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Boundary blow-up solutions to the Monge-Ampère equation: Sharp conditions and asymptotic behavior

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Abstract: Consider the boundary blow-up Monge-Ampère problem

$$M[u] = K(x)f(u) \text{ for } x \in \Omega,\ \ u(x) \to +\infty \text{ as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

Here $M[u] = \det(u_{x_i x_j})$ is the Monge-Ampère operator, and Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^N$ $(N \geq 2)$. Under $K(x)$ satisfying appropriate conditions, we first prove that the boundary blow-up Monge-Ampère problem has a strictly convex solution if and only if f satisfies Keller-Osserman type condition. Then the asymptotic behavior of strictly convex solutions to the boundary blow-up Monge-Ampère problem is considered under weaker condition with respect to previous references. Finally, if f does not satisfy Keller-Osserman type condition, we show the existence of strictly convex solutions under different conditions on $K(x)$. The proof combines standard techniques based upon the sub-supersolution method with non-standard arguments, such as the Karamata regular variation theory.

Keywords: Monge-Ampère equation; boundary blow up; sub-supersolution method; sharp conditions; strictly convex solution; asymptotic behavior

MSC: 35J60, 35J96

1 Introduction

Monge-Ampère problems are fully nonlinear problems, which can describe Weingarten curvature, or reflector shape design (see [1]). In the past years, increasing attention has been paid to the study of Monge-Ampère problems by various approaches. We list here, for example, papers [2-14]. The other recent results concerning fully nonlinear uniformly elliptic equations can be found in [15-20]

In this article, we consider the boundary blow-up problem for the Monge-Ampère equation

$$M[u] = K(x)f(u) \text{ in } \Omega,\ \ u = +\infty \text{ on } \partial\Omega, \tag{1.1}$$

where $M[u] = \det(u_{x_i x_j})$ is the Monge-Ampère operator, Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^N (N \geq 2)$, and $K(x)$, $f(u)$ are smooth positive functions. The boundary blow-up condition $u = +\infty$ on $\partial\Omega$ means

$$u(x) \to +\infty \text{ as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

We aim to study the existence and asymptotic behavior of strictly convex solution to (1.1). Suppose $K(x)$ and $f(u)$ satisfy

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(**K**): $K \in C^\infty(\Omega)$ and $K(x) > 0$ in Ω;
(**f1**) : there exists $\eta \in \mathbb{R}^1 \cup \{-\infty\}$ such that
(i) $f \in C^\infty(\eta, \infty)$ is positive and strictly increasing in (η, ∞),
(ii) if $\eta \in \mathbb{R}^1$ then additionally $f(\eta) := \lim_{s\to\eta} f(s) = 0$.

The boundary blow-up problems were first studied by Cheng and Yau [21, 22] with $f(u)$ an exponential function of u, due to their applications in geometry. The case $f(u) = u^p$ $(p > 0)$ and $K(x)$ is a smooth positive function over $\overline{\Omega}$ was considered by Lazer and McKenna [23]. Further results can be found in [24-32], and especially papers [33-36] which have mainly motivated us.

Mohammed [33] proved that if $K(x)$ satisfies (**K**) and is such that the Dirichlet problem

$$M[u] = K(x) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \tag{1.2}$$

has a strictly convex solution, then (1.1) has a strictly convex solution if f satisfies (**f1**) and the Keller-Osserman ([37] and [38]) type condition

$$\Psi(r) = \int_r^\infty [(N+1)F(s)]^{-1/(N+1)} ds < \infty, \forall r > \eta. \tag{1.3}$$

Here

$$F(s) = \int_\eta^s f(t)dt \text{ if } \eta \in \mathbb{R}^1,\ F(s) = \int_0^s f(t)dt \text{ if } \eta = -\infty.$$

In [34], the authors showed that, in the case that $\eta > -\infty$, (1.3) alone does not guarantee the existence of a strictly convex solution to (1.1). One needs additionally

$$\int_{\eta^+} [(N+1)F(s)]^{-1/(N+1)} ds = \infty. \tag{1.4}$$

Here $\int_{\eta^+} \Phi(s)ds = \infty$ means that

$$\int_\eta^{\eta+\epsilon} \Phi(s)ds = \infty \text{ for all small positive } \epsilon.$$

Obviously, (1.4) is equivalent to $\lim_{r\to\eta^+} \Psi(r) = \infty$.

From Theorem 1.1 and Theorem 1.2 of [34] we know, if $K(x)$ satisfies (**K**) and $K \in L^\infty(\Omega)$, the Keller-Osserman type condition is necessary and sufficient (combing with (1.4) if $\eta \in \mathbb{R}^1$) for the existence of strictly convex solution, but if $K(x)$ satisfies (**K**) and is such that (1.2) has a strictly convex solution, it is only sufficient. We would like to prove the necessity in this paper.

So the first main result of this paper is the following.

Theorem 1.1. *Suppose that $K(x)$ satisfies* (**K**) *and is such that* (1.2) *has a strictly convex solution. Suppose that $f(u)$ satisfies* (**f1**), *and when $\eta \in \mathbb{R}^1$, it satisfies additionally* (1.4). *Then* (1.1) *has a strictly convex solution if and only if* (1.3) *holds.*

At the same time, in [34], the authors did not consider the boundary asymptotic behavior of the strictly convex solution. The study of boundary asymptotic behavior of blow-up solutions is also a hot topic, see [37-51], and the references therein. Recently, in [35], Zhang studied the boundary behavior of the strictly convex solution to (1.1) with $K(x) \in C(\bar{\Omega})$. Very recently, in [36], Zhang studied the boundary behavior of the strictly convex solution to (1.1) with $f(u)$ including gradient terms and $K(x)$ is in general case or borderline case.

In [35] and [36], there is an important condition on f, i.e.

(**f2**) there exists $C_f \in (0, \infty]$ such that

$$\lim_{s\to\infty} H'(s) \int_s^\infty \frac{d\tau}{H(\tau)} = C_f, \quad H(s) := [(N+1)F(s)]^{1/(N+1)},\ \forall s > 0.$$

However, we find that it is not necessary. It is implied by other conditions. By Lemma AP.2 in Appendix we can see (**f1**) and (1.3) imply (**f2**).

Let

$$I(s) = \frac{\Psi''(s)\Psi(s)}{(\Psi'(s))^2}, \quad I_\infty = \lim_{s\to\infty} I(s). \tag{1.5}$$

We can see C_f has the same meaning of I_∞.

By Theorem 1.1 the existence of strictly convex solution to problem (1.2) is the key point for the existence of strictly convex solution to problem (1.1). If $K(x)$ is bounded on Ω, according to Theorem 1 of [52], (1.2) always has a strictly convex solution. If $K(x)$ is unbounded near $\partial\Omega$, problem (1.2) is not always having solution. The existence of solution depends on the increasing speed of $K(x)$ when x approaches $\partial\Omega$. In [34], the authors gave a sufficient condition for the existence of strictly convex solutions. For ease of composition, we first introduce some notations.

For a positive function $p(t)$ in $C^1(0,\infty)$ satisfying $p'(t) < 0$ and $\lim_{t\to0^+} p(t) = +\infty$, to distinguish its behavior near $t = 0$ we set $P(\tau) = \int_\tau^1 p(t)dt$. We say such a function $p(t)$ is of class $\mathcal{P}_{finite}$ if

$$\int_{0^+} [P(\tau)]^{\frac{1}{N}} d\tau < \infty,$$

and is of class $\mathcal{P}_\infty$ if

$$\int_{0^+} [P(\tau)]^{\frac{1}{N}} d\tau = \infty.$$

In Theorem 1.5 of [34], the author proved that, if $K(x)$ satisfies (**K**), then (1.2) has no strictly convex solution if there exists a function $p(t)$ of class $\mathcal{P}_\infty$ such that $K(x) \geq p(d(x))$ near $\partial\Omega$, and has a strictly convex solution if there exists a function $p(t)$ of class $\mathcal{P}_{finite}$ such that $K(x) \leq p(d(x))$ near $\partial\Omega$.

If $p(t)$ is of class $\mathcal{P}_{finite}$, we may modify $p(t)$ for large t and assume that $p(t) = c_0 e^{-t}$ for some positive constant c_0 and all large t, say $t \geq M_0$. With $p(t)$ modified as above, if we define

$$\tilde{P}(\tau) = \int_\tau^\infty p(t)dt,$$

then we still have

$$\int_{0^+} [\tilde{P}(\tau)]^{\frac{1}{N}} d\tau < \infty. \tag{1.6}$$

Moreover,

$$\tilde{P}(t) = c_0 e^{-t}, \ \tilde{P}(t)/p(t) = 1 \text{ for } t \geq M_0, \ \tilde{P}(t)/p(t) \to 0 \text{ as } t \to 0. \tag{1.7}$$

Set

$$\omega(t) := \int_0^t (N\tilde{P}(\tau))^{\frac{1}{N}} d\tau \text{ for } t > 0. \tag{1.8}$$

Let

$$J(s) = -\frac{\omega(s)\omega''(s)}{(\omega'(s))^2}, J_0 = \lim_{s\to0^+} J(s). \tag{1.9}$$

The second main result of this paper is the following.

Theorem 1.2. *Suppose that $K(x)$ satisfies* (**K**) *and there exists a function $p(t)$ of class $\mathcal{P}_{finite}$ such that*

$$k_2 p(d(x)) \leq K(x) \leq k_1 p(d(x)) \text{ near } \partial\Omega, \tag{1.10}$$

where k_1, k_2 are positive constants. Suppose that $f(u)$ satisfies (**f1**), (1.3) *and when $\eta \in \mathbb{R}^1$, it satisfies additionally* (1.4). *f, K are such that $I_\infty = \infty$ and $J_0 = \infty$ can not hold at same time, then for any strictly convex solution $u(x)$ of* (1.1), *it holds*

$$1 \leq \liminf_{\substack{x\in\Omega,\\ d(x)\to0}} \frac{u(x)}{\psi[\underline{\xi}(\omega(d(x)))^{\frac{N}{N+1}}]}, \limsup_{\substack{x\in\Omega,\\ d(x)\to0}} \frac{u(x)}{\psi[\overline{\xi}(\omega(d(x)))^{\frac{N}{N+1}}]} \leq 1,$$

where ψ is the inverse of Ψ, i.e. ψ satisfies

$$\int_{\psi(t)}^{\infty} [(N+1)F(s)]^{-1/(N+1)} ds = t, \forall t > \eta \tag{1.11}$$

and

$$\overline{\xi} = \left\{ \frac{(N+1)^N k_2}{N^N M_0[\frac{N}{N+1}\frac{1}{J_0} + \frac{1}{(N+1)J_0 I_\infty} + \frac{1}{I_\infty}]} \right\}^{\frac{1}{N+1}},$$

$$\underline{\xi} = \left\{ \frac{(N+1)^N k_1}{N^N m_0[\frac{N}{N+1}\frac{1}{J_0} + \frac{1}{(N+1)J_0 I_\infty} + \frac{1}{I_\infty}]} \right\}^{\frac{1}{N+1}}.$$

Here

$$M_0 = \max_{\bar{x}\in\partial\Omega} \kappa_1(\bar{x})\kappa_2(\bar{x})\dots\kappa_{N-1}(\bar{x}),$$

$$m_0 = \min_{\bar{x}\in\partial\Omega} \kappa_1(\bar{x})\kappa_2(\bar{x})\dots\kappa_{N-1}(\bar{x}),$$

$\kappa_1(\bar{x}), \kappa_2(\bar{x}), \dots \kappa_{N-1}(\bar{x})$ *are the principal curvatures of $\partial\Omega$ at $\bar{x}$.*

Corollary 1.3. *In Theorem 1.2, if Ω is a ball of radius R, $I_\infty \in [1,\infty)$ and $k_1 = k_2 = \bar{k}$, then*

$$\lim_{\substack{x\in\Omega,\\ d(x)\to 0}} \frac{u(x)}{\psi[(\omega(d(x)))^{\frac{N}{N+1}}]} = \xi_0^{1-I_\infty},$$

where

$$\xi_0 = \left\{ \frac{\bar{k}R^{N-1}}{\left(\frac{N}{N+1}\right)^N \left[\frac{N}{N+1}\frac{1}{J_0} + \frac{1}{N+1}\frac{1}{J_0 I_\infty} + \frac{1}{I_\infty}\right]} \right\}^{\frac{1}{N+1}}.$$

For more articles about boundary blow-up solutions in a ball, please see [53-57].

Remark 1.4. *We can determine that the condition imposed on $b(x)$ in [36] is equivalent to*

$$\int_{0^+} [(N+1)p(\tau)]^{\frac{1}{N+1}} d\tau < \infty. \tag{1.12}$$

We can see that (1.6) *is a weaker condition than* (1.12).

For example, letting $p(s) = s^{-N-1}(-\ln s)^{-\beta}, 0 < s < s_0 < 1$, then $\tilde{P}(s) \cong s^{-N}(-\ln s)^{-\beta}$ as $s \to 0^+$. Then we have

$$\begin{aligned} &\int_{0^+} [(N+1)p(s)]^{\frac{1}{N+1}} ds \\ &= \int_{0^+} (N+1)^{\frac{1}{N+1}} s^{-1}(-\ln s)^{-\frac{\beta}{N+1}} ds \\ &= (N+1)^{\frac{1}{N+1}} \frac{N+1}{N+1-\beta}(-\ln s)^{\frac{N+1-\beta}{N+1}}|_{0^+} \\ &\to \begin{cases} 0, & \beta > N+1, \\ \infty, & \beta < N+1, \end{cases} \quad \text{as } s \to 0^+. \end{aligned}$$

$$\begin{aligned} &\int_{0^+} [N\tilde{P}(s)]^{\frac{1}{N}} ds \\ &\cong \int_{0^+} N^{\frac{1}{N}} s^{-1}(-\ln s)^{-\frac{\beta}{N}} ds \\ &= N^{\frac{1}{N}} \frac{N}{N-\beta}(-\ln s)^{\frac{N-\beta}{N}}|_{0^+} \\ &\to \begin{cases} 0, & \beta > N, \\ \infty, & \beta < N, \end{cases} \quad \text{as } s \to 0^+. \end{aligned}$$

Meanwhile, by Theorem 1.5 of [34] we know (1.6) *is sharper than* (1.12) *for the existence of strictly convex solutions of* (1.2) *and the existence of strictly convex solutions of* (1.1).

If $K(x)$ is such that (1.2) has no strictly convex solution, then (1.1) may have or have no strictly convex solution, depending on the behavior of f. In [34], the authors only examined some such cases for the radially symmetric situation. In this paper, we'll consider the general case. But we have to impose some sufficient condition such that (1.2) has no strictly convex solution. It is

(K1) there exists a function $p(t)$ of class $\mathcal{P}_\infty$ such that $k_4 p(d(x)) \le K(x) \le k_3 p(d(x))$ near $\partial\Omega$, where k_3, k_4 are positive constants.

Suppose that $K(x)$ satisfies (**K**) and **(K1)**, $f(u)$ satisfies (**f1**). We'll prove that (1.1) has strictly convex solution if (1.3) does not hold.

If (1.3) does not hold, there exists $c_0 > 0$ such that

$$G(t) := \int_{c_0}^{t} [(N+1)F(\tau)]^{-\frac{1}{N+1}} d\tau \to \infty \text{ as } t \to \infty. \tag{1.13}$$

Let

$$R(s) = -\frac{G''(s)G(s)}{(G'(s))^2}, R_\infty = \lim_{s\to\infty} R(s). \tag{1.14}$$

We have

Theorem 1.5. *Suppose that $K(x)$ satisfies* (**K**) *and* **(K1)**. *Suppose that $f(u)$ satisfies* (**f1**) *and $R_\infty \neq \infty$. Then* (1.1) *has strictly convex solution if* (1.3) *does not hold.*

The rest of the paper is organized in the following way. In Section 2 we will collect some known results to be used in the subsequent sections. Section 3 is devoted to the proofs of the Theorem 1.1 and Theorem 1.2. In Section 4 we prove that Theorem 1.5 holds. In Appendix we will introduce the theory of regular variation for the proof of Corollary 1.3.

2 Some preliminary results

In this section, we collect some results for the convenience of later use and reference.

Lemma 2.1. (Lemma 2.1 of [23]) *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \ge 2$, and let $u^k \in C^2(\Omega) \cap C(\overline{\Omega})$ for $k = 1, 2$. Let $f(x, u)$ be defined for $x \in \Omega$ and u in some interval containing the ranges of u^1 and u^2 and assume that $f(x, u)$ is strictly increasing in u for all $x \in \Omega$. Suppose*
(i) the matrix $(u^1_{x_i x_j})$ is positive definite in Ω,
(ii) $M[u^1](x) \ge f(x, u^1(x)), \forall x \in \Omega$,
(iii) $M[u^2](x) \le f(x, u^2(x)), \forall x \in \Omega$,
(iv) $u^1(x) \le u^2(x), \forall x \in \partial\Omega$.

Then $u^1(x) \le u^2(x)$ in Ω.

Remark 2.2. From the proof in [23], it is easily seen that the condition "$f(x, u)$ is strictly increasing in u for all $x \in \Omega$" in Lemma 2.1 can be relaxed to "$f(x, u)$ is nondecreasing in u for all $x \in \Omega$" provided that one of the inequalities in (ii) and (iii) is replaced by a strict inequality. This observation will be used later in the paper.

Lemma 2.3. (Proposition 2.1 of [24]) *Let $u \in C^2(\Omega)$ be such that the matrix $(u_{x_i x_j})$ is invertible for $x \in \Omega$, and let g be a C^2 function defined on an interval containing the range of u. Then*

$$M[g(u)] = M[u]\Big\{[g'(u)]^N + [g'(u)]^{N-1} g''(u)(\nabla u)^T B(u) \nabla u\Big\}, \tag{2.1}$$

where A^T denotes the transpose of the matrix A, $B(u)$ denotes the inverse of the matrix $(u_{x_i x_j})$, and

$$\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_N})^T.$$

If $u = d(x)$, then

$$M[g(d(x))] = [-g'(d(x))]^{N-1} g''(d(x)) \prod_{i=1}^{N-1} \frac{\kappa_i(\overline{x})}{1 - d(x)\kappa_i(\overline{x})}, \; x \in \Omega_{\delta_1}, \tag{2.2}$$

where

$$\Omega_{\delta_1} = \{x \in \Omega : 0 < d(x) < \delta_1\}, \quad \kappa_1(\bar{x}), \kappa_2(\bar{x}), \dots \kappa_{N-1}(\bar{x})$$

are the principal curvatures of $\partial\Omega$ at $\bar{x}$.

The following interior estimate for derivatives of smooth solutions of Monge-Ampère equations is a simple variant of Lemma 2.2 in [23], which follows from [58, 59].

Lemma 2.4. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $\partial\Omega \in C^\infty$. Let $\eta \in [-\infty, +\infty)$ and $f \in C^\infty(\overline{\Omega} \times (\eta, \infty))$ with $f(x, u) > 0$ for $(x, u) \in \overline{\Omega} \times (\eta, \infty)$. Let $u \in C^\infty(\overline{\Omega})$ be a solution of the Dirichlet problem*

$$\begin{cases} M[u](x) = f(x, u), & x \in \Omega, \\ u(x) = c = constant, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

with $\eta < u(x) < c$ in Ω. Let Ω' be a subdomain of Ω with $\overline{\Omega'} \subset \Omega$ and assume that $\eta < a \leq u(x) \leq b$ for $x \in \overline{\Omega'}$ and let $k \geq 1$ be an integer. Then there exists a constant C which depends only on k, a, b, bounds for the derivatives of $f(x, u)$ for $(x, u) \in \overline{\Omega'} \times [a, b]$, and $\operatorname{dist}(\Omega', \partial\Omega)$ *such that*

$$||u||_{C^k(\overline{\Omega'})} \leq C.$$

The existence result below is a variant of Lemma 2.3 in [23], which is a special case of Theorem 7.1 in [52].

Lemma 2.5. *Let Ω be a strictly convex, bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $\partial\Omega \in C^\infty$. Let $f(x, u)$ be a positive C^∞ function on $\overline{\Omega} \times (\eta, c]$, where $c > \eta \geq -\infty$. If there exists a function $u_\star \in C^2(\overline{\Omega})$, which is convex on $\overline{\Omega}$, such that $u_\star > \eta$ and*

$$\begin{cases} M[u_\star](x) \geq f(x, u_\star(x)), & x \in \Omega, \\ u_\star(x) = c, & x \in \partial\Omega, \end{cases}$$

then there exists a solution u of (2.3) *with $u \in C^\infty(\overline{\Omega})$ and u strictly convex. Moreover, $u(x) \geq u_\star(x)$ on $\overline{\Omega}$.*

Let Ω be a smooth, bounded, strictly convex domain in $\mathbb{R}^N$, by Theorem 1.1 of [52], there exists $u_0 \in C^\infty(\overline{\Omega})$ which is the unique strictly convex solution to

$$M[u_0] = 1 \text{ in } \Omega, \; u_0 = 1 \text{ on } \partial\Omega.$$

Set $z(x) := 1 - u_0(x)$. Then $z(x) > 0$ in Ω and it is the unique strictly concave solution to

$$(-1)^N M[z] = 1 \text{ in } \Omega, \; z = 0 \text{ on } \partial\Omega. \tag{2.4}$$

Since $(z_{x_i x_j})$ is negative definite on $\overline{\Omega}$, its trace is negative, that is $\Delta z < 0$, and hence one can apply the Hopf boundary lemma to conclude that $|\nabla z| > 0$ for $x \in \partial\Omega$. It follows that there exist positive constants b_1 and b_2 such that

$$b_1 d(x) \leq z(x) \leq b_2 d(x) \text{ for } x \in \Omega. \tag{2.5}$$

3 Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1.
Sufficiency. It was proved in [34].
Necessity. Assume to the contrary that (1.1) has a strictly convex solution u. We aim to derive a contradiction.

Denote by $g(t)$ the inverse of $G(t)$, i.e.,

$$\int_{c_0}^{g(t)} [(N+1)F(\tau)]^{-\frac{1}{N+1}} d\tau = t, \forall t > 0, \tag{3.1}$$

where $G(t)$ is defined by (1.13).

Then

$$g(0) = c_0, \ \lim_{t\to\infty} g(t) = \infty$$

and

$$g'(t) = [(N+1)F(g(t))]^{\frac{1}{N+1}}, \quad g''(t) = \frac{f(g(t))}{[(N+1)F(g(t))]^{\frac{N-1}{N+1}}},$$

$$(g'(t))^{N-1}g''(t) = f(g(t)), \quad \frac{g'(t)}{g''(t)} = \frac{[(N+1)F(g(t))]^{\frac{N}{N+1}}}{f(g(t))}. \tag{3.2}$$

Since Ω is bounded in R^N, there exists R_0 such that $\Omega \subset B(0, R_0)$. Then define

$$y(x) := \sum_{i=1}^{N} \frac{1}{2}(x_i + R_0 + K_1)^2 \text{ for } x \in \Omega,$$

where K_1 is a positive constant to be determined.

Clearly

$$[\nabla y(x)]^T = (x_1 + R_0 + K_1, \dots, x_N + R_0 + K_1) > 0,$$

$(y_{x_i x_j})$ is the identity matrix, and $M[y] = 1$.

Let $w_1(x)$ be a strictly convex solution of (1.2) and $K_1 = \max_{x\in\bar\Omega} |\nabla w_1| + 1$. Let

$$w(x) = w_1(x) + y(x) + H,$$

where $H = \max_{x\in\bar\Omega}(-w_1(x)) + 1$. Then

$$|\nabla w| = |\nabla w_1 + \nabla y| > 1, \ M[w] > M[w_1].$$

For $c > 0$ define

$$v(x) := g\left(cw(x)\right), \ x \in \Omega.$$

Then we obtain, for $x \in \Omega$,

$$\begin{aligned} M[v] = \ & M[cw]\Big\{\left[g'(cw)\right]^N + (g'(cw))^{N-1}g''(cw)(\nabla(cw))^T B(cw)\nabla(cw)\Big\} \\ = \ & c^N M[w](g'(cw))^{N-1}g''(cw)\left\{\frac{g'(cw)}{g''(cw)} + c(\nabla w)^T B(w)\nabla w\right\} \\ > \ & c^N M[w_1]f(v)\left\{\frac{g'(cw)}{g''(cw)} + c\lambda_1|\nabla w|^2\right\} \\ > \ & c^{N+1}\lambda_1 K(x)f(v), \end{aligned}$$

where $B(w)$ is the inverse matrix of $(w_{x_i x_j})$, λ_1 is the minimal eigenvalue of $B(w)$. Since w is strictly convex, all the eigenvalue of $B(w)$ is positive. We thus obtain

$$M[v] > K(x)f(v) \text{ in } \Omega$$

provided that c is chosen large enough.

Fix $x_1 \in \Omega$ and by further enlarging c if necessary we may assume that

$$v(x_1) > u(x_1) \text{ and } M[v] > K(x)f(v) \text{ in } \Omega.$$

Since $u(x) \to \infty$ as $d(x) \to 0$, while $v(x)$ is continuous on $\overline{\Omega}$, there exists an open connected set D such that

$$x_1 \in D,\ \overline{D} \subset \Omega,\ u(x) < v(x) \text{ in } D \text{ and } u(x) = v(x) \text{ on } \partial D.$$

On the other hand, since

$$M[u] = K(x)f(u) \text{ in } D \text{ and } v = u \text{ on } \partial D,$$

and the matrix $(v_{x_i x_j})$ is positive definite on $\overline{D}$ (since $w_1(x),\ y(x)$ are strictly convex in Ω and $g', g'' > 0$), we can apply Lemma 3.1 to conclude that $v(x) \le u(x)$ in D. This contradiction completes our proof. □

Proof of Theorem 1.2.

For small $\delta_1 > 0$, let

$$\Omega_{\delta_1} = \{x \in \Omega | 0 < d(x) < \delta_1\}.$$

For an arbitrary $\varepsilon \in (0, \min\{1/4, \underline{k_2}\})$, let

$$\overline{\xi}_\varepsilon = \left\{\frac{k_2(1-\varepsilon)^2}{(\frac{N}{N+1})^N M_0[\frac{N}{N+1}\frac{1}{J_0} + \frac{1}{N+1}\frac{1}{J_0 I_\infty} + \frac{1}{I_\infty}]}\right\}^{\frac{1}{N+1}},$$

$$\underline{\xi}_\varepsilon = \left\{\frac{k_1(1+\varepsilon)^2}{(\frac{N}{N+1})^N m_0[\frac{N}{N+1}\frac{1}{J_0} + \frac{1}{N+1}\frac{1}{J_0 I_\infty} + \frac{1}{I_\infty}]}\right\}^{\frac{1}{N+1}}.$$

where m_0, M_0, k_1, k_2 are given in Theorem 1.2, I_∞, J_0 are given (1.5) and (1.9).

From the definition of $J_0, I_\infty, \overline{\xi}_\varepsilon, \underline{\xi}_\varepsilon$ we see that

$$\lim_{d(x)\to 0} \frac{[N\tilde{P}(d(x))]^{\frac{N+1}{N}}}{p(d(x)) \int_0^{d(x)} (N\tilde{P}(\tau))^{\frac{1}{N}}\, d\tau} = \frac{1}{J_0},$$

$$\lim_{s\to\infty} \frac{[(N+1)F(s)]^{\frac{N}{N+1}}}{f(s)\Psi(s)} = \frac{1}{I_\infty},$$

$$\lim_{d(x)\to 0} \prod_{i=1}^{N-1} (1 - d(x)\kappa_i(\bar{x})) = 1,$$

$$\overline{\xi}_\varepsilon^{N+1}(\frac{N}{N+1})^N[\frac{N}{N+1}\frac{1}{J_0} + \frac{1}{N+1}\frac{1}{J_0 I_\infty} + \frac{1}{I_\infty}]\frac{M_0}{k_2(1-\varepsilon)} - 1 = -\varepsilon,$$

$$\underline{\xi}_\varepsilon^{N+1}(\frac{N}{N+1})^N[\frac{N}{N+1}\frac{1}{J_0} + \frac{1}{N+1}\frac{1}{J_0 I_\infty} + \frac{1}{I_\infty}]\frac{m_0}{k_1(1+\varepsilon)} - 1 = \varepsilon.$$

For $x \in \Omega_{\delta_1}$, define

$$\bar{u}_\varepsilon(x) = \psi[\overline{\xi}_\varepsilon[\bar{\omega}(d(x))]],\ \bar{\omega}(d(x)) = [\omega(d(x))]^{\frac{N}{N+1}} - [\omega(\sigma)]^{\frac{N}{N+1}}, d(x) > \sigma > 0,$$

$$\underline{u}_\varepsilon(x) = \psi[\underline{\xi}_\varepsilon[\underline{\omega}(d(x))]],\ \underline{\omega}(d(x)) = [\omega(d(x))]^{\frac{N}{N+1}} + [\omega(\sigma)]^{\frac{N}{N+1}}, \sigma \in (0, \delta_\varepsilon),$$

where $\delta_\varepsilon \in (0, \min\{1, \delta_1/2\})$ is sufficiently small such that for $x \in \Omega_{2\delta_\varepsilon}$

$$1 - \varepsilon < \prod_{i=1}^{N-1} (1 - d(x)\kappa_i(\bar{x})) < 1 + \varepsilon,$$

$$\overline{\xi}_\varepsilon^{N+1}(\frac{N}{N+1})^N[\frac{N}{N+1}\frac{1}{J(d(x))} + \frac{1}{N+1}\frac{1}{I(\overline{u}_\varepsilon)}\frac{1}{J(d(x))} + \frac{1}{I(\overline{u}_\varepsilon)}]\frac{M_0}{k_2(1-\varepsilon)} - 1 < 0,$$

$$\underline{\xi}_\varepsilon^{N+1}(\frac{N}{N+1})^N[\frac{N}{N+1}\frac{1}{J(d(x))} + \frac{1}{N+1}\frac{1}{I(\underline{u}_\varepsilon)}\frac{1}{J(d(x))} + \frac{1}{I(\underline{u}_\varepsilon)}]\frac{m_0}{k_1(1+\varepsilon)} - 1 > 0.$$

Let

$$D_\sigma^- = \Omega_{2\delta_\varepsilon/\bar{\Omega}_\sigma},\quad D_\sigma^+ = \Omega_{2\delta_\varepsilon - \sigma}.$$

By (2.2) we have for $x \in D_\sigma^-$

$$
\begin{aligned}
&M[\bar{u}_\varepsilon] - K(x)f(\bar{u}_\varepsilon) \\
&= \Big(\frac{N}{N+1}\Big)^N \bar{\xi}_\varepsilon^{N+1} (-1)^{N-1} \psi'^{N-1}\psi''\omega'^{N-1}\omega'' \Big[\frac{N}{N+1}\frac{\omega'^2}{\omega\omega''} - \frac{1}{N+1}\frac{\psi'}{\psi''\bar{\xi}_\varepsilon\omega^{\frac{N}{N+1}}}\frac{\omega'^2}{\omega'' w} + \frac{\psi'}{\psi''\bar{\xi}_\varepsilon\omega^{\frac{N}{N+1}}}\Big] \\
&\prod_{i=1}^{N-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})} - K(x)f(\bar{u}_\varepsilon) \\
&= \Big(\frac{N}{N+1}\Big)^N \bar{\xi}_\varepsilon^{N+1} f(\psi)p(d(x))\Big[-\frac{N}{N+1}\frac{\omega'^2}{\omega\omega''} + \frac{1}{N+1}\frac{\psi'}{\psi''\bar{\xi}_\varepsilon\omega^{\frac{N}{N+1}}}\frac{\omega'^2}{\omega'' w} - \frac{\psi'}{\psi''\bar{\xi}_\varepsilon\omega^{\frac{N}{N+1}}}\Big] \\
&\prod_{i=1}^{N-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})} - K(x)f(\bar{u}_\varepsilon) \\
&< \Big(\frac{N}{N+1}\Big)^N \bar{\xi}_\varepsilon^{N+1} f(\psi)p(d(x))\Bigg[\frac{N}{N+1}\frac{[N\tilde{P}(d(x))]^{\frac{N+1}{N}}}{p(d(x))\int_0^{d(x)}(N\tilde{P}(\tau))^{\frac{1}{N}}\,d\tau} \\
&+\frac{1}{N+1}\frac{[(N+1)F(\psi)]^{\frac{N}{N+1}}}{f(\psi)\Psi(\psi)}\frac{[N\tilde{P}(d(x))]^{\frac{N+1}{N}}}{p(d(x))\int_0^{d(x)}(N\tilde{P}(\tau))^{\frac{1}{N}}\,d\tau} + \frac{[(N+1)F(\psi)]^{\frac{N}{N+1}}}{f(\psi)\Psi(\psi)}\Bigg]\prod_{i=1}^{N-1}\frac{\kappa_i(\bar{x})}{1-d(x)\kappa_i(\bar{x})} - K(x)f(\bar{u}_\varepsilon) \\
&= \Bigg[\bar{\xi}_\varepsilon^{N+1}\Big(\frac{N}{N+1}\Big)^N\Big[\frac{N}{N+1}\frac{1}{J(d(x))} + \frac{1}{N+1}\frac{1}{I(\bar{u})}\frac{1}{J(d(x))} + \frac{1}{I(\bar{u})}\Big]\frac{M_0}{k_2(1-\varepsilon)} - 1\Bigg]K(x)f(\bar{u}_\varepsilon) < 0,
\end{aligned}
$$

i.e. $\bar{u}_\varepsilon$ is a supersolution to (1.1) in D_σ^-.

Similarly, we can prove $\underline{u}_\varepsilon$ is a subsolution to (1.1) in D_σ^+.

By (1.10) and Theorem F, (1.2) has a strictly convex solution. It follows from Theorem 1.1 that (1.1) has a strictly convex solution u. Let A be large enough such that

$$u \le \bar{u}_\varepsilon + A \ \text{ on } \ d(x) = 2\delta_\varepsilon$$

and

$$\underline{u}_\varepsilon \le u + A \ \text{ on } \ d(x) = 2\delta_\varepsilon - \sigma.$$

By the definition of $\bar{u}_\varepsilon$ and $\underline{u}_\varepsilon$, we know that $\bar{u}_\varepsilon(x) \to \infty$ as $d(x) \to \sigma$ and $\underline{u}_\varepsilon|_{\partial\Omega} < u|_{\partial\Omega}$. By Lemma 2.1 we have

$$u \le \bar{u}_\varepsilon + A \ \text{ in } \ D_\sigma^-$$

and

$$\underline{u}_\varepsilon \le u + A \ \text{ in } \ D_\sigma^+.$$

Then

$$\frac{u(x)}{\psi[\bar{\xi}_\varepsilon[\bar{\omega}(d(x))]]} \le 1 + \frac{A}{\psi[\bar{\xi}_\varepsilon[\bar{\omega}(d(x))]]}, \ x \in D_\sigma^-,$$

and

$$1 - \frac{A}{\psi[\underline{\xi}_\varepsilon[\underline{\omega}(d(x))]]} \le \frac{u(x)}{\psi[\underline{\xi}_\varepsilon[\underline{\omega}(d(x))]]}, \ x \in D_\sigma^+.$$

Then, for $x \in D_\sigma^- \cap D_\sigma^+$, the two formulas above hold. Letting $\sigma \to 0$, then we obtain

$$\frac{u(x)}{\psi[\bar{\xi}_\varepsilon[\omega(d(x))]^{\frac{N}{N+1}}]} \le 1 + \frac{A}{\psi[\bar{\xi}_\varepsilon[\omega(d(x))]^{\frac{N}{N+1}}]},$$

and

$$1 - \frac{A}{\psi[\underline{\xi}_\varepsilon[\omega(d(x))]^{\frac{N}{N+1}}]} \le \frac{u(x)}{\psi[\underline{\xi}_\varepsilon[\omega(d(x))]^{\frac{N}{N+1}}]}.$$

Letting $d(x) \to 0$, $\varepsilon \to 0$, we get

$$\limsup_{\substack{x\in\Omega,\\ d(x)\to 0}} \frac{u(x)}{\psi[\bar{\xi}(\omega(d(x)))^{\frac{N}{N+1}}]} \le 1,$$

and

$$1 \le \liminf_{\substack{x\in\Omega,\\ d(x)\to 0}} \frac{u(x)}{\psi[\underline{\xi}(\omega(d(x)))^{\frac{N}{N+1}}]}. \qquad \square$$

Proof of Corollary 1.3.

By (1.11) we have

$$\psi'(t) = -[(N+1)F(\psi(t))]^{\frac{1}{N+1}}.$$

Then

$$\begin{aligned}\lim_{t\to 0}\frac{t\psi'(t)}{\psi(t)} &= -\lim_{t\to 0}\frac{t[(N+1)F(\psi(t))]^{\frac{1}{N+1}}}{\psi(t)}\\ &= \lim_{s\to\infty}\frac{\frac{\Psi(s)}{\Psi'(s)}}{s} = 1-\lim_{s\to\infty}\frac{\Psi(s)\Psi''(s)}{\Psi'^2(s)} = 1-I_\infty.\end{aligned}$$

It follows from Proposition AP.3 of Appendix that $\psi \in NRV_{1-I_\infty}$.

Combing this with Theorem 1.2 we obtain

$$\begin{aligned}&\lim_{\substack{x\in\Omega,\\ d(x)\to 0}}\frac{u(x)}{\psi[(\omega(d(x)))^{\frac{N}{N+1}}]}\\ &= \lim_{\substack{x\in\Omega,\\ d(x)\to 0}}\frac{u(x)}{\psi[\xi_0(\omega(d(x)))^{\frac{N}{N+1}}]}\frac{\psi[\xi_0(\omega(d(x)))^{\frac{N}{N+1}}]}{\psi[(\omega(d(x)))^{\frac{N}{N+1}}]}\\ &= \xi_0^{1-I_\infty}. \qquad \square\end{aligned}$$

4 Proof of Theorem 1.5

For the proof of Theorem 1.5, we first introduce a lemma which is about radial solutions. Let $K(x) = K_1(r)$, $u(x) = v(r)$, $r = |x|$, B is a ball with radius R in $\mathbb{R}^N$ ($N \geq 2$), then

$$\begin{cases} M[u] = K(x)f(u), & x \in B,\\ u = \infty, & x \in \partial B,\end{cases} \tag{4.1}$$

is equivalent to

$$\begin{cases} (v')^{N-1}v'' = r^{N-1}K_1(r)f(v), \ r \in (0,R),\\ v'(0) = 0, \ v(R) = \infty.\end{cases} \tag{4.2}$$

In the radially symmetric setting, the smoothness requirements for K and f can be greatly relaxed. But for convenience, we still use (**K**), (**K1**) and (**f1**). In the case (**K1**) can be state as:

there exist constants $d_3, d_4 > 0$ and a function $p(t)$ of class $\mathcal{P}_\infty$ such that

$$d_4 p(R-r) \leq K_1(r) \leq d_3 p(R-r) \text{ for all } r < R \text{ close to R.}$$

We modify $p(t)$ as in Section 1 and define $\sigma(t)$ by

$$\sigma(s) = \int_s^\infty (N\tilde{P}(\tau))^{\frac{1}{N}}\,d\tau, \tag{4.3}$$

we have

$$\sigma'(s) = -[N\tilde{P}(s)]^{\frac{1}{N}}, \ \sigma''(s) = [N\tilde{P}(s)]^{\frac{1}{N}-1}p(s). \tag{4.4}$$

It follows that

$$-\frac{1}{\sigma'(s)} = [N\tilde{P}(s)]^{-\frac{1}{N}}.$$

Let

$$T(s) = \frac{\sigma(s)\sigma''(s)}{\sigma'^2(s)}. \tag{4.5}$$

Lemma 4.1. *Suppose that K satisfies* (**K**) *and* (**K1**). *Suppose that f satisfies* **(f1)** *and $R_\infty \neq \infty$. If* (1.3) *does not hold, then* (4.2) *has infinitely many strictly convex solutions.*

Proof. Let $w(r) = g(c\sigma^{\frac{N}{N+1}}(y(r)))$ for $r \in [0, R)$, where $y(r)$ satisfies

$$\begin{cases} (-1)^N y'^{N-1} y'' = r^{N-1}, \ r \in (0, R), \\ y'(0) = 0, \ y(R) = 0. \end{cases}$$

and g, σ is defined by (3.1),(4.3).

Similar to the proof of Theorem 5.3 of [34], we can prove the Lemma 4.1. So we omit it here. □

Proof of Theorem 1.5.

Step1. Let

$$w(x) = g(c\sigma^{\frac{N}{N+1}}(\frac{1}{b_2} z(x))), x \in \Omega,$$

where g, σ, z is defined by (3.1),(4.3),(2.4), respectively, and b_2 is defined in (2.5). Then by (1.14), (4.4), (4.5), (3.2) we have

$$\begin{aligned} M[w] &= c^{N+1} b_2^{-N} (\tfrac{N}{N+1})^N g'^{N-1} g'' \sigma'^{N-1} \sigma'' M[z] \Big\{ \tfrac{g'}{c\sigma^{\frac{N}{N+1}} g''} \tfrac{\sigma'}{\sigma''} \\ &- b_2^{-1} (\tfrac{N}{N+1} \tfrac{\sigma'^2}{\sigma\sigma''} - \tfrac{1}{N+1} \tfrac{g'}{c\sigma^{\frac{N}{N+1}} g''} \tfrac{\sigma'^2}{\sigma\sigma''} + \tfrac{g'}{c\sigma^{\frac{N}{N+1}} g''}) (\nabla z)^T B(z) \nabla z \Big\} \\ &= c^{N+1} b_2^{-N} (\tfrac{N}{N+1})^N f(w) p(\tfrac{1}{b_2} z) \Big\{ \tfrac{1}{R(w)} (-\tfrac{\sigma'}{\sigma''}) \\ &- b_2^{-1} (\tfrac{N}{N+1} \tfrac{1}{T(z)} - \tfrac{1}{N+1} \tfrac{1}{R(w)} \tfrac{1}{T(z)} + \tfrac{1}{R(w)}) (\nabla z)^T B(z) \nabla z \Big\}, \end{aligned} \tag{4.6}$$

where R, T is defined by (1.14) and (4.5) respectively.

Since

$$\begin{aligned} \frac{\sigma'(t)^2}{\sigma(t)\sigma''(t)} &= \frac{[N\tilde{P}(t)]^{\frac{N+1}{N}}}{p(t) \int_t^\infty [N\tilde{P}(\tau)]^{1/N} d\tau} \\ &= \frac{\int_t^\infty (N+1)[N\tilde{P}(s)]^{1/N} p(s) ds}{\int_t^\infty \Big\{ -p'(s) \int_s^\infty [N\tilde{P}(\tau)]^{1/N} d\tau + p(s)[N\tilde{P}(s)]^{1/N} \Big\} ds} \\ &\le (N+1). \end{aligned}$$

We have

$$\frac{1}{R(w)} - \frac{1}{N+1} \frac{1}{R(w)} \frac{1}{T(z)} \ge 0.$$

Then

$$\frac{N}{N+1} \frac{1}{T(z)} - \frac{1}{N+1} \frac{1}{R(w)} \frac{1}{T(z)} + \frac{1}{R(w)} > 0.$$

Let

$$\triangle_1 = \frac{1}{R(w)} (-\frac{\sigma'}{\sigma''}) - b_2^{-1} (\frac{N}{N+1} \frac{1}{T(z)} - \frac{1}{N+1} \frac{1}{R(w)} \frac{1}{T(z)} + \frac{1}{R(w)}) (\nabla z)^T B(z) \nabla z.$$

By the definition of z we have $(z_{x_i x_j})$ is negative definite. It follows that there exist $e_1, e_2 > 0$ such that

$$-e_1 ||\nabla z||^2 \le (\nabla z)^T B(z) \nabla z \le -e_2 ||\nabla z||^2,$$

and trace$(z_{x_i x_j}) = \Delta z < 0$. Therefore, since $\Delta(-z) > 0$ on Ω and $-z$ attains its maximum on $\bar{\Omega}$ at each point of $\partial\Omega$, it follows from the maximum principle that there exists an open set U containing $\partial\Omega$ such that

$$||\nabla z|| \ge e > 0.$$

On the other hand, it is easy to see that z is bounded below by a positive constant on $\Omega - \mathrm{U}$. Then

$$w \to \infty \text{ as } c \to \infty \text{ on } \Omega - \mathrm{U}.$$

Combining this with the fact that $R_\infty \neq \infty$, we can conclude that Δ_1 is positive on Ω. By (4.6) we have

$$\begin{aligned} M[w] &\geq c^{N+1} b_2^{-N} (\tfrac{N}{N+1})^N f(w) p(d(x)) \Delta_1 \\ &\geq c^{N+1} b_2^{-N} (\tfrac{N}{N+1})^N f(w) \tfrac{1}{K_3} K(x) \Delta_1 \\ &\geq K(x) f(w) \end{aligned} \tag{4.7}$$

for large c_1, i.e. $w_1(x) = g(c_1 \sigma^{\frac{N}{N+1}} (\frac{1}{b_2} z(x)))$ is a subsolution of (1.1).

Step2. The existence of a solution $u(x) \in C^\infty(\Omega)$.

Let $\{\sigma_n\}_1^\infty$ be a strictly increasing sequence of positive numbers such that $\sigma_n \to \infty$ as $n \to \infty$, and let $\Omega_n = \{x \in \Omega | w_1(x) < \sigma_n\}$. Since any level surface of w_1 is a level surface of z, for each $n \geq 1$, $\partial\Omega_n$ is a strictly convex C^∞-submanifold of R^N of dimension $N-1$.

By Lemma 2.5 there exists $u_n \in C^\infty(\bar{\Omega}_n)$ for $n \geq 1$ such that

$$\begin{cases} M[u_n] = K(x) f(u_n), \ x \in \Omega_n, \\ u_n \|_{\partial\Omega_n} = \sigma_n = w_1 \|_{\partial\Omega_n}. \end{cases} \tag{4.8}$$

By Lemma 2.1 and (4.7)

$$u_n(x) \geq w_1(x) \quad x \in \bar{\Omega}_n.$$

Clearly, for $n > 1$

$$\bar{\Omega}_n \subset \Omega_{n+1}$$

and

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

We claim that

$$u_n(x) \leq u_{n+1}(x), \quad \forall x \in \Omega_n. \tag{4.9}$$

Indeed, since u_n and u_{n+1} are both positive solutions of $M[u] = K(x) f(u)$ on $\bar{\Omega}_n$, u_n is strictly convex in Ω_n and for $x \in \partial\Omega_n \subset \Omega_{n+1}$,

$$u_{n+1}(x) \geq w_1(x) = u_n(x),$$

the inequality (4.9) is a consequence of Lemma 2.1.

Fix m. For each $x_0 \in \bar{\Omega}_m$, let R be small to have $\bar{B}(x_0; R) \subset \Omega_{m+1}$. By Lemma 4.1, there exists a solution v of (4.2). Let $d(x, \partial B) = d(x, \partial B(x_0; R))$. Define

$$w_2(x) = v(R - d(x, \partial B)), \ x \in B(x_0; R).$$

Then by (2.2) and (4.2) we have

$$\begin{aligned} M[w_2] &= (v')^{N-1} v'' \prod_{i=1}^{N-1} \tfrac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \\ &= r^{N-1} K_1(r) f(w_2) \prod_{i=1}^{N-1} \tfrac{\kappa_i(\bar{x})}{1 - d(x)\kappa_i(\bar{x})} \\ &\leq K(x) f(w_2) \end{aligned}$$

for small R.

It follows from Lemma 2.1 $u_n(x) \leq w_2(x)$, $x \in B(x_0; R)$ for all $n \geq m+1$. Then u_n have an uniform bound from above in $B(x_0; R/2)$ for $n \geq m+1$. Covering Ω_m with finite ball of this kind, one gets the uniform bound C_m, i.e. there exists $C_m > 0$ such that

$$u_n(x) \leq C_m \text{ for } x \in \overline{\Omega}_m, \ n \geq m+1.$$

This implies that, for every $x \in \Omega$,

$$u(x) := \lim_{n \to \infty} u_n(x) \text{ exists}$$

and

$$u_m(x) \leq u(x) \leq C_m \text{ for } x \in \overline{\Omega}_m.$$

As we also have $u_n(x) \geq c_0 > 0$ in $\overline{\Omega}_m$ for $n \geq m+1$, and for such n, $\overline{\Omega}_m \subset \Omega_n$,

$$0 < \operatorname{dist}(\overline{\Omega}_m, \partial\Omega_{m+1}) \leq \operatorname{dist}(\overline{\Omega}_m, \partial\Omega_n) < \operatorname{dist}(\Omega_m, \partial\Omega),$$

we are in a position to apply Lemma 2.4 to conclude that, for any fixed integer $k \geq 1$, there exists a constant $C = C_{k,m}$ independent of n such that for all $n > m$,

$$\|u_n\|_{C^k(\overline{\Omega}_m)} \leq C.$$

It follows that the convergence $u_n(x) \to u(x)$ holds in $C^k_{loc}(\Omega)$ for every $k \geq 1$, and $u \in C^\infty(\Omega)$. Moreover, for $x \in \Omega$,

$$M[u](x) = \lim_{n\to\infty} M[u_n](x) = K(x) \lim_{n\to\infty} f(u_n(x)) = K(x)f(u(x)).$$

Since each u_n is strictly convex, $u(x)$ is strictly convex in Ω. Thus u is a strictly convex solution of (1.1).

□

Appendix

We present some basic facts of Karamata regular variation theory (refer to [60], [61]) here.

Definition AP.1. A positive measurable function f defined on $[A, \infty)$, for some $A > 0$, is called **regularly varying at infinity** with index $\rho \in R$, written $f \in RV_\rho$, if for all $\xi > 0$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{AP1}$$

In particular, when $\rho = 0$, f is called **slowly varying at infinity**.

Clearly, if $f \in RV_\rho$, then $L(s) = \frac{f(s)}{s^\rho}$ is slowly varying at infinity.

Definition AP.2. A positive measurable function f defined on $[A, \infty)$, for some $A > 0$, is called **rapidly varying at infinity** if for each $\rho > 1$

$$\lim_{s\to\infty} \frac{f(s)}{s^\rho} = \infty. \tag{AP2}$$

or

A positive measurable function f defined on $[A, \infty)$, for some $A > 0$, is called **rapidly varying at infinity** if for each $\xi > 1$

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \infty.$$

Proposition AP.1. (Uniform convergence theorem). If $f \in RV_\rho$, then (AP1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals (c_1, ∞) with $c_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals $(0, c_2]$ provided f is bounded on $(0, c_2]$ with $c_2 > 0$.

Proposition AP.2. (Representation theorem). A function L is slowly varying at infinity if and only if it may be written in the form

$$L(s) = \psi(s) exp\left(\int_{A_1}^{s} \frac{y(\tau)}{\tau} d\tau\right), s \geq A_1,$$

for some $A_1 \geq A$, where the function ψ and y are continuous and for $s \to \infty$, $y(s) \to 0$ and $\psi(s) \to c_0$, with $c_0 > 0$.

We say that

$$\hat{L}(s) = c_0 exp\left(\int_{A_1}^{s} \frac{y(\tau)}{\tau} d\tau\right)$$

is **normalized slowly varying at infinity** and

$$f(s) = s^{\rho}\hat{L}(s),\ s \geq A_1,$$

is **normalized regularly varying at infinity** with index ρ and write $f \in NRV_{\rho}$.

Proposition AP.3. A function $f \in RV_{\rho}$ belongs to NRV_{ρ} if and only if

$$f \in C^1[A_1, \infty),\ for\ some\ A_1 > 0\ and\ \lim_{s\to\infty}\frac{sf'(s)}{f(s)} = \rho.$$

Proposition AP.4. If function f, g, L are slowly varying at infinity,then
(1) f^p for every $p \in R$, $c_1f + c_2g(c_1, c_2 \geq 0)$,$f \circ g(if\ g(s) \to 0\ as\ s \to 0^+)$ are also slowly varying at infinity.
(2) For every $\rho > 0$ and $s \to \infty$,

$$s^{-\rho}L(s) \to 0,\ s^{\rho}L(s) \to \infty.$$

(3) For $\rho \in R$ and $s \to \infty$, $\frac{\ln(L(s))}{\ln s} \to 0$ and $\frac{\ln(s^{\rho}L(s))}{\ln s} \to \rho$.
Proposition AP.5. If $f_1 \in RV_{\rho_1}, f_2 \in RV_{\rho_2}$, then $f_1f_2 \in RV_{\rho_1+\rho_2}$ and $f_1 \circ f_2 \in RV_{\rho_1\rho_2}$.
Proposition AP.6. (Asymptotic behavior) If a function L is slowly varying at infinity, then for $a \geq 0$ and $t \to \infty$,
(1) $\int_a^t s^{\rho}L(s)ds \cong (1+\rho)^{-1}t^{1+\rho}L(t)$, for $\rho > -1$;
(2) $\int_t^{\infty} s^{\rho}L(s)ds \cong (-1-\rho)^{-1}t^{1+\rho}L(t)$, for $\rho < -1$.
Remark AP.1. The result of Proposition AP.6 remains true for $\rho = -1$ in the sense that

$$\frac{\int_a^t s^{-1}L(s)ds}{L(t)} \to \infty \text{ as } t \to \infty.$$

The way to remember Proposition AP.6 is that $L(s)$ can be taken out of the integral as if it were $L(t)$, thus

$$\int_a^t s^{\rho}L(s)ds \sim L(t)\int_a^t s^{\rho}ds (t \to \infty).$$

When $\rho = -1$, let $z(s) = s^{-1}L(s)$, we have
Proposition AP.7. (Asymptotic behavior)(See Karamata's Theorem 1.5.9b in [60].) If a function $z \in RV_{-1}$ and $\int_s^{\infty} z(\tau)d\tau < \infty$, $s > 0$, then $\int_s^{\infty} z(\tau)d\tau$ is slowly varying at infinity and

$$\lim_{s\to\infty}\frac{sz(s)}{\int_s^{\infty} z(\tau)d\tau} = 0.$$

By (1.3) we have

$$\Psi'(s) = -[(N+1)F(s)]^{-\frac{1}{N+1}},\ \Psi''(s) = [(N+1)F(s)]^{-\frac{1}{N+1}-1}f(s).$$

It follows that

$$-\frac{1}{\Psi'(s)} = [(N+1)F(s)]^{\frac{1}{N+1}}. \tag{AP3}$$

By (1.5) we have

$$I(s) = \frac{f(s)\Psi(s)}{[(N+1)F(s)]^{\frac{N}{N+1}}}.$$

Then we have
Lemma AP.1. Let f satisfy (**f1**) and (1.3). Then $I_{\infty} \geq 1$.
Proof. It is easy to prove by integrating $I(s)$ from $a(a > 0)$ to v. So we omit it. □
Lemma AP.2. Let f satisfy (**f1**), (1.3). We have
(1) $I_{\infty} \in (1, \infty)$ if and only if $F \in NRV_{q+1}$ with $q > N$. In this case, $f \in RV_q$;
(2) $I_{\infty} = 1$ if and only if F is rapidly varying at infinity;
(3) $I_{\infty} = \infty$ if and only if $F \in NRV_{N+1}$. In this case, $f \in RV_N$.
Proof. It can be proved by the definition of regularly varying and rapidly varying at infinity. So we omit it. □
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