Research Article

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Ground states for fractional Schrödinger equations involving a critical nonlinearity

Abstract: This paper is aimed to study ground states for a class of fractional Schrödinger equations involving the critical exponents:

\[ (-\Delta)^{\alpha} u + u = \lambda f(u) + |u|^{2^{*}_\alpha - 2} u \quad \text{in } \mathbb{R}^N, \]

where \( \lambda \) is a real parameter, \((-\Delta)^{\alpha}\) is the fractional Laplacian operator with \( 0 < \alpha < 1 \), \( 2^{*}_\alpha = \frac{2N}{N - 2\alpha} \) with \( 2 \leq N \), \( f \) is a continuous subcritical nonlinearity without the Ambrosetti–Rabinowitz condition. Based on the principle of concentration compactness in the fractional Sobolev space and radially decreasing rearrangements, we obtain a nonnegative radially symmetric minimizer for a constrained minimization problem which has the least energy among all possible solutions for the above equations, i.e., a ground state solution.

Keywords: Fractional Schrödinger equations, fractional Sobolev space, critical Sobolev exponent, ground states

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1 Introduction and main result

In this paper, we study ground state solutions for the following fractional Schrödinger equations with a critical nonlinearity:

\[ (-\Delta)^{\alpha} u + u = \lambda f(u) + |u|^{2^{*}_\alpha - 2} u \quad \text{in } \mathbb{R}^N, \]

where \( 2^{*}_\alpha = \frac{2N}{N - 2\alpha} \) with \( N \geq 2 \), \( \lambda > 0 \), \( \alpha \in (0, 1) \) and \((-\Delta)^{\alpha}\) is the fractional Laplacian operator, which (up to normalization constants) may be defined as

\[ (-\Delta)^{\alpha} u(x) := \text{PV} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N + 2\alpha}} \, dy, \quad x \in \mathbb{R}^N, \]

where PV stands for the principal value.

The fractional Laplacian operator \((-\Delta)^{\alpha}\) can be seen as the infinitesimal generators of Lévy stable diffusion processes (see [1]). The Lévy processes occur widely in physics, chemistry, biology and finance, see for example [11, 22]. Some interesting topics concerning the fractional Laplacian such as the nonlinear fractional Schrödinger equation (see [17, 18, 22, 23, 26, 51]), the nonlinear fractional Kirchhoff equation (see [20, 29, 34, 35, 48–50]), the fractional porous medium equation (see [13, 46]) and so on, have attracted recently much research interest. Indeed, the literature on fractional operators and their applications to partial differential equations is quite large. Here we would like to mention a few, see for instance [4, 9, 15, 26, 27, 30, 31, 36] for recent results.

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In the celebrated paper [8], Berestycki and Lions studied the following classical nonlinear scalar field equation:

\[-\Delta u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}\]

where \( N \geq 3 \). Using certain assumptions on \( g \), which are now named Berestycki–Lions conditions, they proved the existence of a ground state solution. Using Pohozaev identity, they also showed that Berestycki–Lions conditions are almost necessary for the existence of a solution for problem (1.2). For \( N = 2 \), Berestycki, Gallouët and Kavian [7] obtained the existence of a radially symmetric positive solution of (1.2) under some appreciate conditions on \( g \). In fact, the authors in [7, 8] just dealt with the subcritical case. However, for the critical case, the problem becomes very difficult due to the loss of the compactness of the embedding \( H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \). About the characterization of ground state solutions corresponding to the Berestycki–Lions (and others) result for the critical growth case, for example, we refer to [2, 52].

In [14], using minimax arguments, Chang and Wang study the following scalar field equation involving the fractional Laplacian:

\[-(-\Delta)^s u = g(u) \quad \text{in } \mathbb{R}^N. \tag{1.3}\]

They obtained a positive ground state under the fractional version of Berestycki–Lions type assumptions, in which \( g \) is subcritical at infinity. On some recent works involving the subcritical case, we refer to, for instance, [17, 37] and references therein.

In [42], Shang and Zhang studied the existence and multiplicity of solutions for the critical fractional Schrödinger equation:

\[\varepsilon^{2\alpha}(-\Delta)^s u + V(x) u = \lambda f (u) + \left| u \right|^{2^*_s-2} u \quad \text{in } \mathbb{R}^N. \tag{1.4}\]

Based on variational methods, they showed that problem (1.4) has a nonnegative ground state solution for all sufficiently large \( \lambda \) and small \( \varepsilon \). In this paper, the following monotone condition was imposed on the continuous subcritical nonlinearity \( f \):

\[ f(t) \text{ is strictly increasing in } (0, +\infty). \tag{1.5}\]

In [44], Shen and Gao obtained the existence of nontrivial solutions for problem (1.4) under various assumptions on \( f(t) \) and the potential function \( V(x) \). Indeed, the authors assumed the well-known Ambrosetti–Rabinowitz condition (AR condition for short) on \( f \):

\[ \text{there exists } \theta > 2 \text{ such that } 0 < \theta F(t) \leq f(t) t \text{ for any } t > 0. \tag{1.6}\]

where \( F(t) = \int_0^t f(s) \, ds \). See also the recent papers [37, 38] on the fractional Schrödinger equations with or without (AR) condition. In [45], Teng was concerned with the following fractional Schrödinger equations involving a critical nonlinearity:

\[-(-\Delta)^s u + u = A(x) \left| u \right|^{p-2} u + B(x) \left| u \right|^{2^*_s-2} u \quad \text{in } \mathbb{R}^N. \tag{1.7}\]

where \( 2 < p < 2^*_s \), and the potential functions \( A(x) \) and \( B(x) \) satisfy certain hypotheses. Using the \( s \)-harmonic extension technique of Caffarelli and Silvestre [12], the concentration-compactness principle of Lions [24] and methods of Brézis and Nirenberg [10], the author obtained the existence of ground state solutions. On fractional Kirchhoff problems involving critical nonlinearity, see for example [3, 33] for some recent results. Last but not least, fractional elliptic problems with critical growth, in a bounded domain, have been studied by some authors in the last years, see [5, 6, 21, 40, 41, 43] and references therein.

On the other hand, Feng in [19] investigated the following fractional Schrödinger equations:

\[-(-\Delta)^s u + V(x) u = \lambda \left| u \right|^{p-2} u \quad \text{in } \mathbb{R}^N, \tag{1.8}\]

where \( 2 < p < 2^*_s \), \( V(x) \) is a positive continuous function. By using the fractional version of the concentration-compactness principle of Lions [24], the author obtained the existence of ground state solutions to problem (1.8) for some \( \lambda > 0 \).

Motivated by the above works, we are interested in the existence of ground state solutions for problem (1.1) via concentration compactness principle in the fractional Sobolev space (see [32, Theorem 1.5]), which is another fractional version of Lions [25]. To this end, we impose the following conditions on \( f \):
(H1) \( f \in C(\mathbb{R}, \mathbb{R}) \) and for any \( t \leq 0 \), \( f(t) = 0 \).

(H2) \( \lim_{t \to 0^-} \frac{f(t)}{t^p} = 0 \) and \( \lim_{t \to \infty} \frac{f(t)}{t^q} = 0 \).

(H3) There exists \( q > 2 \) such that for any \( t \geq 0 \), \( f(t) \geq t^{q-1} \).

(H4) For any \( t > 0 \), \( 0 < 2F(t) \leq f(t) \).

In order to obtain a nonnegative solution, we assume that \( f(t) = 0 \) for any \( t \leq 0 \) throughout the paper. From (H2) we know that \( f \) is subcritical. Moreover, the solution to problem (1.1) is obtained without assuming the classical condition (1.5) or (1.6). Clearly, we employ the weaker condition (H4) on \( f \) to replace (AR) condition. A typical example for \( f \) is given by \( f(t) = t^{q-1} \) for \( t \geq 0 \) with \( q > 2 \).

Now, we give the definition of weak solutions for problem (1.1).

**Definition 1.1.** We say that \( u \) is a weak solution of (1.1) if for any \( \phi \in H^a(\mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N} (-\Delta)^{\frac{a}{2}} u \cdot (-\Delta)^{\frac{a}{2}} \phi \, dx + \int_{\mathbb{R}^N} u \phi \, dx = \int_{\mathbb{R}^N} (\lambda f(u) + |u|^{2^*_a - 2} u) \phi \, dx,
\]

i.e.,

\[
\int_{\mathbb{R}^N} \frac{(u(x) - u(y))((\phi(x) - \phi(y))}{|x-y|^{N+2a}} \, dx \, dy + \int_{\mathbb{R}^N} u \phi \, dx = \int_{\mathbb{R}^N} (\lambda f(u) + |u|^{2^*_a - 2} u) \phi \, dx,
\]

where \( H^a(\mathbb{R}^N) \) is the fractional Sobolev space which is a Hilbert space (see [16]), see Section 2 for more details.

We define the following functionals on \( H^a(\mathbb{R}^N) \):

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{a}{2}} u|^2 \, dx, \quad K(u) = \int_{\mathbb{R}^N} \left( \lambda F(u) + \frac{1}{2} |u|^{2^*_a} - \frac{1}{2} u^2 \right) \, dx
\]

and

\[
I(u) = J(u) - K(u).
\]

It is easy to check that \( I \in C^1(H^a(\mathbb{R}^N), \mathbb{R}) \) and the weak solution for problem (1.1) coincides with the critical point of \( I \).

**Definition 1.2.** We say that a weak solution \( w \) of (1.1) is a ground state solution if

\[
I(w) = \inf\{I(u) : u \in H^a(\mathbb{R}^N) \text{ is a nontrivial weak solution of (1.1)}\}.
\]

Note that \( I \) is neither bounded from above nor from below on \( H^a(\mathbb{R}^N) \), it is difficult to look directly for critical points of \( I \). In this paper, we will first consider a constrained minimization problem and obtain its minimizer in the fractional radially symmetric function space \( H^a_0(\mathbb{R}^N) \). Then, we verify that the minimizer under a scale change is a ground state solution for problem (1.1). Now we are ready to give our main result as follows.

**Theorem 1.3.** Assume hypotheses (H1)–(H4) are fulfilled. Then, there exists \( \lambda_* > 0 \) such that for any \( \lambda \in (\lambda_*, \infty) \), problem (1.1) has a ground state solution \( w \in H^a(\mathbb{R}^N) \) which is nonnegative and radially symmetric.

**Remark 1.4.** In [21], the authors studied the existence of ground state solutions for a critical fractional Laplacian equation in a bounded domain. Using the \( \alpha \)-harmonic extension introduced by Caffarelli and Silvestre [12], they transformed the nonlocal problem into a local problem. While in this paper we propose a completely different approach. Namely, in our approach we search directly for ground state solutions for problem (1.1) in the whole space and give a characterization of the least energy \( I(w) \) (see the proof of Theorem 1.3). To the best of our knowledge, it seems that it is the first time to investigate ground state solutions for problem (1.1) by using the concentration-compactness principle in the fractional Sobolev space which is different with the version used in [19].

This paper is organized as follows. In Section 2, we will give some necessary definitions and properties of fractional Sobolev spaces. In Section 3, by using the concentration-compactness principle and radially decreasing rearrangements, we give the proof of Theorem 1.3.
2 Preliminaries

For the convenience of the reader, in this part we recall some definitions and basic properties of fractional Sobolev spaces $H^s(\mathbb{R}^N)$. For a deeper treatment on these spaces and their applications to fractional Laplacian problems of elliptic type, we refer to [16, 28] and references therein.

We consider the Schwartz space $\mathcal{S}$ of rapidly decaying $C^\infty$ functions in $\mathbb{R}^N$, with the corresponding topology generated by the seminorms

$$p_M(\varphi) = \sup_{x \in \mathbb{R}^N} (1 + |x|)^M \sum_{|\alpha| \leq M} |D^\alpha \varphi(x)|, \quad M = 0, 1, 2, \ldots,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^N)$. Let $\mathcal{S}^\prime(\mathbb{R}^N)$ be the set of all tempered distributions, that is the topological dual of $\mathcal{S}(\mathbb{R}^N)$. As usual, for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$, we denote by

$$\mathcal{F}(\varphi)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \varphi(x) \, dx$$

the Fourier transform of $\varphi$ and we recall that one can extend $\mathcal{F}$ from $\mathcal{S}(\mathbb{R}^N)$ to $\mathcal{S}^\prime(\mathbb{R}^N)$.

For any $\alpha \in (0, 1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ is defined by

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy \right)^{\frac{1}{2}} + \|u\|_{L^2(\mathbb{R}^N)},$$

where the term

$$[u]_{H^\alpha(\mathbb{R}^N)} = \|(-\Delta)^{\alpha} u\|_{L^2(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \, dx \, dy \right)^{\frac{1}{2}}$$

is the so-called Gagliardo seminorm of $u$. The space $\dot{H}^\alpha(\mathbb{R}^N)$ is defined as the completion of $C^\infty_0(\mathbb{R}^N)$ under the norm $[u]_{\dot{H}^\alpha(\mathbb{R}^N)}$.

Using the Fourier transform, the fractional Laplacian $(-\Delta)^{\alpha}$ can also be seen as a pseudo-differential operator of $|\xi|^{2\alpha}$:

$$\mathcal{F}((-\Delta)^{\alpha} \phi)(\xi) = |\xi|^{2\alpha} \mathcal{F}(\phi)(\xi) \quad \text{for any} \ \xi \in \mathbb{R}^N, \tag{2.1}$$

where $\phi \in \mathcal{S}$.

Evidently, (2.1) means that the fractional Laplacian is nonlocal, which is a distinguished feature, and hence makes it difficult to deal with. It is worth mentioning that in a bounded domain, the Fourier definition of the fractional laplacian does not agree with its local Caffarelli–Silvestre interpretation (see [12]), we refer to [39] for a detailed discussion.

3 Proof of Theorem 1.3

Throughout this section, we assume that conditions (H1)–(H4) are satisfied. In this part, rather than looking for critical points of $I$, we will first consider the following constrained minimization problem.

We define

$$\mathcal{M} = \left\{ u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\}$$
and
\[
A = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx : u \in \mathcal{M} \right\},
\]
where \( H^s_0(\mathbb{R}^N) = \{ u \in H^s(\mathbb{R}^N) : u(x) = u(|x|) \} \) is the fractional radially symmetric function space and
\[
G(t) = \lambda F(t) + \frac{1}{2a} |t|^{2s} - \frac{1}{2} t^2.
\]

The main difficulties here are that the embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N) \) is not compact and we do not have a similar radial lemma (see [8]) in \( H^s_0(\mathbb{R}^N) \). To get the compactness of bounded minimizing sequence in \( H^s(\mathbb{R}^N) \), we assume that \( \lambda \) in (1.1) is large. Based on the concentration-compactness principle in \( H^s(\mathbb{R}^N) \) (see [32]) and radially decreasing rearrangements in [17], we obtain a nonnegative radially symmetric minimizer for (3.1).

**Lemma 3.1.** We have that \( A > 0 \), and any minimizing sequence for (3.1) is bounded in \( H^s(\mathbb{R}^N) \).

**Proof.** First, we will verify that the set \( \mathcal{M} \) is not empty. By the definition of \( G \) in (3.2), there exists \( \zeta > 0 \) such that \( G(\zeta) > 0 \). Let \( R > 0 \), we define
\[
w_R(x) = \begin{cases} 
\zeta & \text{for } |x| \leq R, \\
\zeta(R + 1 - |x|) & \text{for } R < |x| < R + 1, \\
0 & \text{for } |x| \geq R + 1,
\end{cases}
\]
thus \( w_R \in H^s_0(\mathbb{R}^N) \). Hence, we have
\[
\int_{\mathbb{R}^N} G(w_R) \, dx = \int_{B(0,R)} G(w_R) \, dx + \int_{B(0,R+1) \setminus B(0,R)} G(w_R) \, dx \
\geq G(\zeta) |B(0,R)| - |B(0,R+1) \setminus B(0,R)| \left( \max_{t \in [0,\zeta]} |G(t)| \right) \
\geq C_1 R^{N_s} - C_2 R^{N-1},
\]
where \( |\cdot| \) denotes the Lebesgue measure and \( C_1, C_2 \) are positive constants. So we could choose \( R > 0 \) large enough such that
\[
\int_{\mathbb{R}^N} G(w_R) \, dx > 0.
\]
Define \( w_{R,\sigma}(x) = w_R(\frac{x}{\sigma}) \), where \( \sigma > 0 \). Note that
\[
\int_{\mathbb{R}^N} G(w_{R,\sigma}) \, dx = \sigma^N \int_{\mathbb{R}^N} G(w_R) \, dx,
\]
so we could choose \( \sigma > 0 \) such that
\[
\int_{\mathbb{R}^N} G(w_{R,\sigma}) \, dx = 1.
\]
Let \( \{u_n\} \) be a minimizing sequence for (3.1), i.e., \( \{u_n\} \subset H^s_0(\mathbb{R}^N) \) such that
\[
\frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx \to A \quad \text{as } n \to \infty
\]
and
\[
\int_{\mathbb{R}^N} G(u_n) \, dx = 1.
\]
Using (H1) and (H2), we get \( F(t) \leq \frac{1}{4\lambda} t^2 + C t^{2s} \) for \( t \geq 0 \) and \( F(t) = 0 \) for \( t \leq 0 \), where \( C \) is a positive constant. Then,
\[
\int_{\mathbb{R}^N} F(u_n) \, dx \leq \frac{1}{4\lambda} \int_{\mathbb{R}^N} u_n^2 \, dx + C \int_{\mathbb{R}^N} |u_n|^{2s} \, dx.
\]
We will show that $A > 0$. Suppose $A = 0$. Then,
\[
\frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u_n \right|^2 \, dx \to 0 \quad \text{as } n \to \infty.
\] (3.4)

Note that
\[
1 = \int_{\mathbb{R}^N} G(u_n) \, dx \leq \int_{\mathbb{R}^N} \left( \frac{1}{q} u_n^q + C|u_n|^{2^*_s} + \frac{1}{2^*_s} |u_n|^{2^*_s} - \frac{1}{2} u_n^2 \right) \, dx \leq C \int_{\mathbb{R}^N} |u_n|^{2^*_s} \, dx
\] (3.5)

and
\[
\int_{\mathbb{R}^N} |u_n|^{2^*_s} \, dx \leq \left( S_{a}^{-1} \right) \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u_n \right|^2 \, dx \right)^{\frac{2^*_s}{s}} \, dx,
\] (3.6)

where $S_{a}$ is the best Sobolev constant of the embedding $H^a(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ (see [16]), i.e.,
\[
S_{a} = \inf_{u \in H^a(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 \, dx}{\|u\|_{L^{2^*_s}(\mathbb{R}^N)}^2}.
\] (3.7)

From (3.4), (3.5) and (3.6), we get a contradiction.

In the following, we will verify that $\{u_n\}$ is bounded in $H^a(\mathbb{R}^N)$. We have
\[
\int_{\mathbb{R}^N} \lambda F(u_n) \, dx = \int_{\mathbb{R}^N} G(u_n) \, dx - \frac{1}{2^*_s} \int_{\mathbb{R}^N} |u_n|^{2^*_s} \, dx + \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 \, dx = 1 - \frac{1}{2^*_s} \int_{\mathbb{R}^N} |u_n|^{2^*_s} \, dx + \frac{1}{2} \int_{\mathbb{R}^N} u_n^2 \, dx.
\]

Then, using (3.3) we get
\[
\frac{1}{q} \int_{\mathbb{R}^N} u_n^q \, dx \leq C \int_{\mathbb{R}^N} |u_n|^{2^*_s} \, dx.
\]

By (3.6), $\{\int_{\mathbb{R}^N} |u_n|^{2^*_s} \, dx\}_n$ is bounded. Thus, $\{\int_{\mathbb{R}^N} |u_n|^2 \, dx\}_n$ is also bounded, which implies that
\[
\left\{ \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u_n \right|^2 \, dx + \int_{\mathbb{R}^N} |u_n|^2 \, dx \right\}_n
\]

is bounded, i.e., $\{u_n\}$ is bounded in $H^a(\mathbb{R}^N)$. \(\square\)

Next, using Pohozaev identity for (1.1) we will give a characterization of $A$.

In [14], using the $\alpha$-harmonic extension, the authors proved the Pohozaev identity for (1.3) with sub-critical nonlinearities. In this paper, although the problem (1.1) involves critical nonlinearities, similarly to the proof of Pohozaev identity in [14], we could also obtain the following Pohozaev identity for (1.1): Let $u \in H^a(\mathbb{R}^N)$ be a weak solution of (1.1), then
\[
(N - 2a) \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 \, dx = 2N \int_{\mathbb{R}^N} G(u) \, dx.
\] (3.8)

We introduce the set $\mathcal{P}$ of nontrivial functions satisfying the Pohozaev identity (3.8), i.e.,
\[
\mathcal{P} = \left\{ u \in H^a_p(\mathbb{R}^N) \setminus \{0\} : (N - 2a) \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 \, dx = 2N \int_{\mathbb{R}^N} G(u) \, dx \right\}
\]

and
\[
P = \inf_{u \in \mathcal{P}} I(u).
\]

**Lemma 3.2.** We have that
\[
P = 2a \left( \frac{N - 2a}{N} \right)^{\frac{s}{2}} \frac{\lambda_{s,a}}{\lambda_{a}} A \frac{A}{N}.
\]
Proof. For any \( u \in \mathcal{M} \), take
\[
t_u = \left( \frac{N - 2\alpha}{2N} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u|^2 \, dx \right)^{\frac{1}{2}}.
\]
Define \( \Phi : \mathcal{M} \to \mathcal{P} \) as follows:
\[
\Phi(u) = u \left( \frac{x}{t_u} \right).
\]
In the following, we will verify that \( \Phi \) is a well-defined one-to-one correspondence. In fact, for any \( u \in \mathcal{M} \),
\[
\int_{\mathbb{R}^N} G(u) \, dx = 1.
\]
Note that
\[
\Phi(t_u) = u \left( \frac{x}{t_u} \right) = u(x) \cdot t_u^{-1} \cdot \left( \frac{t_u}{x} \right).
\]
Then, by the definition of \( t_u \) we have
\[
2N \int_{\mathbb{R}^N} G \left( u \left( \frac{x}{t_u} \right) \right) \, dx = 2Nt_u^N \int_{\mathbb{R}^N} G(u) \, dx = 2Nt_u^N
\]
and
\[
(N - 2\alpha) \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} u \left( \frac{x}{t_u} \right) \right|^2 \, dx = (N - 2\alpha)t_u^{N-2\alpha} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} u \right|^2 \, dx.
\]
Thus,
\[
2N \int_{\mathbb{R}^N} G \left( u \left( \frac{x}{t_u} \right) \right) \, dx = (N - 2\alpha) \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} u \left( \frac{x}{t_u} \right) \right|^2 \, dx,
\]
which implies \( u \left( \frac{x}{t_u} \right) \in \mathcal{P} \).

For any \( v \in \mathcal{P} \), i.e.,
\[
(N - 2\alpha) \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} v \right|^2 \, dx = 2N \int_{\mathbb{R}^N} G(v) \, dx,
\]
we set
\[
t_v = \left( \frac{N - 2\alpha}{2N} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} v \right|^2 \, dx \right)^{\frac{1}{2}}
\]
and
\[
u(x) = v(t_v x).
\]
Then, \( u \left( \frac{x}{t_v} \right) = \nu(x) \). Note that
\[
t_v = \left( \frac{N - 2\alpha}{2N} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{2N}} |v(x) - v(y)|^2 \, dx \, dy \right)^{\frac{1}{2}} = \left( \frac{N - 2\alpha}{2N} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 \, dx \, dy \right)^{\frac{1}{2}},
\]
which implies
\[
t_v = \left( \frac{N - 2\alpha}{2N} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 \, dx \, dy \right)^{\frac{1}{2}} = t_u
\]
with \( \nu(x) = u \left( \frac{x}{t_v} \right) \). We have
\[
\int_{\mathbb{R}^N} G(u) \, dx = \frac{1}{t_v^N} \int_{\mathbb{R}^N} G \left( u \left( \frac{x}{t_u} \right) \right) \, dx = \frac{1}{t_v^N} \left( \frac{N - 2\alpha}{2N} \right)^{\frac{1}{2}} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} v \right|^2 \, dx = 1,
\]
i.e., \( u \in \mathcal{M} \). Thus, \( \Phi(\mathcal{M}) = \mathcal{P} \).

For any \( u \in \mathcal{M} \), we obtain
\[
I(\Phi(u)) = \frac{1}{2} t_u^{N-2\alpha} \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 \, dx \, dy - t_u^N \int_{\mathbb{R}^N} G(u) \, dx = \frac{\alpha}{N} \left( \frac{N - 2\alpha}{2N} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} u \right|^2 \, dx \right)^{\frac{N}{2N}}.
\]
By Lemma three.tf. two.tf, we obtain Lemma three.tf. three.tf.

and which implies that for any

Proof. Define

Thus,

Set

Lemma three.tf. four.tf.

Thus,

Note that

We have that

|\((\Delta)^{\frac{1}{2}}u\)| \(\leq \left\{ \begin{array}{ll}
\max_{\gamma \in \mathcal{Y}(0,1)} \gamma(1) \\
\max_{\gamma \in \mathcal{Y}(0,1)} \gamma(1) < 0
\end{array} \right.

Lemma three.tf. five.tf.

Then there exists \(\rho_0 > 0\) such that for any \(0 < \|u\|_{H^s(\mathbb{R}^N)} < \rho_0\), we have \(P(u) > 0\).

Note that

which implies that for any \(\gamma \in \Gamma\), we have \(P(\gamma(1)) \leq 2NI(\gamma(1)) < 0\). Then, there exists \(t_0 \in (0,1)\) such that \(\|\gamma(t_0)\|_{H^s(\mathbb{R}^N)} > \rho_0\) and \(P(\gamma(t_0)) = 0\). We get \(\gamma(t_0) \in \mathcal{Y} \cap \mathcal{Y}(0,1)\), i.e.,

Thus, \(P \leq I(\gamma(t_0)) \leq \max_{\gamma \in \mathcal{Y}(0,1)} I(\gamma(t))\), which implies

By Lemma three.tf. two.tf, we obtain

Note that the embedding

is compact (see [14]). Hence, there exists \(0 \leq \psi \in H^s(\mathbb{R}^N)\) such that \(\|\psi\|_{H^s(\mathbb{R}^N)} = 1\) and \(\|\psi\|_{L^q(\mathbb{R}^N)} = C_q^{-1}\), where \(q \in (2, 2a^*)\) and \(C_q\) is the best Sobolev constant of the above embedding, i.e.,

\[
C_q = \inf_{u \in H^s(\mathbb{R}^N)} \frac{\|u\|_{H^s(\mathbb{R}^N)}}{\|u\|_{L^q(\mathbb{R}^N)}}.
\]

Lemma three.tf. six.tf.

b \leq \left(\frac{1}{2} - \frac{1}{q}\right) \lambda^{\frac{1}{2}} C_q^{-\frac{2a}{q}}.
Proof. By (H3), for any $t > 0$, we get
\[ I(t\psi) \leq \frac{1}{2} t^2 \|\psi\|_{H^q}(\mathbb{R}^N)^2 - \lambda \int_{\mathbb{R}^N} F(t\psi) \, dx \leq \frac{1}{2} t^2 \left(1 - \frac{\lambda}{q} t^{q-2} \right) - \frac{\lambda}{q} t^{q-2}. \]

Note that $q > 2$ and $I(t\psi) < 0$ when $t$ is large enough. Hence, there exists $t_0 > 0$ such that $I(t_0\psi) < 0$.

Take $\gamma(t) = tt_0\psi$. Then, $\gamma(0) = 0$, $\gamma(1) = t_0\psi$ and $I(\gamma(1)) < 0$, which implies $\gamma \in \Gamma$. Thus, we obtain
\[ b \leq \max_{t \in [0,1]} I(\gamma(t)) = \max_{t \in [0,1]} I(tt_0\psi) \leq \max_{t \in [0,1]} t^2 \left(1 - \frac{\lambda}{q} t^{q-2} \right) \leq \left(1 - \frac{1}{q}\right) \lambda \frac{a}{r+1} \frac{1}{t^{q-2}}. \]

In the following, we will take a special minimizing sequence for $A$ and get its compactness. Then, we could obtain a minimizer for (3.1).

From Ekeland’s variational principle (see [47, Theorem 8.5]), there exist $\{u_n\} \subset M$ and $\{\lambda_n\} \subset \mathbb{R}$ such that
\[ \lim_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \, dx = A \]
and
\[ f'(u_n) - \lambda_n K'(u_n) \to 0 \quad \text{in } (H^q_0(\mathbb{R}^N))^* \quad \text{as } n \to \infty. \]

By Lemma 3.1, $\{u_n\}$ is bounded in $H^q(\mathbb{R}^N)$. Passing to a subsequence, still denoted by $\{u_n\}$, we may assume that $u_n \to u$ weakly in $H^q_0(\mathbb{R}^N)$, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and there exist $\mu, \nu \in M(\mathbb{R}^N)$ such that $|(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \to \mu$ and $|u_n|^{2^*} \to \nu$ weakly* in $M(\mathbb{R}^N)$ as $n \to \infty$. It follows from the concentration-compactness principle (see [32, Theorem 1.5]) that $u_n \to u$ in $L^{2^*}_\text{loc}(\mathbb{R}^N)$ or $\nu = |u|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j}$ as $n \to \infty$, where $J$ is a countable set, $\{\nu_j\} \subset [0, \infty)$ and $\{x_j\} \subset \mathbb{R}^N$.

The concentration-compactness principle in [32] does not provide any information about the possible loss of mass at infinity of $\{u_n\}$. The following results expresses this fact in quantitative terms.

**Lemma 3.5.** Define
\[ \mu_{\infty} = \lim_{n \to \infty} \limsup_{R \to \infty} \int_{\{x \in \mathbb{R}^N ; |x| > R\}} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \, dx, \]
\[ \nu_{\infty} = \lim_{n \to \infty} \limsup_{R \to \infty} \int_{\{x \in \mathbb{R}^N ; |x| > R\}} |u_n|^{2^*} \, dx. \]

The quantities $\mu_{\infty}$ and $\nu_{\infty}$ are well defined and satisfy
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \, dx = \int_{\mathbb{R}^N} d\mu + \mu_{\infty}, \quad (3.9) \]
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx = \int_{\mathbb{R}^N} d\nu + \nu_{\infty}. \quad (3.10) \]

**Proof.** Let $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \chi \leq 1$ with $\chi \equiv 1$ in $\mathbb{R}^N \setminus B(0, 2)$ and $\chi \equiv 0$ in $B(0, 1)$. For any $R > 0$, define $\chi_R(x) = \chi(\frac{x}{R})$. We have
\[ \int_{\{x \in \mathbb{R}^N ; |x| > 2R\}} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \, dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \chi_R \, dx \leq \int_{\{x \in \mathbb{R}^N ; |x| > R\}} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \, dx. \]
Then,
\[ \mu_{\infty} = \lim_{n \to \infty} \limsup_{R \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} u_n|^2 \chi_R \, dx. \]
Similarly, we obtain
\[
\nu_{\infty} = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{R^N} |u_n|^{2^*} \chi_R \, dx = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{R^N} |u_n|^{2^*} \chi_R^2 \, dx.
\]

Note that
\[
\int_{R^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx = \int_{R^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \chi_R \, dx + \int_{R^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 (1 - \chi_R) \, dx.
\]

It is easy to verify that
\[
\int_{R^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 (1 - \chi_R) \, dx \to \int_{R^N} (1 - \chi_R) \, d\mu \quad \text{as} \ n \to \infty.
\]

Thus, we have
\[
\mu(\mathbb{R}^N) = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{R^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 (1 - \chi_R) \, dx.
\]

Then,
\[
\limsup_{n \to \infty} \int_{R^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx = \lim_{R \to \infty} \left( \limsup_{n \to \infty} \int_{R^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \chi_R \, dx + \int_{R^N} (1 - \chi_R) \, d\mu \right)
\]
\[
= \lim_{R \to \infty} \limsup_{n \to \infty} \int_{R^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \chi_R \, dx + \mu(\mathbb{R}^N)
\]
\[
= \mu_{\infty} + \mu(\mathbb{R}^N).
\]

Similarly, we obtain that \(\limsup_{n \to \infty} \int_{R^N} |u_n|^{2^*} \, dx = \nu(\mathbb{R}^N) + \nu_{\infty}\).

In the following, we derive some results involving \(\nu_i\) for any \(i \in J\) and \(\nu_{\infty}\) to obtain that \(u_n \rightharpoonup u\) in \(L^{2^*}(\mathbb{R}^N)\) as \(n \to \infty\).

**Lemma 3.6.** For any \(i \in J\), we have that \(\mu(|x_i|) \leq A\nu_i\) and \(\mu_{\infty} \leq A\nu_{\infty}\).

**Proof.** As \(|u_n| \subset H^s(\mathbb{R}^N)\) is bounded, we obtain
\[
\langle J'(u_n) - \lambda_n K'(u_n), u_n \rangle = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx - \lambda_n \int_{\mathbb{R}^N} (\lambda f(u_n)u_n + |u_n|^{2^*} - u_n^2) \, dx \to 0 \quad \text{as} \ n \to \infty,
\]

which implies
\[
2A = \lim_{n \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx = \lim_{n \to \infty} \lambda_n \int_{\mathbb{R}^N} (\lambda f(u_n)u_n + |u_n|^{2^*} - u_n^2) \, dx.
\]

It follows from (H4) that
\[
\int_{\mathbb{R}^N} (\lambda f(u_n)u_n + |u_n|^{2^*} - u_n^2) \, dx \geq \int_{\mathbb{R}^N} \left( 2\lambda f(u_n) + \frac{2}{2^*} |u_n|^{2^*} - u_n^2 \right) \, dx = 2.
\]

Thus,
\[
0 \leq \limsup_{n \to \infty} \lambda_n \leq A.
\]

Let \(\varphi \in C_0^\infty(\mathbb{R}^N)\) such that \(0 \leq \varphi \leq 1\) with \(\varphi \equiv 1\) in \(B(0, 1)\) and \(\varphi \equiv 0\) in \(\mathbb{R}^N \setminus B(0, 2)\). For any \(\varepsilon > 0\), define \(\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})\), where \(i \in J\). Note that \(|u_n \varphi_\varepsilon|\) is bounded in \(H^s(\mathbb{R}^N)\), hence
\[
\langle J'(u_n) - \lambda_n K'(u_n), u_n \varphi_\varepsilon \rangle \to 0 \quad \text{as} \ n \to \infty,
\]

which implies
\[
\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n : (-\Delta)^{\frac{s}{2}} (u_n \varphi_\varepsilon) \, dx = \lambda_n \int_{\mathbb{R}^N} (\lambda f(u_n)u_n + |u_n|^{2^*} - u_n^2) \varphi_\varepsilon \, dx + o(1). \quad (3.11)
\]
For any $\eta > 0$, by (H2) there exist $r \in (2, 2^*_n)$ and $C > 0$ such that
\[
tf(t) \leq \frac{1}{2}t^2 + \eta t^{2^*_n} + Ct',
\] (3.12)
where $t \geq 0$. Then,
\[
\int_{\mathbb{R}^n} \left( C|u_n|^r - \frac{1}{2}u_n^2 \right) \varphi \, dx = \int_{\mathbb{R}^n} \left( C|u_n|^r - \frac{1}{2}u_n^2 \right) \varphi \, dx \to \int_{\mathbb{R}^n} \left( C|u|^r - \frac{1}{2}u^2 \right) \varphi \, dx
\]
as $n \to \infty$ and
\[
\int_{\mathbb{R}^n} \left( C|u|^r - \frac{1}{2}u^2 \right) \varphi \, dx \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Then,
\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \lambda_n \int_{\mathbb{R}^n} \left( \lambda f(u_n)u_n + |u_n|^{2^*_n} - u_n^2 \right) \varphi \, dx
\]
\[
\leq A(\eta + 1) \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \int_{\mathbb{R}^n} |u_n|^{2^*_n} \varphi \, dx = A(\eta + 1)v_1.
\]
If we let $\eta \to 0$, then we obtain
\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \lambda_n \int_{\mathbb{R}^n} \left( \lambda f(u_n)u_n + |u_n|^{2^*_n} - u_n^2 \right) \varphi \, dx \leq Av_1.
\] (3.13)

Notice that
\[
\int_{\mathbb{R}^n} (-\Delta)^{\frac{a}{2}} u_n \cdot (-\Delta)^{\frac{a}{2}} (u_n \varphi) \, dx
\]
\[
= \iint_{\mathbb{R}^{2n}} \frac{(u_n(x) - u_n(y))^2 \varphi \, dy \, dx + \iint_{\mathbb{R}^{2n}} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))u_n(x) \, dy \, dx}{|x - y|^{N+2a}}
\]
It is easy to verify that
\[
\iint_{\mathbb{R}^{2n}} \frac{(u_n(x) - u_n(y))^2 \varphi \, dy \, dx \to \iint_{\mathbb{R}^n} \varphi \, d\mu \quad \text{as} \quad n \to \infty
\]
and
\[
\int_{\mathbb{R}^n} \varphi \, d\mu \to \mu(\{x_i\}) \quad \text{as} \quad \varepsilon \to 0.
\]
Moreover, Hölder’s inequality implies that
\[
\iint_{\mathbb{R}^{2n}} \frac{|u_n(x) - u_n(y)||\varphi(x) - \varphi(y)||u_n(x)| \, dy \, dx \leq \iint_{\mathbb{R}^{2n}} \frac{|u_n(x) - u_n(y)|\cdot|\varphi(x) - \varphi(y)| \cdot |u_n(x)| \, dy \, dx}{|x - y|^{N+2a}}
\]
\[
\leq C \left( \iint_{\mathbb{R}^{2n}} \frac{u_n^2(x)|\varphi(x) - \varphi(y)|^2 \, dy \, dx}{|x - y|^{N+2a}} \right)^{\frac{1}{2}}.
\]
In the following, we claim that
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^{2n}} \frac{u_n^2(x)(\varphi(x) - \varphi(y))^2 \, dy \, dx}{|x - y|^{N+2a}} = 0.
\]
Note that
\[ R^N \times R^N = \left( (R^N \setminus B(x_i, 2\varepsilon)) \cup B(x_i, 2\varepsilon) \right) \times \left( (R^N \setminus B(x_i, 2\varepsilon)) \cup B(x_i, 2\varepsilon) \right) \]
\[ \quad = \left( (R^N \setminus B(x_i, 2\varepsilon)) \times (R^N \setminus B(x_i, 2\varepsilon)) \right) \cup \left( B(x_i, 2\varepsilon) \times R^N \right) \cup \left( (R^N \setminus B(x_i, 2\varepsilon)) \times B(x_i, 2\varepsilon) \right). \]

**Case 1:** If \((x, y) \in (R^N \setminus B(x_i, 2\varepsilon)) \times (R^N \setminus B(x_i, 2\varepsilon))\), then \(\varphi_\varepsilon(x) = \varphi_\varepsilon(y) = 0\).

**Case 2:** \((x, y) \in B(x_i, 2\varepsilon) \times R^N\). If \(|x - y| \leq \varepsilon\), then
\[ |y - x| \leq |x - y| + |x - x_i| \leq 3\varepsilon, \]
which implies
\[
\int_{B(x_i, 2\varepsilon)} \int_{\{y \in R^N : |x - y| \leq \varepsilon\}} \frac{u_n^2(x)(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x - y|^{N+2a}} \, dy \, dx = \int_{B(x_i, 2\varepsilon)} \int_{\{y \in R^N : |x - y| \leq \varepsilon\}} \frac{u_n^2(x)|\nabla \varphi_\varepsilon(\xi)|^2 \varepsilon^{-y^2}}{|x - y|^{N+2a}} \, dy \, dx \leq C\varepsilon^{-2} \int_{B(x_i, 2\varepsilon)} \int_{\{y \in R^N : |x - y| \leq \varepsilon\}} \frac{u_n^2(x)}{|x - y|^{N+2a-2}} \, dy \, dx = C\varepsilon^{-2a} \int_{B(x_i, 2\varepsilon)} u_n^2(x) \, dx,
\]
where \(\xi = \frac{y - x_i}{\varepsilon} + \frac{y(x - x_i)}{\varepsilon^2}\) and \(r \in (0, 1)\).

If \(|x - y| > \varepsilon\), we have
\[
\int_{B(x_i, 2\varepsilon)} \int_{\{y \in R^N : |x - y| > \varepsilon\}} \frac{u_n^2(x)(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x - y|^{N+2a}} \, dy \leq C \int_{B(x_i, 2\varepsilon)} \int_{\{y \in R^N : |x - y| > \varepsilon\}} \frac{u_n^2(x)}{|x - y|^{N+2a}} \, dy \leq C\varepsilon^{-2a} \int_{B(x_i, 2\varepsilon)} u_n^2(x) \, dx.
\]

**Case 3:** \((x, y) \in (R^N \setminus B(x_i, 2\varepsilon)) \times B(x_i, 2\varepsilon)\). If \(|x - y| \leq \varepsilon\), then
\[ |x - x_i| \leq |x - y| + |y - x_i| \leq 3\varepsilon. \]
Thus,
\[
\int_{R^N \setminus B(x_i, 2\varepsilon)} \int_{\{y \in B(x_i, 2\varepsilon) : |x - y| \leq \varepsilon\}} \frac{u_n^2(x)(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x - y|^{N+2a}} \, dy \leq C\varepsilon^{-2} \int_{B(x_i, 3\varepsilon)} \int_{\{y \in B(x_i, 2\varepsilon) : |x - y| \leq \varepsilon\}} \frac{u_n^2(x)}{|x - y|^{N+2a-2}} \, dy \leq C\varepsilon^{-2a} \int_{B(x_i, 3\varepsilon)} u_n^2(x) \, dx.
\]
There exist \(k > 4\) such that
\[ (R^N \setminus B(x_i, 2\varepsilon)) \times B(x_i, 2\varepsilon) \subset (B(x_i, k\varepsilon) \times B(x_i, 2\varepsilon)) \cup \left( (R^N \setminus B(x_i, k\varepsilon)) \times B(x_i, 2\varepsilon) \right). \]
If \(|x - y| > \varepsilon\), we obtain
\[
\int_{B(x_i, k\varepsilon)} \int_{\{y \in B(x_i, 2\varepsilon) : |x - y| > \varepsilon\}} \frac{u_n^2(x)(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x - y|^{N+2a}} \, dy \leq C \int_{B(x_i, k\varepsilon)} \int_{\{y \in B(x_i, 2\varepsilon) : |x - y| > \varepsilon\}} \frac{u_n^2(x)}{|x - y|^{N+2a}} \, dy \leq C\varepsilon^{-2a} \int_{B(x_i, k\varepsilon)} u_n^2(x) \, dx.
\]
If \((x, y) \in (R^N \setminus B(x_i, k\varepsilon)) \times B(x_i, 2\varepsilon)\), we get
\[ |x - y| \geq |x - x_i| - |y - x_i| \geq \frac{|x - x_i|}{2} + \frac{k}{2}\varepsilon - 2\varepsilon > \frac{|x - x_i|}{2}, \]
which implies
\[
\int_{\mathbb{R}^N \setminus B(x, k\varepsilon)} dx \int_{\{y \in B(x, 2\varepsilon) : |x - y| > \varepsilon\}} \frac{u_n^2(x)(\varphi_n(x) - \varphi_n(y))^2}{|x - y|^{N + 2a}} dy \leq C \int_{\mathbb{R}^N \setminus B(x, k\varepsilon)} dx \int_{\{y \in B(x, 2\varepsilon) : |x - y| > \varepsilon\}} \frac{u_n^2(x)}{|x - y|^{N + 2a}} dy
\]
\[
\leq C\varepsilon^{-2a} \int_{B(x, \varepsilon)} u_n^2(x) dx + C\varepsilon^{-2a} \int_{B(x, 3\varepsilon)} u_n^2(x) dx + C\varepsilon^{-2a} \int_{B(x, \varepsilon)} u_n^2(x) dx + C \kappa^{-N} \left( \int_{\mathbb{R}^N \setminus B(x, k\varepsilon)} |u_n(x)|^{2^*_a} dx \right)^{\frac{2}{2^*_a}}
\]
\[
\leq C\varepsilon^{-2a} \int_{B(x, \varepsilon)} u_n^2(x) dx + C \kappa^{-N} \left( \int_{\mathbb{R}^N \setminus B(x, k\varepsilon)} |u_n(x)|^{2^*_a} dx \right)^{\frac{2}{2^*_a}}
\]
\[
\leq C\varepsilon^{-2a} \int_{B(x, \varepsilon)} u_n^2(x) dx + C \kappa^{-N}
\]
From cases 1–3, we have
\[
\int_{\mathbb{R}^N} u_n^2(x)(\varphi_n(x) - \varphi_n(y))^2 dx dy
\]
\[
= \int_{B(x, \varepsilon)} u_n^2(x) dx + \int_{B(x, 3\varepsilon)} u_n^2(x) dx + \int_{B(x, \varepsilon)} u_n^2(x) dx + C \kappa^{-N} \left( \int_{\mathbb{R}^N \setminus B(x, k\varepsilon)} |u_n(x)|^{2^*_a} dx \right)^{\frac{2}{2^*_a}}
\]
\[
\leq C\varepsilon^{-2a} \int_{B(x, \varepsilon)} u_n^2(x) dx + C \kappa^{-N} \left( \int_{\mathbb{R}^N \setminus B(x, k\varepsilon)} |u_n(x)|^{2^*_a} dx \right)^{\frac{2}{2^*_a}}
\]
Note that $u_n \to u$ weakly in $H^a(\mathbb{R}^N)$, thus $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, which implies
\[
Ce^{-2a} \int_{B(x, \varepsilon)} u_n^2(x) dx + C \kappa^{-N} \to Ce^{-2a} \int_{B(x, \varepsilon)} u^2(x) dx + C \kappa^{-N} \quad \text{as } n \to \infty.
\]
Then,
\[
Ce^{-2a} \int_{B(x, \varepsilon)} u^2(x) dx + C \kappa^{-N} \leq C\varepsilon^{-2a} \left( \int_{B(x, \varepsilon)} |u(x)|^{2^*_a} dx \right)^{\frac{2}{2^*_a}} \left( \int_{B(x, \varepsilon)} dx \right)^{1 - \frac{2}{2^*_a}} + C \kappa^{-N}
\]
\[
= C \kappa^{2a} \left( \int_{B(x, \varepsilon)} |u(x)|^{2^*_a} dx \right)^{\frac{2}{2^*_a}} + C \kappa^{-N} \to C \kappa^{-N} \quad \text{as } \varepsilon \to 0.
\]
We get
\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{u_n^2(x)(\varphi_n(x) - \varphi_n(y))^2}{|x - y|^{N + 2a}} dx dy = 0.
\]
Combining this with (3.11) and (3.13), we obtain that for any $i \in I$,
\[
\mu((x_i)) \leq Av_i,
\]
which proves the first assertion of the lemma.
For the second assertion, note that \(|u_n\chi_R|\) is bounded in \(H^a(\mathbb{R}^N)\), where \(\chi_R\) is from Lemma 3.5, thus

\[ \langle f'(u_n) - \lambda_n R'(u_n), u_n\chi_R \rangle \to 0 \quad \text{as} \quad n \to \infty, \]

which implies

\[ \int_{\mathbb{R}^N} (\Delta)^{\frac{a}{2}} u_n (\Delta)^{\frac{a}{2}} (u_n\chi_R) \, dx = \lambda_n \int_{\mathbb{R}^N} \left( \frac{1}{2} u_n^2 + \eta \lambda |u_n|^{2^*_a} + C\lambda |u_n|^r + |u_n|^{2^*_a} - u_n^2 \right) \chi_R \, dx + o(1). \quad (3.15) \]

By (3.12),

\[ \int_{\mathbb{R}^N} (\lambda f(u_n) u_n + |u_n|^{2^*_a} - u_n^2) \chi_R \, dx \leq \int_{\mathbb{R}^N} \left( \frac{1}{2} u_n^2 + \eta \lambda |u_n|^{2^*_a} + C\lambda |u_n|^r + |u_n|^{2^*_a} - u_n^2 \right) \chi_R \, dx \]

\[ \leq \int_{\mathbb{R}^N} (\eta \lambda |u_n|^{2^*_a} + |u_n|^{2^*_a} + C\lambda |u_n|^r) \chi_R \, dx. \]

Note that the embedding

\[ H^a(\mathbb{R}^N) \hookrightarrow L'(\mathbb{R}^N) \]

is compact, where \(r \in (2, 2^*_a)\). Hence, we get

\[ \int_{\mathbb{R}^N} C\lambda |u_n|^r \chi_R \, dx \to \int_{\mathbb{R}^N} C\lambda |u|^r \chi_R \, dx \quad \text{as} \quad n \to \infty \]

and

\[ \int_{\mathbb{R}^N} C\lambda |u| \chi_R \, dx \to 0 \quad \text{as} \quad R \to \infty. \]

Then,

\[ \limsup_{R \to \infty} \limsup_{n \to \infty} \lambda_n \int_{\mathbb{R}^N} (\lambda f(u_n) u_n + |u_n|^{2^*_a} - u_n^2) \chi_R \, dx \]

\[ \leq A(\eta \lambda + 1) \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*_a} \chi_R \, dx = A(\eta \lambda + 1) \nu_{\infty}. \]

If we let \(\eta \to 0\), then we get

\[ \limsup_{R \to \infty} \limsup_{n \to \infty} \lambda_n \int_{\mathbb{R}^N} (\lambda f(u_n) u_n + |u_n|^{2^*_a} - u_n^2) \chi_R \, dx \leq A\nu_{\infty}. \quad (3.16) \]

It is easy to verify that

\[ \int_{\mathbb{R}^N} (\Delta)^{\frac{a}{2}} u_n \cdot (\Delta)^{\frac{a}{2}} (u_n\chi_R) \, dx \]

\[ = \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))^2 \chi_R(y)}{|x - y|^{N+2a}} \, dx \, dy + \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\chi_R(x) - \chi_R(y))u_n(x)}{|x - y|^{N+2a}} \, dx \, dy \]

and

\[ \limsup_{R \to \infty} \limsup_{n \to \infty} \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))^2 \chi_R(y)}{|x - y|^{N+2a}} \, dx \, dy = \mu_{\infty}. \]

Moreover, we obtain

\[ \left| \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\chi_R(x) - \chi_R(y))u_n(x)}{|x - y|^{N+2a}} \, dx \, dy \right| \leq \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)| \cdot |\chi_R(x) - \chi_R(y)| \cdot |u_n(x)|}{|x - y|^{N+2a}} \, dx \, dy \]

\[ \leq C \left( \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)|\chi_R(x) - \chi_R(y)|^2}{|x - y|^{N+2a}} \, dx \, dy \right)^{\frac{1}{2}}. \]
Note that
\[
\limsup_{R \to \infty} \limsup_{n \to \infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\chi_R(x) - \chi_R(y))^2}{|x - y|^{N+2a}} \, dx \, dy = \limsup_{R \to \infty} \limsup_{n \to \infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)((1 - \chi_R(x)) - (1 - \chi_R(y)))^2}{|x - y|^{N+2a}} \, dx \, dy.
\]

Then, similarly to the proof of (3.14), we obtain
\[
\limsup_{R \to \infty} \limsup_{n \to \infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)((1 - \chi_R(x)) - (1 - \chi_R(y)))^2}{|x - y|^{N+2a}} \, dx \, dy = 0.
\]
Combining this with (3.15) and (3.16), we have
\[
\mu_{\infty} \leq A\nu_{\infty}.
\]

Lemma 3.7. For any \( i \in I \), we have that \( \psi_i \leq (S_a^{-1} \mu(x_i))^{\frac{2}{s}} \) and \( \nu_{\infty} \leq (S_a^{-1} \mu_{\infty})^{\frac{2}{s}} \).

Proof. It follows from (3.7) that
\[
\int_{\mathbb{R}^N} |u_n \varphi_\varepsilon|^2 \, dx \leq \left( S_a^{-1} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) \varphi_\varepsilon(x) - u_n(y) \varphi_\varepsilon(y)|^2}{|x - y|^{N+2a}} \, dx \, dy \right)^{\frac{2}{s}},
\]
where \( \varphi_\varepsilon \) is from Lemma 3.6. We have
\[
\int_{\mathbb{R}^N} |u_n \varphi_\varepsilon|^2 \, dx \to \int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, dv \quad \text{as } n \to \infty
\]
and
\[
\int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, dv \to \psi_i(x_i) = \psi_i \quad \text{as } \varepsilon \to 0.
\]
We obtain
\[
\iiint_{\mathbb{R}^{2N}} \frac{|u_n(x) \varphi_\varepsilon(x) - u_n(y) \varphi_\varepsilon(y)|^2}{|x - y|^{N+2a}} \, dx \, dy = \iiint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x - y|^{N+2a}} \, dx \, dy + \iiint_{\mathbb{R}^{2N}} \frac{\varphi_\varepsilon^2(y)(u_n(x) - u_n(y))^2}{|x - y|^{N+2a}} \, dx \, dy

+ \iiint_{\mathbb{R}^{2N}} 2u_n(x) \varphi_\varepsilon(y)(u_n(x) - u_n(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y)) \, dx \, dy.
\]
Note that
\[
\iiint_{\mathbb{R}^{2N}} \frac{\varphi_\varepsilon^2(y)(u_n(x) - u_n(y))^2}{|x - y|^{N+2a}} \, dx \, dy \to \int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, d\mu \quad \text{as } n \to \infty,
\]
\[
\left( \int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, d\mu \right) \to \mu(x_i) \quad \text{as } \varepsilon \to 0,
\]
and
\[
\left( \iiint_{\mathbb{R}^{2N}} \frac{u_n(x) \varphi_\varepsilon(y)(u_n(x) - u_n(y))(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))}{|x - y|^{N+2a}} \, dx \, dy \right) \leq C \left( \iiint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x - y|^{N+2a}} \, dx \, dy \right)^{\frac{1}{2}}.
\]
Similar to the proof of (3.14) in Lemma 3.6, we obtain
\[
\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \iiint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x - y|^{N+2a}} \, dx \, dy = 0.
\]
Thus, for any \( i \in I \), we obtain
\[
\psi_i \leq (S_a^{-1} \mu(x_i))^{\frac{2}{s}}.
\]
It follows from (3.7) that
\[
\int_{\mathbb{R}^N} |u_n\chi_R|^{2^*} \, dx \leq \left( S_a^{-1} \int_{\mathbb{R}^N} \frac{|u_n(x)\chi_R(x) - u_n(y)\chi_R(y)|^2}{|x - y|^{N+2a}} \, dx \, dy \right)^{\frac{2^*}{2}}.
\]
Hence, we have
\[
\limsup_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n\chi_R|^{2^*} \, dx = \nu_{\infty}.
\]
Note that
\[
\left( \int_{\mathbb{R}^{2N}} \frac{|u_n(x)\chi_R(x) - u_n(y)\chi_R(y)|^2}{|x - y|^{N+2a}} \, dx \, dy \right)^{\frac{2^*}{2}} = \int_{\mathbb{R}^N} \frac{u_n^2(x)(\chi_R(x) - \chi_R(y))^2}{|x - y|^{N+2a}} \, dx \, dy + \int_{\mathbb{R}^N} \frac{\chi_R^2(y)(u_n(x) - u_n(y))^2}{|x - y|^{N+2a}} \, dx \, dy
\]
\[
+ \int_{\mathbb{R}^N} \frac{2u_n(x)\chi_R(y)(u_n(x) - u_n(y))(\chi_R(x) - \chi_R(y))}{|x - y|^{N+2a}} \, dx \, dy.
\]
We obtain
\[
\limsup_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{\chi_R^2(y)(u_n(x) - u_n(y))^2}{|x - y|^{N+2a}} \, dx \, dy = \mu_{\infty}
\]
and
\[
\left| \int_{\mathbb{R}^{2N}} \frac{u_n(x)\chi_R(y)(u_n(x) - u_n(y))(\chi_R(x) - \chi_R(y))}{|x - y|^{N+2a}} \, dx \, dy \right| \leq C \left( \int_{\mathbb{R}^N} \frac{u_n^2(x)(\chi_R(x) - \chi_R(y))^2}{|x - y|^{N+2a}} \, dx \, dy \right)^{\frac{1}{2}}.
\]
Similarly, we obtain
\[
\limsup_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\chi_R(x) - \chi_R(y))^2}{|x - y|^{N+2a}} \, dx \, dy = 0.
\]
Then,
\[
\nu_{\infty} \leq (S_a^{-1} \mu_{\infty})^{\frac{2^*}{2}}. \tag{3.17}
\]
In the following theorem, by assuming that $\lambda$ is large, we obtain a nontrivial radially symmetric minimizer for problem (3.1).

**Theorem 3.8.** If
\[
\lambda > \left( \frac{N - 2a}{2N} \right)^{\frac{2}{2}} \frac{\alpha}{N} 2^{\frac{2a}{q}} a_\varepsilon \frac{2}{q - 2} C_\varepsilon^{\frac{2}{q}} \left( S_a \frac{2}{q} - 2 \right)^{\frac{q}{2}} \frac{q}{2},
\]
then problem (3.1) has a nontrivial minimizer $u \in H^a_0(\mathbb{R}^N)$, i.e.,
\[
A = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{a}{2}} u|^2 \, dx \quad \text{and} \quad \int_{\mathbb{R}^0} G(u) \, dx = 1.
\]

**Proof.** For any $i \in J$, we have $v_i = 0$ and $v_{\infty} = 0$. Suppose that there exists $i_0 \in J$ such that $v_{i_0} > 0$ or $v_{\infty} > 0$. Using Lemmas 3.6 and 3.7, we obtain
\[
v_{i_0} \leq (S_a^{-1} \mu_{(i_0)})^{\frac{2}{2}} \leq (S_a^{-1} A v_{i_0})^{\frac{2}{2}} \quad \text{or} \quad v_{\infty} \leq (S_a^{-1} \mu_{\infty})^{\frac{2}{2}} \leq (S_a^{-1} A v_{\infty})^{\frac{2}{2}},
\]
which implies
\[
v_{i_0} \geq (S_a A^{-1})^{\frac{2}{2^* - 2}} \quad \text{or} \quad v_{\infty} \geq (S_a A^{-1})^{\frac{2}{2^* - 2}}, \tag{3.18}
\]
As Lemma /three.tf./four.tf, we have
\[
\int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \leq \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \leq \left( S_a^{-1} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2a}} \, dx \, dy \right)^{\frac{2^*}{2}}
\]
and
\[
\int_{\mathbb{R}^N} |u_n|^{2^*} \chi_R \, dx \leq \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \leq \left( S_a^{-1} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2a}} \, dx \, dy \right)^{\frac{2^*}{2}},
\]
which implies
\[
v_{i_0} = \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \leq (2S_a^{-1} A)^{\frac{N}{2N+2a}}
\]
and
\[
v_{\infty} = \limsup_{n \to \infty} \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} |u_n|^{2^*} \chi_R \, dx \leq (2S_a^{-1} A)^{\frac{N}{2N+2a}}.
\]
Combining with (3.17) and (3.18), we get
\[
(S_a A^{-1})^{\frac{N}{2N+2a}} \leq (2S_a^{-1} A)^{\frac{N}{2N+2a}},
\]
i.e.,
\[
2S_a^{-1} A^{\frac{N}{2N+2a}} \geq 1. \tag{3.19}
\]
By Lemma 3.4, we have
\[
b \leq \frac{q - 2}{2q} \lambda^{\frac{2a}{q}} C_q^{\frac{2\lambda}{q}}.
\]
If
\[
\lambda > \left( \frac{N - 2a}{2N} \right)^{\frac{2\lambda}{q}} C_q^{\frac{2\lambda}{q}} A^{\frac{\lambda}{N}} \geq \left( \frac{N - 2a}{2N} \right)^{\frac{2\lambda}{q}} C_q^{\frac{2\lambda}{q}} S_a^{-1} \frac{2q}{q - 2} C_q^{-\frac{2\lambda}{q}}^{1} \lambda^\frac{2\lambda}{q},
\]
then it follows that
\[
b \left( \frac{N - 2a}{2N} \right)^{\frac{2\lambda}{q}} C_q^{\frac{2\lambda}{q}} A^{\frac{\lambda}{N}} \geq \left( \frac{N - 2a}{2N} \right)^{\frac{2\lambda}{q}} C_q^{\frac{2\lambda}{q}} S_a^{-1} \frac{2q}{q - 2} C_q^{-\frac{2\lambda}{q}}^{1} \lambda^\frac{2\lambda}{q},
\]
which implies
\[
b \geq \left( \frac{N - 2a}{2N} \right)^{\frac{2\lambda}{q}} C_q^{\frac{2\lambda}{q}} A^{\frac{\lambda}{N}} \geq \left( \frac{N - 2a}{2N} \right)^{\frac{2\lambda}{q}} C_q^{\frac{2\lambda}{q}} S_a^{-1} \frac{2q}{q - 2} C_q^{-\frac{2\lambda}{q}}^{1} \lambda^\frac{2\lambda}{q},
\]
From Lemma 3.3 and (3.19),
\[
b \geq \frac{2\lambda}{N} \left( \frac{N - 2a}{2N} \right)^{\frac{2\lambda}{q}} A^{\frac{\lambda}{N}} \geq \left( \frac{N - 2a}{2N} \right)^{\frac{2\lambda}{q}} A^{\frac{\lambda}{N}} \frac{2q}{q - 2} C_q^{-\frac{2\lambda}{q}}^{1} \lambda^\frac{2\lambda}{q}.
\]
That is a contradiction. Thus, for any \( i \in J \), we have \( v_i = 0 \) and \( v_{\infty} = 0 \).
Using (3.10) we obtain
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx = \int_{\mathbb{R}^N} |u|^{2^*} \, dx.
\]
As \( |u_n - u|^{2^*} \leq 2^{2^*} (|u_n|^{2^*} + |u|^{2^*}) \), it follows from Fatou's lemma that
\[
\int_{\mathbb{R}^N} 2^{2^*+1} |u|^{2^*} \, dx = \int_{\mathbb{R}^N} \liminf_{n \to \infty} (2^{2^*} |u_n|^{2^*} + 2^{2^*} |u|^{2^*} - |u_n - u|^{2^*}) \, dx
\]
\[
\leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} (2^{2^*} |u_n|^{2^*} + 2^{2^*} |u|^{2^*} - |u_n - u|^{2^*}) \, dx
\]
\[
= 2^{2^*+1} \int_{\mathbb{R}^N} |u|^{2^*} \, dx - \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n - u|^{2^*} \, dx,
\]
which implies
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n - u|^{2s} \, dx = 0. \]

Thus,
\[ u_n \to u \quad \text{in} \quad L^{2s}(\mathbb{R}^N) \quad \text{as} \quad n \to \infty. \]

Passing to a subsequence, still denoted \{u_n\}, we may assume that there exists \(0 \leq h \in L^{2s}(\mathbb{R}^N)\) such that
\[ |u_n(x)| \leq h(x) \ \text{a.e.} \ x \in \mathbb{R}^N. \]
Using (3.3),
\[ G(u_n) = AF(u_n) + \frac{1}{2a} |u_n|^{2s} - \frac{1}{2} u_n^2 \leq C |u_n|^{2s} \leq Ch^{2s}. \]

It follows from the Lebesgue’s dominated convergence theorem that
\[ \int_{\mathbb{R}^N} G(u_n) \, dx \to \int_{\mathbb{R}^N} G(u) \, dx \quad \text{as} \quad n \to \infty. \]

Then,
\[ \int_{\mathbb{R}^N} G(u) \, dx = 1, \]
which implies
\[ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx \geq A. \]

Note that
\[ \lim_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx, \]
then \( \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx = A > 0. \)

Next, using radially decreasing rearrangements of \{u_n\}, we will verify that the minimizer in \(H^s(\mathbb{R}^N)\) for \(A\) is also a minimizer in \(H^s(\mathbb{R}^N)\).

**Lemma 3.9.** Define
\[ B = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx : u \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\}. \]

Then, \(A = B\) and \(u\) from Theorem 3.8 is also a nontrivial minimizer of \(B\).

**Proof.** It is easy to verify that \(B \leq A\). We will verify that \(A \leq B\). Let \{u_n\} \(\subset H^s(\mathbb{R}^N)\) such that
\[ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx \to B \quad \text{as} \quad n \to \infty \]
and
\[ \int_{\mathbb{R}^N} G(u_n) \, dx = 1. \]

Let \(u_n^*\) be the symmetric radial decreasing rearrangement of \(u_n\). Using [17, Lemma 2.3], we have
\[ \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n^*|^2 \, dx \leq \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx \]
and
\[ \int_{\mathbb{R}^N} G(u_n^*) \, dx = \int_{\mathbb{R}^N} G(u_n) \, dx = 1, \]
which implies
\[ A \leq \liminf_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n^*|^2 \, dx \leq \limsup_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx \leq \limsup_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx = B. \]

Thus, \(A = B\).  

\[ \square \]
Finally, we obtain that the minimizer for \( A \) under a scale change is a ground state solution for (1.1).

**Proof of Theorem 1.3.** We claim that problem (1.1) has a nonnegative radially symmetric ground state solution \( w \) and

\[
I(w) = P = \frac{2\alpha}{N} \left( \frac{N - 2\alpha}{N} \right)^{\frac{N-2\alpha}{2\alpha}} A^\frac{N}{2}.
\]

The proof is similar to that of [8, Theorem 3]. Here we would like to give a detailed account for the reader’s convenience.

Suppose \( u \) is the minimizer of \( B \), then there exists \( \theta \in \mathbb{R} \) such that

\[
J'(u) = \theta K'(u) \quad \text{in} \quad (H^s_0(\mathbb{R}^N)',
\]

where \((H^s_0(\mathbb{R}^N)')\) is the dual space of \((H^s_0(\mathbb{R}^N))\). First, we will verify that \( \theta > 0 \). In fact, if \( \theta = 0 \), then \( J'(u) = 0 \), which implies

\[
\langle J'(u), u \rangle = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy = 0.
\]

That is a contradiction. If \( \theta < 0 \), then using (H4) we have

\[
\langle K'(u), u \rangle \geq \int_{\mathbb{R}^N} \left( 2F(u) + \frac{2\alpha}{N} |u|^{2_*} - u^2 \right) \, dx = 2,
\]

which implies \( K'(u) \neq 0 \). Hence, there exists \( w_0 \in C^\infty_0(\mathbb{R}^N) \) such that

\[
\langle K'(u), w_0 \rangle > 0.
\]

If on the contrary \( \langle K'(u), w \rangle \leq 0 \) for any \( w \in C^\infty_0(\mathbb{R}^N) \), then we take \( w_1, w_2 \in C^\infty_0(\mathbb{R}^N) \), \( t_1 < 0 \) and \( t_2 > 0 \), and have \( \langle K'(u), t_1 w_1 \rangle + \langle K'(u), t_2 w_2 \rangle > 0 \) for \( t_1 \) small enough. That is a contradiction. Then,

\[
K(u + \varepsilon w_0) - K(u) = \int_{\mathbb{R}^N} G(u + \varepsilon w_0) \, dx - \int_{\mathbb{R}^N} G(u) \, dx = \varepsilon \int_{\mathbb{R}^N} g(u + \tau_1 \varepsilon w_0) w_0 \, dx,
\]

where \( 0 < \tau_1 < 1 \). It follows from Lebesgue’s dominated convergence theorem that

\[
\int_{\mathbb{R}^N} g(u + \tau_1 \varepsilon w_0) w_0 \, dx \to \int_{\mathbb{R}^N} g(u) w_0 \, dx \quad \text{as} \ \varepsilon \to 0.
\]

Then, there exists \( \varepsilon_1 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_1 \),

\[
\int_{\mathbb{R}^N} g(u + \tau_1 \varepsilon w_0) w_0 \, dx > 0,
\]

i.e., \( K(u + \varepsilon w_0) > K(u) \), which implies

\[
K(u + \varepsilon w_0) > 1.
\]

We obtain

\[
J(u + \varepsilon w_0) - J(u) = \langle J'(u + \tau_2 \varepsilon w_0), \varepsilon w_0 \rangle,
\]

where \( 0 < \tau_2 < 1 \) and

\[
\langle J'(u + \tau_2 \varepsilon w_0), \varepsilon w_0 \rangle = \varepsilon \langle J'(u), w_0 \rangle + \tau_2 \varepsilon^2 \int_{\mathbb{R}^N} \frac{(w_0(x) - w_0(y))^2}{|x - y|^{N+2\alpha}} \, dx \, dy.
\]

Then,

\[
J(u + \varepsilon w_0) - J(u) = \varepsilon \theta \langle K'(u), w_0 \rangle + \tau_2 \varepsilon^2 \int_{\mathbb{R}^N} \frac{(w_0(x) - w_0(y))^2}{|x - y|^{N+2\alpha}} \, dx \, dy.
\]

Using (3.21), there exists \( 0 < \varepsilon_2 < \varepsilon_1 \) such that for any \( 0 < \varepsilon < \varepsilon_2 \), we have \( J(u + \varepsilon w_0) - J(u) < 0 \) and thus

\[
J(u + \varepsilon w_0) < J(u) = A.
\]
Denote \( v = u + \varepsilon w_0 \) and \( v_0 = v(\frac{\varepsilon}{\alpha}) \), where \( \alpha > 0 \). We have \( K(v) = \int_{\mathbb{R}^N} G(v(x)) \, dx > 1 \) and \( J(v) < A \). If

\[
K(v_0) = \int_{\mathbb{R}^N} G(v_0) \, dx = \int_{\mathbb{R}^N} G(v(x)) \sigma^N \, dx = 1,
\]

we get \( 0 < \sigma < 1 \). Then,

\[
J(v_0) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|v_0(x) - v_0(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy = \sigma^{N-2\alpha} J(v) < A.
\]

That is a contradiction, hence \( \theta > 0 \).

Next, we will verify that under a scale change \( u \) is a ground state solution for (1.1). Using (3.20), we obtain that \( u \) is a positive solution of

\[
(-\Delta)^{\frac{\alpha}{2}} u = \theta g(u),
\]

which implies

\[
\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} \, dx \, dy = \frac{\theta}{N-2\alpha} J(u) = \frac{\theta}{N-2\alpha} K(u).
\]

By the Pohozaev identity (3.8), we have \( J(u_0) = \frac{N}{N-2\alpha} K(u_0) \). Thus

\[
\theta^{\frac{N-2\alpha}{N}} J(u) = \frac{N}{N-2\alpha} \theta^{\frac{N}{N-2\alpha}} K(u) = \frac{N}{N-2\alpha} \theta^{\frac{N}{N-2\alpha}},
\]

i.e., \( \theta = \frac{N-2\alpha}{N} J(u) \), which implies

\[
I(u_0) = J(u_0) - K(u_0) = \frac{2\alpha}{N} \theta^{\frac{N-2\alpha}{N}} J(u) = \frac{2\alpha}{N} \left( \frac{N-2\alpha}{N} \right)^{\frac{N-2\alpha}{N}} J(u)^{\frac{N}{N-2\alpha}}.
\]

Let \( v \) be the solution of (3.22). Then,

\[
J(v) = \frac{N}{N-2\alpha} K(v).
\]

Taking \( \sigma_1 = K(v)^{-\frac{1}{\alpha}} = (\frac{N-2\alpha}{N} J(v))^{-\frac{1}{\alpha}} \) and \( v_{\sigma_1} = v(\frac{1}{\sigma_1}) \), we obtain

\[
J(v_{\sigma_1}) = \sigma_1^{N-2\alpha} J(v) = \left( \frac{N-2\alpha}{N} \right)^{\frac{N-2\alpha}{N}} (J(v))^\frac{N}{N-2\alpha}
\]

and

\[
K(v_{\sigma_1}) = \sigma_1^N K(v) = 1.
\]

Then,

\[
I(v) = \frac{2\alpha}{N} \left( \frac{N-2\alpha}{N} \right)^{\frac{N-2\alpha}{N}} J(v_{\sigma_1})^{\frac{N}{N-2\alpha}} \geq \frac{2\alpha}{N} \left( \frac{N-2\alpha}{N} \right)^{\frac{N-2\alpha}{N}} (J(u))^\frac{N}{N-2\alpha} = I(u_0),
\]

which implies \( u_0 \) is the least energy solution. By Lemma 3.2, we get

\[
I(u_0) = \frac{2\alpha}{N} \left( \frac{N-2\alpha}{N} \right)^{\frac{N-2\alpha}{N}} A^{\frac{N}{N-2\alpha}} = P.
\]

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