{3^n}\right)}{2/3^n} = +\infty$; therefore,

$$\lim_{n \to \infty} \frac{D_0\left(\frac{1}{2}\right) - D_0\left(\frac{1}{2} - \frac{1}{3^n}\right)}{1/3^n} = \lim_{n \to \infty} \frac{D_0\left(\frac{1}{2}\right) - D_0\left(\frac{1}{2} + \frac{1}{3^n}\right)}{1/3^n} = +\infty.$$ 

By the way to construct $D$, at points in the form $1/2, 1/2 - 1/3^n$, and $1/2 + 1/3^n$, the functions $D$ and $D_0$ coincide. Therefore, the above limits remain true for $D$. In particular, for every open interval containing $1/2$, it is impossible to find $m \in \mathbb{R}$ satisfying

$$D_0\left(\frac{1}{2} - \frac{1}{3^n}\right) + m\left(\frac{1}{2} - \frac{1}{3^n}\right) \leq D_0\left(\frac{1}{2}\right) + \frac{m}{2} \leq D_0\left(\frac{1}{2} + \frac{1}{3^n}\right) + m\left(\frac{1}{2} + \frac{1}{3^n}\right),$$

for all $n$ such that $1/2 - 1/3^n$ and $1/2 + 1/3^n$ are in that interval. In consequence, $D$ is MTNI in these intervals.

The way $D$ is defined ensures that, for a general open interval $I \subset I$, there exist closed subintervals $J$ with non-empty interior, where $D$ and $D_{I,J}$ are obtained in the same way (but multiplicative factors). Therefore, the above reasoning can be applied on $J$ to conclude that $D$ is MTNI. □

Remark 14. We have directly proved that there is no interval where $D$ is a monotone function. Note that, for each real interval in $I$, we can find a subinterval of the form $[a_{k,n}, b_{k,n}]$. But, $D\left(a_{k,n}\right) = D\left(b_{k,n}\right) < D\left(c_{k,n}\right)$. Consequently, there is no possibility of monotonicity there.
4. A continuous nowhere differentiable function

**Definition 15.** Let us consider the interval \([a_{k,n}, b_{k,n}]\) of length \(3^{-r_{k,n}}\). We consider the function \(h\) given by:

\[
h(x) := D_0(x) + \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2^{r_{n,k}}} D_{a_{k,n}b_{k,n}}^0(x).
\]

Then, we set the new function \(f(x) := h(x/2)\).

The way function \(f\) has been constructed shows that it satisfies this system of functional equations:

\[
\begin{align*}
f \left( \frac{x}{4} \right) &= \frac{f(x)}{2} \\
f \left( \frac{x+2}{6} \right) &= \frac{1}{2} + \frac{f(x)}{2} \\
f \left( \frac{x+3}{6} \right) &= \frac{1}{2} + \frac{f(1-x)}{2} \\
f \left( \frac{x+2}{3} \right) &= \frac{1}{2} + \frac{f(x)}{2}.
\end{align*}
\]

The contraction mapping principle of Banach ensures its uniqueness. With the goal of obtaining a function with a relatively simple expression, we study the continuous function satisfying the following functional equations:

\[
\begin{align*}
f \left( \frac{x}{4} \right) &= \frac{f(x)}{2} \\
f \left( \frac{x+1}{4} \right) &= \frac{1}{2} + \frac{f(x)}{2} \\
f \left( \frac{x+2}{4} \right) &= \frac{1}{2} + \frac{f(1-x)}{2} \\
f \left( \frac{x+3}{4} \right) &= \frac{1}{2} + \frac{f(x)}{2}.
\end{align*}
\]

This function is similar to the previous one but, when creating its corresponding Cantor set, the removed intervals have a length of \(1/2^n\) instead of \(1/3^n\). This allows us to obtain a 4-base representation for this function.

Here we need a brief introduction to sequential machines and sequential functions according to Eilenberg (see [10, Chap.XI]). A set \(S = \{s_1, \ldots, s_n\}\) containing finite symbols is called an *alphabet*, and its elements are called *letters*. For an alphabet \(S\) we denote with \(S^* = \{a_1 \ldots a_k : k \in \mathbb{Z}^+, a_j \in S, 1 \leq j \leq k\}\) the set of all the words with letters in \(S\).

A *sequential function* is a function between the words from different alphabets \(A\) and \(B\) (see [21]). Moreover, we show how sequential functions can be applied to generate certain functions used in our construction.
A sequential machine (or transducers) $T = (Q, A, B, \sigma, \tau)$ consists of a finite set of states $Q$, an input alphabet $A$, an output alphabet $B$, a transition function $\sigma : Q \times A \rightarrow Q$, and an output function $\tau : Q \times A \rightarrow B$.

The transition and output functions can be uniquely extended to $\sigma : Q \times A^* \rightarrow Q$ and $\tau : Q \times A^* \rightarrow B^*$, respectively, satisfying that $\sigma(q, \theta) = q$, for all $q \in Q$, and $\sigma(q; vw) = \sigma(\sigma(q, v), w)$, $\tau(q; vw) = \tau(q, v) \tau(\sigma(q, v), w)$, for all words $v, w \in A^*$ and all $q \in Q$.

For each $q \in Q$, the sequential machine $T = (Q, A, B, \sigma, \tau)$ defines a function $f^*_q : A^* \rightarrow B^*$ that is called the sequential function corresponding to $T$ and $q$. It is defined by $f^*_q(a_1, \ldots, a_n) = \tau(q, a_1, \ldots, a_n)$. If $A = \{0, \ldots, n - 1\}$ and $B = \{0, \ldots, m - 1\}$, then each sequential function $f^*_q$ generally leads to multi-valued functions $f_q : I \rightarrow I$ as follows. For $x \in I$, let $x = \sum_{k=0}^{\infty} x_k n^{-k}$, $x_k \in \{0, \ldots, n - 1\}$ for all $k \in \mathbb{Z}^+ \cup \{0\}$, and $f^*_q(x_1 \ldots x_r) = b_1 \ldots b_r$ for all $r \in \mathbb{Z}^+$. Then $f_q(x) := \sum_{j=1}^{\infty} \frac{b_j}{m^j}$.

**Theorem 16.** Let $T = (Q, A, B, \sigma, \tau)$ be a sequential machine with $Q = \{\alpha, \beta\}$, $A = \{0, 1, 2, 3\}$, and $B = \{0, 1\}$; with

\[
\tau : Q \times A \rightarrow Q
\]
such that

- $\alpha, 0) \rightarrow 0$; $\beta, 0) \rightarrow 1$
- $\alpha, 1) \rightarrow 1$; $\beta, 1) \rightarrow 1$
- $\alpha, 2) \rightarrow 1$; $\beta, 2) \rightarrow 1$
- $\alpha, 3) \rightarrow 1$; $\beta, 3) \rightarrow 0$

and

\[
\sigma : Q \times A \rightarrow B
\]
such that

- $\alpha, 0) \rightarrow \alpha$; $\beta, 0) \rightarrow \beta$
- $\alpha, 1) \rightarrow \alpha$; $\beta, 1) \rightarrow \alpha$
- $\alpha, 2) \rightarrow \beta$; $\beta, 2) \rightarrow \beta$
- $\alpha, 3) \rightarrow \alpha$; $\beta, 3) \rightarrow \beta$.

Then, $f$ coincides with $f_\alpha$ and $f_\beta(x) = f(1 - x)$.

**Proof.** Let us observe that $f_\alpha$ is well defined on points with two different dyadic representations. It is enough to prove the statement for $1/4, 1/2$ and $3/4$. The equality for the functions will be established through functional equations.
If \( x = \sum_{k=1}^{\infty} \frac{x_k}{4^k} \), then \( \frac{x}{4} = \sum_{k=1}^{\infty} \frac{x_k}{4^k} \) (primes are used when it is related with \( \frac{1}{4} \)), satisfying that \( x'_k = x_{k-1} \) for \( k \geq 2 \), and \( x'_1 = 0 \). As a result, \( b'_1 = 0 \), and for the other values we obtain \( b'_k = b_{k-1} \), which produces the identity \( f_\alpha \left( \frac{x}{4} \right) = \frac{f_\alpha(x)}{2} \).

In the same way, it follows that \( f_\alpha \left( \frac{x+1}{4} \right) = \frac{1}{2} + \frac{f_\alpha(x)}{2} \) and \( f_\alpha \left( \frac{x+3}{4} \right) = \frac{1}{2} + \frac{f_\alpha(x)}{2} \).

Since \( x = \sum_{k=1}^{\infty} \frac{x_k}{4^k} \), then \( 1-x = \sum_{k=1}^{\infty} \frac{4-x_k}{4^k} \), and if we proceed as above, then we obtain \( f_\alpha \left( \frac{x+2}{4} \right) = \frac{1}{2} + \frac{f_\alpha(1-x)}{2} \). □

**Theorem 17.** \( f \) is a continuous and nowhere differentiable function.

**Proof.** Fix \( x \) in \( I \). For each open interval containing \( x \), it is possible to obtain \( x' \) satisfying that \( |x-x'| \gg \frac{1}{4^k} \) and \( |f(x) - f(x')| \gg \frac{1}{4^k} \). Therefore, if the derivatives exist, then they must be equal to infinity. We conclude that the assertion is true. □

Observe that \( f \) is not a Besicovitch function (i.e. a continuous function that has no one-sided derivatives finite or infinite anywhere). In fact, there are points (e.g. the origin) where we can obtain one-sided derivatives with values equal to infinity. But these points are exceptional points, as we see below.

**Proposition 18.** There exists a set of measure 1 where \( f \) does not admit one-sided derivatives, finite or infinite, at any of its points.

**Proof.** If \( x \in I \) has infinite 1s among its digits in its 4-base expansion then, for each \( \varepsilon > 0 \), it is possible to find \( x_{1,\varepsilon} > x \) satisfying that \( |x-x_{1,\varepsilon}| < \varepsilon \) and \( f(x) = f(x') \). Therefore, if the right-sided derivative exists at \( x \), it must be zero.

If \( x \in I \) has infinite 2s among its digits in its 4-base expansion then, for each \( \varepsilon > 0 \), there is an \( x_{2,\varepsilon} > x \) satisfying that \( |x-x_{2,\varepsilon}| < \varepsilon \) and \( f(x) = f(x') \). Therefore, if the left-sided derivative exists, it must be zero.

If \( x \in I \) has infinite 0s among its digits in its 4-base expansion, then, for an infinite number of \( n \in \mathbb{Z}^+ \) it is possible to find \( x_{1,n} > x \) satisfying that \( |x-x_{1,n}| \gg \frac{1}{4^n} \) and \( |f(x) - f(x_{1,n})| \gg \frac{1}{4^n} \). Therefore, if the right-sided derivative exists at \( x \), it cannot be finite.

If \( x \in I \) has infinite 3s among its digits in its 4-base expansion then, for an infinite number of \( n \in \mathbb{Z}^+ \), it is possible to find \( x_{2,n} > x \) satisfying that \( |x-x_{2,n}| \gg \frac{1}{4^n} \) and \( |f(x) - f(x_{2,n})| \gg \frac{1}{4^n} \). Therefore, where the left-sided derivative exists at \( x \), it cannot be finite.
As is known from Number Theory, the set of points in 4-base with infinite 0s, 1s, 2s, and 3s among their digits has a measure of 1 (see [12, 18]). Therefore, there are no one-sided derivatives (finite nor infinite) for \( f \) at any point in this set.

**Proposition 19.** i. For each \( x \in \mathbb{I} \), the function \( f \) satisfies Hölder condition for index \( 1/2 \); that is, \( |f(x) - f(y)| = O(|x - y|^{1/2}) \) for \( y \) near \( x \).

   ii. There exists no point where the statement above is true for \( 1/2 - \varepsilon \).

**Proof.** i. If \( x = \sum_{k=1}^{\infty} \frac{x_k}{4^k} \), and \( y = \sum_{k=1}^{\infty} \frac{y_k}{4^k} \), then we consider two cases.

   i.a) \( x = \sum_{k=1}^{n} \frac{x_k}{4^k} + \frac{1}{4^n + r} + \sum_{k=m}^{\infty} \frac{x_k}{4^k} \), and \( y = \sum_{k=1}^{n} \frac{x_k}{4^k} + \sum_{k=n+2}^{\infty} \frac{3}{4^k} + \sum_{k=m}^{\infty} \frac{y_k}{4^k} \), for \( x_m \neq 0 \). In this case, \( |x - y| \asymp \frac{1}{3^m} \) and \( |f(x) - f(y)| = O \left( \frac{1}{3^m} \right) \). Therefore, the statement is true.

   i.b) \( x = \sum_{k=1}^{n} \frac{x_k}{4^k} + \frac{1}{4^n + r} + \sum_{k=m}^{\infty} \frac{x_k}{4^k} \), and \( y = \sum_{k=1}^{n} \frac{x_k}{4^k} + \sum_{k=n+2}^{\infty} \frac{3}{4^k} + \sum_{k=p}^{\infty} \frac{y_k}{4^k} \), for \( y_p \neq 3 \) \( y_p < m \). In this case, \( |x - y| \asymp \frac{1}{4^p} \) and \( |f(x) - f(y)| = O \left( \frac{1}{4^p} \right) \). Therefore, the statement is true in this case as well.

   i.c) In another case, if \( x_k = y_k \) for the first \( n \) terms, then \( |x - y| \asymp \frac{1}{3^m} \) and \( |f(x) - f(y)| = O \left( \frac{1}{3^m} \right) \), and the result again follows.

   ii. The second part is implicit in the proof of Theorem 17 because for each \( n \), we can always find a value \( y \) for a given \( x \) such that \( |x - y| \asymp \frac{1}{3^m} \) and \( |f(x) - f(y)| \asymp \frac{1}{3^m} \). Then, the index cannot be changed by \( 1/2 - \varepsilon \).

**Corollary 20.** The graph of \( f \) has box-counting dimension 1/2.

Now, we show that the strange function \( f \) is a nowhere approximately differentiable function. Below, \( \lambda \) denotes the Lebesgue measure on the real line. We need some notation.

Let us recall that a real \( x \) is called a density point in a measurable set \( A \) (see [30]) if it satisfies that \( \lim_{n \to \infty} \frac{\lambda(\mathcal{A}^n[x-h,x+h])}{2n} = 1 \). (Almost all points in \( A \) are density points in it.) A function \( f \) is said to be approximately differentiable at \( x \) if there exists a set \( A \) for which \( x \) is a density point and there exists the limit \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \). (If such a set exists, it is easy to check that the limit does not depend on the set \( A \).) This limit is said to be the approximate derivative of \( f \) at \( x \).

**Theorem 21.** The function \( f \) has no approximate derivatives at any point.
Proof. For each interval \([x - \frac{1}{4^{n-1}}, x + \frac{1}{4^{n-1}}]\), we can find subintervals of the form
\[x \in \left[\frac{t}{4^{n+1}}, \frac{t+1}{4^{n+1}} \right] \subset \left[\frac{k}{4^{n}}, \frac{k+1}{4^{n}} \right].\]
If \(t \equiv 0\), then we consider \(B_n := \left[\frac{4(t+1)+1}{4^{n+2}}, \frac{4(t+1)+2}{4^{n+2}}\right]\). In another case, we consider \(B_n := \left[\frac{4t}{4^{n+2}}, \frac{4t+1}{4^{n+2}}\right]\).

In both cases, if \(y \in B_n\), then \(|x - y| \leq \frac{1}{4^{n+1}}\) and \(\frac{1}{2^{n+2}} \leq |f(x) - f(y)|\); therefore \(2^{n-2} \leq \frac{|f(x) - f(y)|}{x - y}|\).

If \(x\) is a density point in \(A\), then there exists an infinity of \(n\)'s such that \(A \cap B_n \neq \emptyset\). If that is not the case, then
\[\lambda \left(A \cap \left[x - \frac{1}{4^{n-1}}, x + \frac{1}{4^{n-1}}\right]\right) \leq 2 \frac{1}{4^{n-1}} - \frac{1}{4^{n+1}},\]
which contradicts that \(\lim_{n \to \infty} \lambda(A \cap \left[x - \frac{1}{4^{n-1}}, x + \frac{1}{4^{n-1}}\right]) = 1\).

Let \((n_k)\) be the sequence of such indices for which \(A \cap B_{n_k} \neq \emptyset\). We can choose \(y_k \in A \cap B_{n_k}\). For each one of them, we have \(\frac{|f(x) - f(y_k)|}{x - y_k} \geq 2^{n_k-2}\).
As a consequence, it can not be approximately differentiable. \(\square\)

Proposition 22. \(\int_0^1 f(x)\,dx = \frac{3}{4}\).

Proposition 23. The Hausdorff dimension of \(\{x \in I : f(x) = 1\}\) is \(\ln 3/\ln 4\).

Proof. The set is self-similar under these three contractions in \(I\):
\[
\begin{align*}
T_1(x) &= \frac{1}{4}x + \frac{1}{4}, \\
T_2(x) &= \frac{1}{4}(1 - x) + \frac{1}{2}, \\
T_3(x) &= \frac{1}{4}x + \frac{3}{4},
\end{align*}
\]
which produces the statement. \(\square\)

Definition 24. The capacity measure for a given function \(g\), which will be denoted by \(v_g\), is given by \(v_g(A) = \lambda \left(\{x : g(x) \in A\}\right)\).

Among the singular functions there exists a widely studied family (see [3, 20, 22, 24, 29]) that we denote by \(S_a\). They are defined on the unit interval \(I\) and characterized by the functional equations system
\[
\begin{align*}
S_a \left(\frac{x}{1+1}\right) &= aS_a (x), \\
S_a \left(\frac{1-x}{2}\right) &= a + (1 - a)S_a (x).
\end{align*}
\]
Proposition 25. For $f$, we have $v_f([0,x]) = S_{1/4}(x)$.

Proof. The function $k(x) := v_f([0,x])$ satisfies the above functional equations. These equalities follow, via the self-affinity of $f$. $\square$

Corollary 26. There are two sets, $A$ and $B$, satisfying that $A$ has Lebesgue measure 1 and $f^{-1}(A)$ is of zero measure. On the other hand, $B$ is of Lebesgue-measure 0 and $f^{-1}(B)$ is of measure 1.

Remark 27. It is possible to explicitly describe the above sets with the aid of the dyadic generalized representation system developed in [4]. In particular, $A$ is the set of points satisfying their series expansion in the form $x = \sum_{k=0}^{+\infty} \left(\frac{3}{4}\right)^k \frac{1}{4^m}$, with $1 \leq m_0 \leq m_1 \leq \cdots \leq m_k \leq \cdots$ and $\lim_{n \to \infty} \frac{m_k}{k} = 1$. Moreover, it is possible to establish that its Hausdorff dimension is $\frac{3}{4} \ln \frac{3}{4} + \frac{1}{4} \ln \frac{1}{2}$.

5. Absolutely continuous function and monotone on no interval (MNI)

Following the above ideas we build an absolutely continuous MNI function. We again use a Cantor-type set. In the unit interval $I$, let us drop the central open interval of length $1/2^2$ (i.e. $[3/8, 5/8]$); then, proceeding by iteration for each new pair of closed intervals, we remove the respective central open subinterval of length $1/2^{2n}$ in each one.

The process yields a set of measure 1/2, which we denote by $h$. The corresponding function $H$ can be obtained as the limit from a sequence of polynomial approximations that is constant on each removed subinterval; however, we introduce it in a different form.

Definition 28. Let $H(x) := 2 \int_0^x \chi_h(t)dt$ where $\chi_h$ denotes the characteristic or indicator function for the set $h$.

This is an absolutely continuous function, and there exists a set of intervals whose union is dense in $I$, with $H$ being constant on each interval of the set and $H(0) = 0$, $H(1) = 1$.

We introduce a function $K_0 : I \to I$ given by

$$K_0(x) = \begin{cases} H(2x), & \text{if } 0 \leq x \leq 1/2 \\ H(2 - 2x), & \text{if } 1/2 \leq x \leq 1. \end{cases}$$
This function is constant on an infinity of intervals we enumerate by $[c_{0,n},d_{0,n}]$. With its aid, we define

$$K_1(x) := K_0(x) + \sum_{n=1}^{\infty} \frac{(d_{0,n} - c_{0,n})}{2^{s+1}} K_0^{c_{0,n}d_{0,n}}(x).$$

The intervals where $K_1$ is constant are denoted by $[c_{1,n},d_{1,n}]$, and

$$K_2(x) := K_1(x) + \sum_{n=1}^{\infty} \frac{(d_{1,n} - c_{1,n})}{2^{s+1}} K_0^{c_{1,n}d_{1,n}}(x).$$

Recurrently, we can state:

$$K_{s+1}(x) := K_s(x) + \sum_{n=1}^{\infty} \frac{(d_{s,n} - c_{s,n})}{2^{s+1}} K_0^{c_{s,n}d_{s,n}}(x).$$

**Definition 29.** Let us set

$$K(x) := \lim_{s \to \infty} K_s(x) = K_0(x) + \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} \frac{(d_{s,n} - c_{s,n})}{2^{s+1}} K_0^{c_{s,n}d_{s,n}}(x).$$

**Remark 30.** The lengths of the intervals make the factor $1/2^{s+1}$ unnecessary. It is possible to define

$$K^*(x) := K_0(x) + \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} (d_{s,n} - c_{s,n}) K_0^{c_{s,n}d_{s,n}}(x),$$

but this new function would have the same properties.

Reasoning as we did for the function $D$ in Section 3, we can state the result below.

**Theorem 31.** $K$ is an absolutely continuous and MNI function. Points where $K$ reaches its relative maxima and minima values is a dense set of rational numbers in $\mathbb{I}$.

**Proof.** We only prove the absolute continuity for $K$ (the other properties follow as for $D$). Because of the integral definition for $H$,

$$K(x) = K_0(x) + \lim_{r \to \infty} \sum_{s=0}^{r} \sum_{n=1}^{r} \frac{1}{2^{s+1}} \int_0^x \phi_{c_{s,n}d_{s,n}}(t)dt.$$
The $\phi_{c, a, d, s, n}$ are bounded functions. Therefore, the Lebesgue-dominated convergence theorem provides commutativity for the limit and integral. This limit is absolutely continuous. □

Although there already exist examples of MNI functions having derivatives at every point, see for example [15] or [16], our construction above seems to be one of the simplest.

5.1. Partitions of $\mathbb{I}$ and MNI functions

The existence of an absolutely continuous function that is monotone on no interval is guaranteed if we build a decomposition of the unit interval $\mathbb{I}$ in two measurable subsets $A$ and $B$ satisfying that $\lambda(J \cap A) > 0$ and $\lambda(J \cap B) > 0$, for each open interval $J$. The function to be considered is:

$$g(x) := \int_0^x [\chi_A(t) - \chi_B(t)] dt.$$ 

These sets $A$ and $B$ cannot be “homogeneously” distributed. For example, it is impossible to find a number $\alpha$ satisfying $\lambda(J \cap A) = \alpha \lambda(J)$ and $\lambda(J \cap B) = (1 - \alpha) \lambda(J)$. In such a case, the function $g(x) = \int_0^x \chi_A(t) dt = \alpha x$ would have null derivatives on a set of positive measure $\lambda(B)$, which is impossible.

We show two ways to construct these sets $A$ and $B$.

a) By removing central subintervals, we build a positive measure Cantor set, say $S$. Let us set $A_1 := S \cap [0, 1/2]$ and $B_1 := S \cap [1/2, 1]$. We proceed by recurrence with the middle points for each removed subinterval: by $A_n$ and $B_n$ we denote, respectively, the points in the first or in the second halves removed in the $n$-th step. We can make $A := \cup_n A_n$ and $B := \cup_n B_n$. With these sets, we obtain the function $g$.

b) Another way to proceed. We can make use of strictly increasing singular functions. Let $f : \mathbb{I} \to \mathbb{I}$ be one such function (having $f(0) = 0$, $f(1) = 1$), $A^*$ a Lebesgue zero measure set concentrating the total mass of $df$, and $B^*$ its complement; that is, $df(A^*) = 1$, $dx(A^*) = 0$, $df(B^*) = 0$, $dx(B^*) = 1$.

Furthermore, let $g : \mathbb{I} \to \mathbb{I}$ be an absolutely continuous and strictly increasing function, with $g(0) = 0$, $g(1) = 1$. If we define $h := \frac{g}{f}$, the corresponding sets are $A = h(A^*)$ and $B = h(B^*)$. For an interval $J$, let us write $J^* := h^{-1}(J)$. Consequently, $\lambda(J \cap A) = dh(J^* \cap A^*) = \frac{df(J^*)+dg(J^* \cap A^*)}{2} > 0$ (the last equality holds via Banach-Zaretzki’s theorem in [14, p.167]). For $B$ the process is similar.
5.2. An absolutely continuous function whose inverse is not absolutely continuous

It is known that a bijection is singular in $I$ if and only if it maps a set of measure 1 onto another of measure zero, and conversely, one of measure zero onto another of measure 1. This implies that if $f$ is singular and strictly increasing, then $f^{-1}$ is as well. However this is not the case for absolutely continuous functions.

The key to the example we give is the Banach-Zaretzki theorem. A function is absolutely continuous if and only if it maps sets of measure zero on sets of the same measure. Therefore, we need to look for a strictly increasing and absolutely continuous function that applies sets of positive measure on another of zero measure. The next method provides a function with the required properties.

Let us consider a set $A$ as described at the end of Subsection 5.1; that is, both its intersections with intervals and intersections of its complement with intervals in the unit interval $I$ are sets of positive measure. Clearly, $G(x) = \frac{1}{\lambda(A)} \int_0^x \chi_A(t) dt$ is an absolutely continuous distribution function. On a set of measure $\lambda(A) > 0$, it satisfies $G'(x) = \frac{1}{\lambda(A)}$ and, on another of measure $\lambda(B) = 1 - \lambda(A)$, it has derivatives $G'(x) = 0$.

Working with increasing functions $f$ on a set $E$ having derivatives at every point, we require this result: for a non-decreasing function $f$ having derivatives at every point in $E$, with value $\alpha$, then $\lambda(f(E)) = \alpha \lambda(f(E))$.

We have that $f$ maps a subset $S \subset B$ of positive measure onto another of zero measure. The inverse $f^{-1}$, maps the set $f(S)$, of zero measure, onto another of positive measure. Therefore, it can be neither absolutely continuous nor purely singular.

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