Research Article

Olivier Goubet* and Emna Hamraoui

Blow-up of solutions to cubic nonlinear Schrödinger equations with defect: The radial case

DOI: 10.1515/anona-2016-0238
Received November 7, 2016; accepted November 7, 2016

Abstract: In this article we investigate both numerically and theoretically the influence of a defect on the blow-up of radial solutions to a cubic NLS equation in dimension 2.

Keywords: Cubic NLS equation, blow-up, defect

MSC 2010: 35Q55, 35B44

1 Introduction

The issue of the existence of blow-up solutions for nonlinear Schrödinger (NLS) equations in $\mathbb{R}^2$ has widely been investigated in the literature (see [3, 16] and the references therein). These equations read

$$iu_t = \Delta u + |u|^{2\sigma} u,$$

supplemented with initial data in $H^1(\mathbb{R}^2)$. For $\sigma < 1$ the solutions are global in time. Then the so-called cubic NLS equation $\sigma = 1$ is critical in $H^1$. In fact, there exists solutions of the cubic NLS equation that blow up in finite time. This can be established for instance by the so-called Glassey’s virial method [9]. Conversely, a famous result of Weinstein [17] asserts that any solution whose mass is less than the mass of the ground state is global in time. For the existence and properties of the ground state see [1, 4, 5, 13]. Actually, consider $C_{GN}$ the best constant in the Gagliardo–Nirenberg inequality

$$\|u\|_{L^6}^6 \leq C_{GN} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$

Then $C_{GN} = 2/\|Q\|_{L^2}^2$, where $\|Q\|_{L^2}$ is the mass of the ground state and if $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution starting from $u_0$ cannot blow up in finite time ($\|Q\|_{L^2}^2(\mathbb{R}^2) = 2\pi \times 1.86225 \ldots$, see [17]). This is easy to check observing that the NLS equation has two invariants that are respectively the mass $\|u(t)\|_{L^2}$ and the energy

$$E(t) = \|\nabla u\|_{L^2}^2 - \frac{1}{2} \|u\|_{L^4}^4. \quad (1)$$

A point defect has been introduced and studied for NLS equations in dimension 2 in [7, 8, 11, 14]. In this article we are concerned with the blow-up of radial solutions to a cubic nonlinear Schrödinger equation with
a radial defect, located on the sphere of radius $r_0$. The equation reads
\[
\frac{1}{i} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial r^2} - \frac{1}{r} \frac{\partial v}{\partial r} - Z v \delta_{r_0} - |v|^2 v = 0, \quad r > 0, \quad t \geq 0, \\
v(0, r) = v_0(r), \quad r \geq 0, \\
\frac{\partial v}{\partial r}(t, 0) = 0, \quad t \geq 0.
\] (2)

Here the unknown $v$ depends only on the distance $r$ to the origin, and the defect is modeled by the term $Z v \delta_{r_0}$. The real number $Z$ is the amplitude of the defect, and $\delta_{r_0}$ is the usual delta measure at $r = r_0$. Moreover, the scaling $v(t, r) = r_0 v(\frac{t}{r_0^2}, \frac{r}{r_0})$ (changing accordingly the value of $|Z|$) allows us to focus on the case $r_0 = 1$.

The rest of the article is organized as follows. In a second section we introduce the mathematical framework associated with equation (2) and we investigate the initial value problem, discussing the role of the Gagliardo–Nirenberg inequality in our case. In a third section we revisit the virial method. In a fourth section, we investigate numerically how a defect can affect the behavior of explosive solutions.

We now introduce some notations. We denote by $L^2_{\text{rad}}$ (or simply $L^2$) the set of functions $v : [0, +\infty[ \to \mathbb{C}$ measurable such that
\[
\|v\|_{L^2}^2 = \int_0^{+\infty} r |v(r)|^2 \, dr < +\infty.
\]

We denote by $H^1_{\text{rad}}$ (or simply $H^1$) the set of radial functions such that
\[
\|v\|_{H^1}^2 = \int_0^{+\infty} r(|v|^2 + |v_t|^2) \, dr < +\infty.
\]

We also define here the invariants respectively for the mass as
\[
M(t) = \int_0^{+\infty} r |v(r, t)|^2 \, dr,
\]
and respectively for the energy as
\[
E(t) = \left\| \frac{\partial v}{\partial r}(t) \right\|_{L^2_{\text{rad}}}^2 - Z |v(t, 1)|^2 - \frac{1}{2} \|v(t)\|_{L^2_{\text{rad}}}^4.
\]

We have divided by $2\pi$ the quantities defined above (see (1)) for the sake of convenience. Let us observe that there is an extra term while $Z \neq 0$. Moreover, if a function $v$ is in $H^1_{\text{rad}}$, then $v$ is continuous in $(0, +\infty)$ and $v(1) = \langle v, \delta_1 \rangle$ makes sense. This is valid due to the following lemma

**Lemma 1.1.** Any $v$ in $H^1_{\text{rad}}(\mathbb{R}^2)$ is a continuous function for $r > 0$ that satisfies
\[
\sqrt{\tau} |v(r)| \leq \|v\|_{H^1_{\text{rad}}}. \tag{3}
\]

**Proof.** Consider first $v$ a smooth compactly supported radial function. Equality (3) holds true integrating $\partial_r |v|^2 = 2 \text{Re}(\bar{v} v_t)$ between $r$ and $+\infty$ and using the Cauchy–Schwarz inequality. We then conclude by a density argument: if $v_k \in C^\infty_{\text{rad}, 0}$ converges towards $v$ in $H^1$, then the sequence $\sqrt{\tau} v_k(r)$ converges uniformly towards $\sqrt{\tau} v(r)$.

## 2 The initial value problem

In this section, we address the issue of the existence of solutions to (2) in $C([0, T), H^1_{\text{rad}}) \cap C^1([0, T); H^{-1}_{\text{rad}})$.

### 2.1 The mathematical framework

We now introduce a mathematical setting that allow us to address the defect as a transmission problem.
Let \( a_1 \) be the bilinear form in \( H^1_{\text{rad}} \) defined as

\[
  a_1(v, w) = \text{Re} \left( \int_0^{+\infty} \frac{\partial v}{\partial t}(t) \frac{\partial w}{\partial r} r \, dr \right) - Z \text{Re}(v(t, 1)\overline{w}(1)) \quad \text{for all } v, w \in H^1_{\text{rad}}.
\]

Then we state and prove the following lemma.

**Lemma 2.1.** The bilinear form \( a_1(\cdot, \cdot) \) is continuous and symmetric in \( H^1_{\text{rad}} \).

**Proof.** For all \( v, w \in H^1_{\text{rad}} \),

\[
  |a_1(v, w)| \leq \left\| \frac{\partial v}{\partial t} \right\|_{L^2_{\text{rad}}} \left\| \frac{\partial w}{\partial t} \right\|_{L^2_{\text{rad}}} + |Z|\|v(t, 1)\|\|w(t, 1)\|.
\]

We recall that for any fixed \( r \) we have

\[
  r|v(r)|^2 \leq \|v\|^2_{H^1_{\text{rad}}}
\]

We then have

\[
  |a_1(v, w)| \leq \left\| \frac{\partial v}{\partial t} \right\|_{L^2_{\text{rad}}} \left\| \frac{\partial w}{\partial t} \right\|_{L^2_{\text{rad}}} + |Z|\|v\|_{H^1_{\text{rad}}} \|w\|_{H^1_{\text{rad}}} \leq (1 + |Z|)\|v\|_{H^1_{\text{rad}}} \|w\|_{H^1_{\text{rad}}}.
\]

This completes the proof of the lemma.

**Proposition 2.2.** There exists \( A_1 \) a unbounded self-adjoint operator in \( H^1_{\text{rad}} \) such that

\[
  a_1(v, w) = \langle A_1 v, w \rangle_{H^{-1}_{\text{rad}}, H^1_{\text{rad}}}.
\]

**Proof.** Due to the proof of Lemma 2.1 for \( \lambda > 0 \) large enough the bilinear for \( b_1(v, w) = a_1(v, w) + \lambda(v, w)_{L^2_{\text{rad}}} \) is coercive, continuous and symmetric. The Lax–Milgram theorem applies and for any \( \lambda \) there exists a unique \( v \in H^1_{\text{rad}} \) such that \( b_1(v, w) = \langle f, w \rangle \) for all \( w \in H^1 \). We define \( B \) as the maximal monotone operator such that \( b_1(v, w) = \langle Bw, v \rangle \) and define \( A_1 \) as \( A_1 = B - \lambda \text{Id} \). Then we also have

\[
  a_1(v, w) = \langle A_1 v, w \rangle_{H^{-1}_{\text{rad}}, H^1_{\text{rad}}}.
\]

This completes the proof of the proposition.

We now characterize the domain of \( A_1 \). We state:

**Proposition 2.3.** The domain of \( A_1 \) is

\[
  D(A_1) = \left\{ v \in H^1_{\text{rad}} : \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \in L^2_{\text{rad}}(0, 1) \cap L^2(1, +\infty) \quad \text{and} \quad \frac{\partial^2 v}{\partial r^2}(t, 1^+) - \frac{\partial v}{\partial r}(t, 1^-) = -Zv(t, 1) \right\}.
\]

**Proof.** Consider a test radial function \( w \in C_0^\infty \). We seek \( v \) in \( H^1 \) such that for any such \( w ",

\[
  \left| \text{Re} \left( \int_0^{+\infty} r \frac{\partial v}{\partial t}(t) \frac{\partial w}{\partial r} r \, dr \right) - Z \text{Re}(v(t, 1)\overline{w}(1)) \right| \leq \|w\|_{L^2_{\text{rad}}}.
\]

Consider first \( w \) that vanishes at a neighborhood of \( r = 1 \). We then have

\[
  \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \in L^2_{\text{rad}}(0, 1) \cap L^2(1, +\infty).
\]

Therefore the derivative of \( v \) has traces at \( r = 1, r < 1 \) and \( r = 1, r > 1 \) (see [2]). We consider now a general \( w \in H^1_{\text{rad}} \). Integrating by parts, we have

\[
  \int_0^1 \frac{\partial v}{\partial t}(t) \frac{\partial w}{\partial r} r \, dr = -\int_0^1 \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial v}{\partial r} \right) r w \, dr + \frac{\partial v}{\partial r}(t, 1^-)w(t, 1),
\]
and
\[
\int_{1}^{+\infty} \frac{\partial v}{\partial t}(t) \frac{\partial w}{\partial r} r \, dr = - \int_{1}^{\infty} \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial v}{\partial r}) w \, dr - \frac{\partial v}{\partial r}(t, 1^+) w(t, 1).
\]

Introducing
\[
[v_r]_1 = \frac{\partial v}{\partial r}(t, 1^+) - \frac{\partial v}{\partial r}(t, 1^-),
\]
we thus obtain
\[
\left| - \text{Re} \left( \int_{0}^{+\infty} \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial v}{\partial r}) w \, dr - \bar{w}(t, 1)(Z v(1) + [v_r]_1) \right) \right| \leq C \| w \|_{L^2_{\text{rad}}}.
\]

Since this is valid for any \( w \), we infer the transmission condition
\[
Z v(1) + [v_r]_1 = 0. \tag{4}
\]

The proof of the proposition is complete. \( \square \)

We now have enough material to handle the Initial Value Problem.

**Proposition 2.4.** For any \( \nu_0 \) in \( H^1_{\text{rad}} \) there exist at \( T > 0 \) and a unique solution of nonlinear Schrödinger equation (2) in \( C([0, T); H^1_{\text{rad}}) \cap C^1([0, T); H^{-1}) \). If moreover \( \nu_0 \) belongs to \( D(A_1) \), then the solution remains in \( D(A_1) \) for \( t < T \).

**Proof.** For the uniqueness of solutions, we rely on a famous argument due to Vladimirov. To begin with, we recall the Trudinger inequality (written here for radial functions) [2]. For \( M > 0 \), there exist \( \mu, K > 0 \) such that if \( \| \nu \|_{H^1_{\text{rad}}(\mathbb{R}^2)} < M \), then
\[
\int_{0}^{+\infty} \left( \exp(\mu |\nu(r)|^2) - 1 \right)^2 r \, dr \leq K^2.
\]

Let \( v(t) \) and \( \bar{v}(t) \) be two solutions of (2) starting from \( v(0) \). Introduce \( M = 8 \sup_{[0, T_0]} (\| v \|_{H^1} + \| \bar{v} \|_{H^1}) \) for \( T_0 < T \). Setting \( w(t) = v(t) - \bar{v}(t) \), we see that \( w(t) \) satisfies
\[
i \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial r^2} - \frac{1}{r} \frac{\partial w}{\partial r} - Z w \delta_{r_0} - (|v|^2 v - |\bar{v}|^2 \bar{v}) = 0, \quad r > 0, \; t \geq 0,
\]
\[
w(0, r) = 0, \quad r \geq 0. \tag{5}
\]

Considering the scalar product of (5) with \( iw(t) \), we then have
\[
\frac{1}{2} \frac{d}{dt} \| w(t) \|_{L^2_{\text{rad}}}^2 = \text{Im} \int_{0}^{+\infty} r(|v|^2 v - |\bar{v}|^2 \bar{v}) w(t) \, dr \leq \int_{0}^{+\infty} r|\bar{v}|(|\bar{v}| + |v|)|w|^2 \, dr,
\]
so
\[
\frac{d}{dt} \| w(t) \|_{L^2_{\text{rad}}}^2 \leq 4 \int_{0}^{+\infty} r(|v|^2 + |\bar{v}|^2)|w|^2 \, dr.
\]

Let us introduce the function \( h^2(x) = |\bar{v}(x)|^2 + |v(x)|^2 \). Thanks to the Hölder inequality for \( p > 2 \), we then have
\[
\frac{d}{dt} \| w(t) \|_{L^2_{\text{rad}}}^2 \leq 4 \int_{0}^{+\infty} \| h \|_{L^2_{\text{rad}}}^2 \| w \|_{L^2_{\text{rad}}}^2 \| w \|_{L^2_{\text{rad}}}^2 \, dr \leq 4 \int_{0}^{+\infty} \left( \| h \|^p_{L^p_{\text{rad}}} \| w \|^2_{L^2_{\text{rad}}} \right)^{\frac{2}{p}} \leq 4 \| w \|_{L^2_{\text{rad}}} \| \bar{w} \|_{L^2_{\text{rad}}} \| h \|_{L^2_{\text{rad}}}^2.
\]

On one hand, using the elementary inequality \( a^2 \leq \left( \frac{b}{p} \right)^p (\exp(\mu a^2) - 1) \), we have
\[
\| h \|_{L^p_{\text{rad}}}^2 \leq \frac{\mu}{\mu} K \frac{2}{p} \int_0^1 \left( \| h \|_{L^2_{\text{rad}}}^2 \right)^2 \, dr,
\]
since the \( H^1 \)-norm of \( h \) is bounded by \( M \). On the other hand, due to the embedding \( H^1_{\text{rad}} \subset L^4_{\text{rad}} \), we have
\[
\| v \|_{L^4_{\text{rad}}} \leq C \| v \|_{H^1_{\text{rad}}}.
\]
So, gathering the previous inequalities, we obtain
\[
\frac{d}{dt} \|w(t)\|^2_{L^2_{\text{rad}}} \leq CK^2 \|w\|^2_{L^2_{\text{rad}}}.
\] (6)
By integrating (6) between 0 and $T$, we have
\[
\|w(t)\|^2_{L^2_{\text{rad}}} \leq (aT)^2.
\]
For $T$ small enough such that $aT < 1$, $T < T_0$ and $p \to \infty$ we have
\[
\|w(t)\|^2_{L^2_{\text{rad}}} = 0
\]
in $[0, T]$, and hence the uniqueness of solutions, since we can iterate this argument on $[T, 2T]$, and then in $[kT, (k + 1)T]$ for any $k$.

For the existence result, the difficulty is that we cannot use Strichartz estimates due to the defect. We use instead the regularization method described in [3, Section 3.3]. Setting $F(v) = |v|^2v$, let us recall that the equation reads in its abstract form
\[
iv_t = Av + F(v).
\]

**First step: Shifting.** The operator $A$ is not positive. We overcome this difficulty considering
\[
w(t) = \exp(-i\lambda t)v(t)
\]
that is solution to
\[
iv_t = (A + \lambda I)w + F(w).
\] (7)
We know that, for $\lambda$ large enough, $B = A + \lambda I$ is a positive symmetric unbounded operator such that
\[
D(B^{\frac{3}{2}}) = H^1_{\text{rad}}.
\]

**Second step: Regularizing the nonlinearity.** We introduce for $\varepsilon > 0$ the operator $J_\varepsilon = (I + \varepsilon B)^{-1}$. We set $F_\varepsilon(v) = J_\varepsilon F(J_\varepsilon v)$. Then we have that $F_\varepsilon$ is a locally Lipschitz map from $H^1_{\text{rad}}$ into $H^1_{\text{rad}}$ uniformly with respect to $\varepsilon$. Actually, if $v$ and $w$ belong to some bounded set of $H^1_{\text{rad}}$
\[
\|F_\varepsilon(v) - F_\varepsilon(w)\|_{H^1} \leq \|F(J_\varepsilon v) - F(J_\varepsilon w)\|_{H^1} \leq C\|J_\varepsilon(v - w)\|_{H^1} \leq C\|v - w\|_{H^1}.
\]

**Third step: Construction of an approximate solution.** We now perform a fixed point in $C([0, T]; H^1_{\text{rad}})$ for the Duhamel’s form of the equation that reads
\[
w^\varepsilon(t) = e^{-iHt}v_0 + \int_0^t e^{-i(s-t)H}F_\varepsilon(w^\varepsilon(s)) \, ds.
\] (8)
This is standard and omitted for the sake of conciseness. It is worth to point out that since the nonlinearity is uniformly locally Lipschitz in $H^1_{\text{rad}} = D(B^{\frac{3}{2}})$, the time $T$ does not depend on $\varepsilon$. Moreover, the solution $w^\varepsilon$ belongs to $C([0, T]; H^1_{\text{rad}}) \cap C^1([0, T]; H^{-1}_{\text{rad}})$ and satisfies, going back to $v^\varepsilon = \exp(i\lambda t)w^\varepsilon$,
\[
iv^\varepsilon_t = Aw^\varepsilon + F_\varepsilon(v^\varepsilon) = 0.
\] (9)

**Fourth step: A priori estimates.** We already know that the sequence $v^\varepsilon$ is uniformly bounded in the space $C([0, T]; H^1_{\text{rad}}) \cap C^1([0, T]; H^{-1}_{\text{rad}})$. Since the embedding $H^1_{\text{rad}} \subset L^4_{\text{rad}}$ is compact (see [12]), we can extract a subsequence still denoted by $v^\varepsilon$ that converges to $v$ in $L^4((0, T); L^4_{\text{rad}})$ weak-star and strongly in $L^6((0, T); L^4_{\text{rad}})$ and such that $v^\varepsilon_t$ converges to $v_t$ in $L^6((0, T); H^{-1}_{\text{rad}})$ weak-star. We also have that some invariants are conserved. Since $\text{Im}(F_\varepsilon(v), v) = 0$, the mass $\|v^\varepsilon(t)\|^2_{L^2_{\text{rad}}} = \|v_0\|^2_{L^2_{\text{rad}}}$ is constant. We also have that the modified energy $E_\varepsilon(v) = (Av, v) - \frac{1}{2}\|J_\varepsilon v\|^4_{L^8_{\text{rad}}}$ is conserved along the trajectories.
Fifth step: Passing to the limit. Observing that for any given \( v \) in \( H^1_{\text{rad}} \),
\[
\| F_c(v) - F(v) \|_{H^{1/2}} \leq c (\| F'(v) - F(v) \|_{L^2} + \| (J \varepsilon - \text{Id}) F(v) \|_{H^{1/2}}) \leq K (\| J \varepsilon v - v \|_{H^{1/2}} + \| (J \varepsilon - \text{Id}) F(v) \|_{H^{1/2}}),
\]
it is standard to pass to the limit either in (9) and (8) to have a solution \( v \) in \( L^\infty([0, T]; H^1_{\text{rad}}) \) (and then continuous in time due to (8)) of the equation. We can also pass to the limit in the invariant.

Sixth step: Miscellaneous results. Proceeding as in [3, Section 3.3], we can prove that the solution depends continuously on the initial data and the existence of a maximal time of existence \( T_{\text{max}} \) such that if \( T < +\infty \), then the solution blows up.

We complete the proof of the theorem by proving that if the initial data belongs to \( D(A_1) \), then the solution remains in \( D(A_1) \). Assume \( v_0 \) in \( D(A_1) \). Consider a solution \( w \) of the equation that remains bounded by \( M \) in \( H^1_{\text{rad}} \) for \( t \in [0, T] \). Due to (3), the \( L^\infty \)-norm of \( w \) outside a ball of radius \( 1/2 \) remains bounded by \( C_M \). We use the so-called Brezis–Gallouet inequality to have
\[
\| w \|_{L^\infty(B(0, 1/2))} \leq C_M (1 + \log(1 + \| \Delta w \|_{L^2(B(0, 1/2))}))^{1/2}.
\]

Going back to the equation, this inequality implies (the constant \( C_M \) varying from one line to one another)
\[
\| w \|_{L^\infty(B(0, 1/2))} \leq C_M (1 + \log(1 + \| w_t \|_{L^2}))^{1/2}.
\]

We now differentiate equation (7) with respect to \( t \) to have a new equation for \( Z = w_t \) that reads
\[
i Z_t = BZ + 2 \text{Re}(\overline{w}Z)w + |w|^2 Z.
\]

Considering the scalar product of (11) with \( iZ \) leads to
\[
\frac{d}{dt} \| Z \|_{L^2}^2 \leq c \| w \|_{L^2}^2 \| Z \|_{L^2}^2.
\]

Using (10), we then have
\[
\frac{d}{dt} \| Z \|_{L^2}^2 \leq C_M \| Z \|_{L^2}^2 (1 + \log(1 + \| Z \|_{L^2}^2)).
\]

We then infer from this that \( \| Z(t) \|_{L^2} \leq c(Z_0) \exp(\exp(C_M T)) \). Going back to the equation, we have that \( Bw \) remains also bounded in \( L^2 \) for \( t \) in \([0, T]\). \( \square \)

2.2 A sufficient condition for a solution to be global

At this stage we have a local solution that takes value in \( H^1 \). As for the case \( Z = 0 \), the solution is global in time if we can prove an inequality that reads
\[
E(t) \geq c \| v_t \|_{L^2}^2 - C.
\]

We now define the generalized Gagliardo–Nirenberg constant as \( C_Z \) such that for all \( v \) in \( H^1_{\text{rad}} \),
\[
\| v \|_{L^2}^4 \leq C_Z (\| v \|_{L^2}^2 - Z \| v(1) \|_{L^2}^2) \| v \|_{L^2}^2.
\]

In the case \( Z < 0 \), if \( C_Z > C_{GN} \), then we can improve the sufficient condition for a solution to be global. This is not the case. We state and prove:

Proposition 2.5. Assume \( Z < 0 \). Then we have \( C_Z = C_{GN} \).

Proof. For \( Z < 0 \), we have \( C_Z > C_{GN} \). Let us take \( v(r) = w(\mu r) \) in (12) with \( w \) in \( H^1_{\text{rad}} \). Then we have, dividing the resulting equality by \( \mu^2 \),
\[
\| w \|_{L^2}^4 \leq C_Z (\| w_t \|_{L^2}^2 - Z |w(\mu)|^2) \| w \|_{L^2}^2.
\]

Due to (3), then \( |w(\mu)|^2 \) converges towards 0 and we are back to the usual Gagliardo–Nirenberg inequality. Then \( C_{GN} \leq C_Z \). \( \square \)

Remark 2.6. It is worth to point out that for the proof of this proposition we have used that \( H^1 \) is invariant by dilations. The paradox is that \( D(A_1) \) and the PDE under consideration are not invariant by dilations.
3 Revisiting the virial's method

We now introduce the very definition of the virial $V$ and of the momentum $q$ (see [6, 9, 15]) in the radial case as

$$V(t) = \text{Im} \int_0^{+\infty} \left( r^2 \frac{\partial}{\partial r} \right) v(t, r) \overline{v}(t, r) \, dr, \quad q(t) = \int_0^{+\infty} r^3 |v(t, r)|^2 \, dr.$$

3.1 The momentum identity

We first state and prove that if the solution above belongs in some weighted space for $t=0$, it remains in the same weighted space.

**Proposition 3.1.** Consider $v \in C([0, T]; H^1_{rad})$ such that

$$\int_0^{+\infty} r^3 |v_0|^2 \, dr < \infty.$$

Then for all $t \in [0, T],$

$$q(t) = \int_0^{+\infty} r^3 |v(t)|^2 \, dr < \infty.$$

**Proof.** We first prove the identity assuming that the initial data is smooth, say in $D(A_1)$, and we then conclude by density. Let then $v \in C([0, T]; D(A_1))$ be a solution of (2). We define

$$q_R = \int_0^{+\infty} r^3 \exp \left( -\frac{r}{R} \right) |v(t, r)|^2 \, dr$$

for $R > 1$. We first compute $\frac{\partial q_R}{\partial t}$, and then let $R \to +\infty$. We have

$$\frac{\partial q_R}{\partial t} = 2 \text{Re} \int_0^{+\infty} r^3 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial t} \overline{v} \, dr$$

$$= 2 \text{Re} \int_0^{+\infty} r^3 \exp \left( -\frac{r}{R} \right) \left( -\frac{i}{R} \frac{\partial v}{\partial r} \right) \overline{v} \, dr$$

$$= 2 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \overline{v} \, dr$$

$$= 2 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \overline{v} \, dr + 2 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \overline{v} \, dr.$$

On one hand

$$2 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \, dr = -4 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \overline{v} \, dr + 2 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \overline{v} \, dr$$

On the other hand

$$2 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \, dr = -4 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \overline{v} \, dr + 2 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \overline{v} \, dr$$

$$- 2 \text{Re} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \overline{v} \, dr.$$
Using the transmission condition (4), we obtain
\[ 2 \text{Im} \left( \exp \left( -\frac{1}{R} \right) |v|_1 \bar{v}(1) \right) = 0. \]

We then infer
\[ \frac{\partial q_R}{\partial t} = -4 \text{Im} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \bar{v} \, dr + 2 \text{Im} \int_0^{+\infty} \frac{r^3}{R} \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \bar{v} \, dr. \]

We now use the Cauchy–Schwarz inequality to obtain
\[ \left| -4 \text{Im} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \bar{v} \, dr \right| \leq 4 \sqrt{q_R} \left( \int_0^{+\infty} r \exp \left( -\frac{r}{R} \right) \left| \frac{\partial v}{\partial r} \right|^2 \, dr \right)^{\frac{1}{2}} \leq 4 \sqrt{q_R} \left\| \frac{\partial v}{\partial r} \right\|_{L^2_{\text{rad}}} \cdot \]

Using once more the Cauchy–Schwarz inequality, we have
\[ \left| 2 \text{Im} \int_0^{+\infty} \frac{r^3}{R} \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r} \bar{v} \, dr \right| \leq \sqrt{q_R} \left( \int_0^{+\infty} \frac{r^3}{R^2} \exp \left( -\frac{r}{R} \right) \left| \frac{\partial v}{\partial r} \right|^2 \, dr \right)^{\frac{1}{2}} \leq 2 \sqrt{q_R} \left\| \frac{\partial v}{\partial r} \right\|_{L^2_{\text{rad}}} \cdot \]

Therefore
\[ \frac{\partial q_R}{\partial t} \leq (4 + 2C) \sqrt{q_R} \left\| \frac{\partial v}{\partial r} \right\|_{L^2_{\text{rad}}} \cdot \]

We then have that for all \( t < T \),
\[ \sqrt{q_R(t)} \leq \sqrt{q_R(0)} + (4 + 2C) \int_0^t \left\| \frac{\partial v}{\partial r} \right\|_{L^2_{\text{rad}}} \, dt. \]

Letting \( R \to +\infty \) provides that for all \( v \in D(A_1) \), \( q(t) < \infty \) since \( q(0) < \infty \). We conclude by the density of \( D(A_1) \) in \( H^1 \).

**Corollary 3.2.** Setting \( q = \lim_{R \to +\infty} q_R \), we have
\[ \frac{\partial q}{\partial t} = -4 \text{Im} \int_0^{+\infty} r^2 \frac{\partial v}{\partial r} \bar{v} \, dr. \]

**Proof.** We go back to (13):
\[ \frac{\partial q_R}{\partial t} = -4 \text{Im} \int_0^{+\infty} \left( 1 - \frac{r}{2R} \right) \exp \left( -\frac{r}{R} \right) r^2 \frac{\partial v}{\partial r} \bar{v} \, dr. \]

The function \( r \to r^2 r^2 \frac{\partial v}{\partial r} \bar{v} \) is integrable since \( \frac{\partial v}{\partial r}, rv \in L^2_{\text{rad}} \). We conclude by the Lebesgue dominated convergence theorem. \( \square \)

### 3.2 The virial identity

To begin with, we recall that the energy \( E(t) = E_0 \) does not depend on \( t \).

**Proposition 3.3.** For any initial data \( v_0 \) in \( H^1 \) such that \( q(0) < +\infty \) we have
\[ \frac{\partial V}{\partial t}(t) = -2E_0 - \left( \left| \frac{\partial v}{\partial r}(1^+) \right|^2 - \left| \frac{\partial v}{\partial r}(1^-) \right|^2 \right). \]

**Proof.** We proceed as above, performing the computations for \( v_0 \) in \( D(A_1) \) and then passing to the limit due to a density argument. We introduce
\[ V_R = \text{Im} \int_0^{+\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial v}{\partial r}(t, r) \bar{v}(t, r) \, dr. \]
We first compute \( \frac{\partial V(t)}{\partial t} \), and then let \( R \to +\infty \). To begin with, we have
\[
\frac{\partial V}{\partial t}(t) = -2 \text{Im} \int_0^\infty r^2 \exp\left(-R \frac{\partial V}{\partial r}\right) \frac{\partial}{\partial t} \left( \frac{\partial V}{\partial r} \right) dr + \text{Im} \int_0^\infty r^2 \exp\left(-R \frac{\partial V}{\partial t}\right) \frac{\partial}{\partial t} \left( \frac{\partial V}{\partial r} \right) dr
\]
\[
= -2 \text{Im} \int_0^\infty r^2 \exp\left(-R \frac{\partial V}{\partial r}\right) \frac{\partial}{\partial t} \left( \frac{\partial V}{\partial r} \right) dr - 2 \text{Im} \int_0^\infty r \exp\left(-R \frac{1}{2R} \frac{\partial V}{\partial t}\right) \frac{\partial V}{\partial r} dr. \tag{14}
\]

We now estimate the first term in the right-hand side of (14),
\[
-2 \text{Im} \int_0^\infty r^2 \exp\left(-R \frac{\partial V}{\partial r}\right) \frac{\partial}{\partial t} \left( \frac{\partial V}{\partial r} \right) dr = 2 \text{Re} \int_0^\infty r^2 \exp\left(-R \frac{\partial V}{\partial r}\right) \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) \right) dr
\]
\[
+ 2 \text{Re} \int_0^\infty r^2 \exp\left(-R \frac{\partial V}{\partial r}\right) r |v|^2 dr. \tag{15}
\]

We then have
\[
2 \text{Re} \int_0^\infty r^2 \exp\left(-R \frac{\partial V}{\partial r}\right) \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) \right) dr = \text{Re} \int_0^\infty \exp\left(-R \frac{\partial V}{\partial r}\right) \left| \frac{\partial V}{\partial r} \right|^2 dr.
\]

Integrating by parts, we infer
\[
\text{Re} \int_0^\infty \exp\left(-R \frac{\partial V}{\partial r}\right) \left| \frac{\partial V}{\partial r} \right|^2 dr = \text{Re} \int_0^1 \exp\left(-R \frac{\partial V}{\partial r}\right) \left| \frac{\partial V}{\partial r} \right|^2 dr + \text{Re} \int_1^\infty \exp\left(-R \frac{\partial V}{\partial r}\right) \left| \frac{\partial V}{\partial r} \right|^2 dr
\]
\[
= \exp\left(-\frac{1}{R}\right) \left( \left| \frac{\partial V}{\partial r} \right|^2 (1) - \left| \frac{\partial V}{\partial r} \right|^2 (1) \right) + \text{Re} \int_0^\infty \exp\left(-R \frac{\partial V}{\partial r}\right) \left| \frac{\partial V}{\partial r} \right|^2 dr.
\]

On one hand, by the Lebesgue dominated convergence theorem, since \( \frac{1}{2} |\frac{\partial V}{\partial r}|^2 \) belongs to \( L^1 \),
\[
\lim_{R \to +\infty} \text{Re} \int_0^\infty \exp\left(-\frac{R}{R}\right) \left| \frac{\partial V}{\partial r} \right|^2 dr = 0.
\]

On the other hand
\[
\lim_{R \to +\infty} \exp\left(-\frac{1}{R}\right) \left( \left| \frac{\partial V}{\partial r} \right|^2 (1) - \left| \frac{\partial V}{\partial r} \right|^2 (1) \right) = \left| \frac{\partial V}{\partial r} \right|^2 (1) - \left| \frac{\partial V}{\partial r} \right|^2 (1).
\]

Therefore
\[
\lim_{R \to +\infty} 2 \text{Re} \int_0^\infty r^2 \exp\left(-\frac{R}{R}\right) \frac{\partial V}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) \right) dr = \left( \left| \frac{\partial V}{\partial r} \right|^2 (1) - \left| \frac{\partial V}{\partial r} \right|^2 (1) \right).
\]

We now compute the second term in (15) as follows:
\[
2 \text{Re} \int_0^\infty r^2 \exp\left(-\frac{R}{R}\right) v |v|^2 dr = \frac{1}{2} \text{Re} \int_0^\infty r^2 \exp\left(-\frac{R}{R}\right) v |v|^4 dr
\]
\[
= \frac{1}{2} \text{Re} \int_0^1 r^2 \exp\left(-\frac{R}{R}\right) v |v|^4 dr + \frac{1}{2} \text{Re} \int_1^\infty r^2 \exp\left(-\frac{R}{R}\right) v |v|^4 dr.
\]

Since the function \( r \to |v(r)|^4 \) is continuous at 1, integrating by parts we have
\[
2 \text{Re} \int_0^\infty r^2 \exp\left(-\frac{R}{R}\right) v |v|^2 dr = - \text{Re} \int_0^\infty r \exp\left(-R \frac{1}{2R}\right) v |v|^4 dr. \tag{16}
\]
Then, using the Lebesgue dominated convergence theorem in (16), we have
\[
\lim_{R \to +\infty} 2 \text{Re} \int_0^{\infty} r^2 \exp \left( -\frac{r}{R} \right) \frac{\partial \overline{v}}{\partial r} |v|^2 \, dr = -\|v\|^4_{L^4_{rad}}.
\]
We now pass to the limit in the second term in the right-hand side of (14). We first have
\[
-2 \text{Im} \int_0^{\infty} r \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \frac{\partial v}{\partial t} \overline{v} \, dr = 2 \text{Re} \int_0^{\infty} r \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) |v|^4 \, dr
\]
\[
+ 2 \text{Re} \int_0^{\infty} \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \left( \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \right) \overline{v} \, dr.
\]
On one hand, using the Lebesgue dominated convergence theorem, we have
\[
\lim_{R \to +\infty} 2 \text{Re} \int_0^{\infty} r \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) |v|^4 \, dr = 2\|v\|^4_{L^4_{rad}}.
\]
On the other hand, the second term reads also
\[
2 \text{Re} \int_0^{\infty} \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \left( \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \right) \overline{v} \, dr = 2 \text{Re} \int_0^{\infty} \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \left( \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \right) \overline{v} \, dr
\]
\[
+ 2 \text{Re} \int_0^{\infty} \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \left( \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \right) \overline{v} \, dr.
\]
Integrating by parts, we have
\[
2 \text{Re} \int_0^{\infty} \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \left( \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \right) \, dr = -2 \exp \left( -\frac{1}{R} \right) \left( 1 - \frac{1}{2R} \right) \text{Re} \left( \overline{v}(1) \frac{\partial v}{\partial r} \right)
\]
\[
- 2 \text{Re} \int_0^{\infty} r \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \frac{\partial \overline{v}}{\partial r} \, dr
\]
\[
+ \text{Re} \int_0^{\infty} r \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \frac{\partial v}{\partial r} \, dr.
\]
Using once again the Lebesgue dominated convergence theorem, we obtain
\[
\lim_{R \to +\infty} 2 \text{Re} \int_0^{\infty} \exp \left( -\frac{r}{R} \right) \left( 1 - \frac{r}{2R} \right) \left( \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) \right) \, dr = 2\|v\|^4_{L^4_{rad}} - 2 \left\| \frac{\partial v}{\partial r} \right\|^2_{L^2_{rad}}.
\]
Gathering these computations we conclude
\[
\lim_{R \to +\infty} \frac{\partial V_R}{\partial t}(t) = \left( \left\| \frac{\partial v}{\partial r} (1^+) \right\|^2 - \left\| \frac{\partial v}{\partial r} (1^-) \right\|^2 \right) + \|v\|^4_{L^4_{rad}} + 2\|v(1)\|^2 - 2 \left\| \frac{\partial v}{\partial r} \right\|^2_{L^2_{rad}} = -2E_0 - \left[ \left\| \frac{\partial v}{\partial r} \right\|^2 \right]_1.
\]
This completes the proof of the proposition. \hfill \Box

### 3.3 Conclusion

In the previous subsection we have proved that
\[
\frac{\partial V}{\partial t}(t) = -2E_0 - \left[ \left\| \frac{\partial v}{\partial r} \right\|^2 \right]_1.
\]
If \(Z = 0\) and \(E_0 < 0\), then the solution blows up in finite time. We assume below that we have a solution with negative energy. Here we are interested in the case \(Z \neq 0\). For general solutions, we do not know the sign of \([\left\| \frac{\partial v}{\partial r} \right\|^2]_1\). In the next section we will investigate this issue using numerics. We shall observe that for
solutions moving from the right to the left (going to the zero) the sign of $|\frac{\partial v}{\partial r}|_1$ is positive and then balance the negative energy.

4 Numerics

We solve our problem using second-order finite differences in $r$ and the second-order implicit Crank–Nicolson scheme in time. We start this section by describing briefly the numerical method, next we discuss the numerical results. We refer to [10] for details.

4.1 The numerical method

We discuss here first the discretization of the delta function. Consider a solution to (2). For the discretization of the transmission condition, we write respectively to the right of $r_0$

$$\frac{\partial v}{\partial r}(t, r_0^+) = \frac{4v(t, r_0 + \Delta r) - v(t, r_0 + 2\Delta r) - 3v(t, r_0)}{2\Delta r},$$

and respectiveld to the left of $r_0$

$$\frac{\partial v}{\partial r}(t, r_0^-) = \frac{v(t, r_0 - 2\Delta r) - 4v(t, r_0 - \Delta r) + 3v(t, r_0)}{2\Delta r}.$$ 

Indeed, this approximation is a second order approximation in space. We have

$$4v(t, r_0 + \Delta r) - v(t, r_0 + 2\Delta r) - 2(3 - 2\Delta r)v(t, r_0) - v(t, r_0 - 2\Delta r) + 4v(t, r_0 - \Delta r) = 0.$$ 

Usual second-order scheme with finite differences is used inside the computational domain, except at the defect, and Crank–Nicolson scheme in time is performed. This reads for $r \neq r_0$:

$$i\frac{v^{n+1}_j - v^n_j}{\Delta t} - \frac{v^{n+\frac{1}{2}}_{j+\frac{1}{2}} - 2v^n_j + v^{n+\frac{1}{2}}_{j-\frac{1}{2}}}{\Delta r^2} - \frac{1}{r_j}\left(\frac{v^{n+1}_j - v^{n+\frac{1}{2}}_{j+\frac{1}{2}}}{\Delta r} - \frac{1}{4}(|v_j^{n+1}|^2 + |v_j^n|^2)(v_j^{n+1} + v_j^n)\right) = 0,$$

where $v_j^{n+\frac{1}{2}} = \frac{v_j^{n+1} + v_j^n}{2}$. Our nonlinear problem is solved using a fixed point method at each time step.

For the boundary conditions, we use a PML method far away to the right of the defect to avoid spurious reflections (see [10] and the references therein). At the left boundary $r = 0$, we solve

$$i\frac{v_0^{n+1} - v_0^n}{\Delta t} - \frac{2v_0^{n+1} - 2v_0^n + v_0^{n+2}}{\Delta r^2} = \frac{2v_0^n - 2v_0^0}{\Delta r^2} + \frac{1}{4}(|v_0^{n+1}|^2 + |v_0^n|^2)(v_0^{n+1} + v_0^n).$$

4.2 The numerical results

In this subsection, we investigate the influence of the defect on the dynamics of traveling Gaussian solution that blows up in the case without defect $Z = 0$. We consider the following initial data (see Figure 1):

$$v_i(r) = 3 \exp(i10r) \exp(-(r - 15)^2),$$

defined on the numerical domain $\Omega = (0, 20)$, that contains a PML band of width $L = 2$. The parameters of the band PML are chosen to absorb the reflected waves at the boundary of the computational domain (see [10, 18]). In our simulation the parameters are $\Delta r = 5 \times 10^{-3}$ and $\Delta t = 2.5 \times 10^{-5}$ for a computation performed with final time $T = 1$. Here we perform some numerical simulations for $Z = 0$, i.e. without defect.

The blow-up structure shows in Figure 2 by the mass concentration of the solution around $r = 1$. To confirm this, we compute in Figure 3 the variation of the $L^2_{\text{rad}}(\mathbb{R}^2)$-norm of the gradient of the solution over time. We note that the solution blows up at $T^* = 0.6512$ and we observe that the norm $\|v_t\|_{L^2_{\text{rad}}}^2$ tends to $\infty$ when $t \to T^*$. 

O. Goubet and E. Hamraoui, Blow-up of solutions to cubic nonlinear Schrödinger equations

Figure 1. Spatial profile of the initial data.

Figure 2. Formation of the singularity at $T = 0.6512$.

Figure 3. Variation of $|v_{r_{rad}}|^2$ versus time for $Z = 0$. 
Let $M_n$ and $E_n$ denote respectively the discrete mass and the energy at $t = t^n$. In Figures 4 and 5 we show the order of magnitude of the relative errors made for $M_n$ and $E_n$ versus time. We observe the conservation of mass and energy over time, and that a singularity appears for $t = T^*$.

![Figure 4](image1.png)

**Figure 4.** Plot of $M_{n+1} - M_n$ versus time.

![Figure 5](image2.png)

**Figure 5.** Plot of $E_{n+1} - E_n$ versus time.

Now, we consider a defect at $r = 10$ and we set $Z = 200$. Is this defect prevent or alter blow-up? After a phase of interaction with the defect Figure 6, we see in Figure 7 that the solution splits into two parts: a transmitted wave $v_t$ and reflected one $v_r$. In our test case, we numerically have

$$
\|v_t\|^2_{L^2(\mathbb{R}^2)} = 9.5828 < \|Q\|^2_{L^2(\mathbb{R}^2)} = 11.7009,
$$

while the reflected part $v_r$ comes out of the computational domain over time (it is absorbed by the PML band). We show in Figure 8 the variation of $(\|v_t\|^2_{L^2(\mathbb{R}^2)})$ versus time. We observe that for $Z = 200$ the $L^2_{\text{rad}}$-norm of the gradient remains bounded along the flow. So, in this case test the defect prevents the blow-up. For this case, we numerically verify the sign of jump $|\frac{\partial v}{\partial r}|$ at $r_0 = 10$ (see the discussion in Section 3.3 above). We observe in Figure 9 that for $Z = 200$ the sign of the jump remains positive. We conclude that the defect splits the incident wave in one reflected part and one transmitted part. It can prevent blow-up if the mass of each part is smaller than the one of the ground state $Q$. 

![Figure 8](image3.png)

![Figure 9](image4.png)
Figure 6. Solution profile then interacting with the defect for $Z = 200$ at $t = 0.2645$.

Figure 7. Solution profile after defect interaction for $Z = 200$ and $t = 0.4320$.

Figure 8. Variation of $|v_r|^2$ versus time for $Z = 200$. 
Figure 9. Evolution of $|\partial v/\partial r|^2$ versus time for $Z = 200$.

**Funding:** This work was supported by PHC Utique ASEO.

**References**


