

Erratum

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Alternative proofs of a formula for Bernoulli numbers in terms of Stirling numbers

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Abstract: In the paper, the authors provide four alternative proofs of an explicit formula for computing Bernoulli numbers in terms of Stirling numbers of the second kind.

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1 Introduction

It is well known that Bernoulli numbers B_k for $k \geq 0$ may be generated by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi. \quad (1.1)$$

In combinatorics, Stirling numbers of the second kind $S(n, k)$ for $n \geq k \geq 0$ may be computed by

$$S(n, k) = \frac{1}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \ell^n$$

and may be generated by

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}, \quad k \in \{0\} \cup \mathbb{N}.$$

In [5, p. 536] and [6, p. 560], the following simple formula for computing Bernoulli numbers B_n in terms of Stirling numbers of the second kind $S(n, k)$ was incidentally obtained.

Theorem 1.1. For $n \in \{0\} \cup \mathbb{N}$, we have

$$B_n = \sum_{k=0}^n (-1)^k \frac{k!}{k+1} S(n, k). \tag{1.2}$$

The aim of this paper is to provide four alternative proofs for the explicit formula (1.2).

2 Four alternative proofs of the formula (1.2)

Considering $S(0, 0) = 1$, it is clear that the formula (1.2) is valid for $n = 0$. Further considering $S(n, 0) = 0$ for $n \geq 1$, it is sufficient to show

$$B_n = \sum_{k=1}^n (-1)^k \frac{k!}{k+1} S(n, k), \quad n \geq 1.$$

First proof. It is listed in [1, p. 230, 5.1.32] that

$$\ln \frac{b}{a} = \int_0^\infty \frac{e^{-au} - e^{-bu}}{u} du. \tag{2.1}$$

Taking $a = 1$ and $b = 1 + x$ in (2.1) yields

$$\frac{\ln(1+x)}{x} = \int_0^\infty \frac{1 - e^{-xu}}{xu} e^{-u} du = \int_0^\infty \left(\int_{1/e}^1 t^{xu-1} dt \right) e^{-u} du. \tag{2.2}$$

Replacing x by $e^x - 1$ in (2.2) results in

$$\frac{x}{e^x - 1} = \int_0^\infty \left(\int_{1/e}^1 t^{ue^x - u - 1} dt \right) e^{-u} du. \tag{2.3}$$

In combinatorics, Bell polynomials of the second kind (also called partial Bell polynomials) $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ are defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}$$

for $n \geq k \geq 1$, see [4, p. 134, Theorem A]. They satisfy

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{2.4}$$

and

$$B_{n,k}(\overbrace{1, 1, \dots, 1}^{n-k+1}) = S(n, k), \tag{2.5}$$

see [4, p. 135], where a and b are any complex numbers. The well-known Faà di Bruno formula may be described in terms of Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{d^n}{dx^n} f \circ g(x) = \sum_{k=1}^n f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)), \tag{2.6}$$

see [4, p. 139, Theorem C].

Applying in (2.6) the function $f(y) = t^y$ and $y = g(x) = ue^x - u - 1$ gives

$$\frac{d^n t^{ue^x}}{dx^n} = \sum_{k=1}^n (\ln t)^k t^{ue^x} B_{n,k}(\overbrace{ue^x, ue^x, \dots, ue^x}^{n-k+1}). \tag{2.7}$$

Making use of the formulas (2.4) and (2.5) in (2.7) reveals

$$\frac{d^n t^{ue^x}}{dx^n} = t^{ue^x} \sum_{k=1}^n S(n, k) u^k (\ln t)^k e^{kx}. \tag{2.8}$$

Differentiating n times on both sides of (2.3) and considering (2.8), we obtain

$$\frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) = \sum_{k=1}^n S(n, k) e^{kx} \int_0^\infty u^k \left(\int_{1/e}^1 (\ln t)^k t^{ue^x - u - 1} dt \right) e^{-u} du. \tag{2.9}$$

On the other hand, differentiating n times on both sides of (1.1) gives

$$\frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) = \sum_{k=n}^\infty B_k \frac{x^{k-n}}{(k-n)!}. \tag{2.10}$$

Equating (2.9) and (2.10) and taking the limit $x \rightarrow 0$, we deduce

$$\begin{aligned} B_n &= \sum_{k=1}^n S(n, k) \int_0^\infty u^k \left(\int_{1/e}^1 \frac{(\ln t)^k}{t} dt \right) e^{-u} du \\ &= \sum_{k=1}^n \frac{(-1)^k}{k+1} S(n, k) \int_0^\infty u^k e^{-u} du \\ &= \sum_{k=1}^n \frac{(-1)^k k!}{k+1} S(n, k). \end{aligned}$$

The first proof of Theorem 1.1 is complete. □

Second proof. In the book [2, p. 386] and in the papers [3, p. 615] and [11, p. 885], it was given that

$$\frac{\ln b - \ln a}{b - a} = \int_0^1 \frac{1}{(1-t)a + tb} dt,$$

where $a, b > 0$ and $a \neq b$. Replacing a by 1 and b by e^x yields

$$\frac{x}{e^x - 1} = \int_0^1 \frac{1}{1 + (e^x - 1)t} dt.$$

Applying the functions $f(y) = \frac{1}{y}$ and $y = g(x) = 1 + (e^x - 1)t$ in the formula (2.6) and simplifying by (2.4) and (2.5) give

$$\begin{aligned} \frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) &= \int_0^1 \frac{d^n}{dx^n} \left[\frac{1}{1 + (e^x - 1)t} \right] dt \\ &= \int_0^1 \sum_{k=1}^n (-1)^k \frac{k!}{[1 + (e^x - 1)t]^{k+1}} B_{n,k}(\overbrace{te^x, te^x, \dots, te^x}^{n-k+1}) dt \\ &= \sum_{k=1}^n (-1)^k k! \int_0^1 \frac{t^k}{[1 + (e^x - 1)t]^{k+1}} B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}) dt \\ &\rightarrow \sum_{k=1}^n (-1)^k k! \int_0^1 t^k B_{n,k}(\overbrace{1, 1, \dots, 1}^{n-k+1}) dt, \quad x \rightarrow 0 \\ &= \sum_{k=1}^n (-1)^k k! S(n, k) \int_0^1 t^k dt \\ &= \sum_{k=1}^n (-1)^k \frac{k!}{k+1} S(n, k). \end{aligned}$$

On the other hand, taking the limit $x \rightarrow 0$ in (2.10) leads to

$$\frac{d^n}{dx^n} \left(\frac{x}{e^x - 1} \right) = \sum_{k=n}^{\infty} B_k \frac{x^{k-n}}{(k-n)!} \rightarrow B_n, \quad x \rightarrow 0.$$

The second proof of Theorem 1.1 is thus complete. □

Third proof. Let $CT[f(x)]$ be the coefficient of x^0 in the power series expansion of $f(x)$. Then

$$\begin{aligned} \sum_{k=1}^n (-1)^k \frac{k!}{k+1} S(n, k) &= \sum_{k=1}^n (-1)^k CT \left[\frac{n!}{x^n} \frac{(e^x - 1)^k}{k+1} \right] \\ &= n! CT \left[\frac{1}{x^n} \sum_{k=1}^{\infty} (-1)^k \frac{(e^x - 1)^k}{k+1} \right] \\ &= n! CT \left[\frac{1}{x^n} \frac{\ln[1 + (e^x - 1)] - (e^x - 1)}{e^x - 1} \right] \\ &= n! CT \left[\frac{1}{x^n} \frac{x}{e^x - 1} \right] \\ &= B_n. \end{aligned}$$

Thus, the formula (1.2) follows. □

Fourth proof. It is clear that the equation (1.1) may be rewritten as

$$\frac{\ln[1 + (e^x - 1)]}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}. \tag{2.11}$$

Differentiating n times on both sides of (2.11) and taking the limit $x \rightarrow 0$ reveal

$$\begin{aligned} B_n &= \lim_{x \rightarrow 0} \sum_{k=n}^{\infty} B_k \frac{x^{k-n}}{(k-n)!} \\ &= \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left(\frac{\ln[1 + (e^x - 1)]}{e^x - 1} \right) \\ &= \lim_{x \rightarrow 0} \sum_{k=1}^n \left[\frac{\ln(1+u)}{u} \right]^{(k)} B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}), \quad u = e^x - 1 \\ &= \lim_{x \rightarrow 0} \sum_{k=1}^n \left[\sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{u^{\ell-1}}{\ell} \right]^{(k)} B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}) \\ &= \lim_{x \rightarrow 0} \sum_{k=1}^n \left[\sum_{\ell=k+1}^{\infty} (-1)^{\ell-1} \frac{(\ell-1)!}{(\ell-k-1)! \ell} u^{\ell-k-1} \right] B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}) \\ &= \sum_{k=1}^n \lim_{u \rightarrow 0} \left[\sum_{\ell=k+1}^{\infty} (-1)^{\ell-1} \frac{(\ell-1)!}{(\ell-k-1)! \ell} u^{\ell-k-1} \right] \lim_{x \rightarrow 0} B_{n,k}(\overbrace{e^x, e^x, \dots, e^x}^{n-k+1}) \\ &= \sum_{k=1}^n (-1)^k \frac{k!}{k+1} B_{n,k}(\overbrace{1, 1, \dots, 1}^{n-k+1}) \\ &= \sum_{k=1}^n (-1)^k \frac{k!}{k+1} S(n, k). \end{aligned}$$

The fourth proof of Theorem 1.1 is thus complete. □

Remark 2.1. In [9, p. 1128, Corollary], among other things, it was found that

$$B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k \sum_{i=1}^{k-1} \frac{A_{2(k-i)}}{2(k-i)+1}$$

for $k \in \mathbb{N}$, where A_m is defined by

$$\sum_{m=1}^n m^k = \sum_{m=0}^{k+1} A_m n^m.$$

It was listed in [6, p. 559] and recovered in [8, Theorem 2.1] that

$$\left(\frac{1}{e^x - 1}\right)^{(k)} = (-1)^k \sum_{m=1}^{k+1} (m-1)! S(k+1, m) \left(\frac{1}{e^x - 1}\right)^m \tag{2.12}$$

for $k \in \{0\} \cup \mathbb{N}$. In [8, Theorem 3.1], by the identity (2.12), it was obtained that

$$B_{2k} = 1 + \sum_{m=1}^{2k-1} \frac{S(2k+1, m+1)S(2k, 2k-m)}{\binom{2k}{m}} - \frac{2k}{2k+1} \sum_{m=1}^{2k} \frac{S(2k, m)S(2k+1, 2k-m+1)}{\binom{2k}{m-1}}, \quad k \in \mathbb{N}.$$

In [12, Theorem 1.4], among other things, it was presented that

$$B_{2k} = \frac{(-1)^{k-1}k}{2^{2(k-1)}(2^{2k}-1)} \sum_{i=0}^{k-1} \sum_{\ell=0}^{k-i-1} (-1)^{i+\ell} \binom{2k}{\ell} (k-i-\ell)^{2k-1}, \quad k \in \mathbb{N}.$$

Remark 2.2. The identities in (2.12) have been generalized and applied in [7, 13].

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