Research Article

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Concentrating solutions for a planar elliptic problem with large nonlinear exponent and Robin boundary condition

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Abstract: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $b(x) > 0$ a smooth function defined on $\partial \Omega$. We study the following Robin boundary value problem:

$$\begin{cases}
\Delta u + u^p = 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} + \lambda b(x) u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\nu$ denotes the exterior unit vector normal to $\partial \Omega$, $0 < \lambda < +\infty$ and $p > 1$ is a large exponent. We construct solutions of this problem which exhibit concentration as $p \to +\infty$ and simultaneously as $\lambda \to +\infty$ at points that get close to the boundary, and show that in general the set of solutions of this problem exhibits a richer structure than the problem with Dirichlet boundary condition.

Keywords: Concentrating solutions, large exponent, Robin boundary condition, finite-dimensional reduction

MSC 2010: Primary 35J25; secondary 35B25, 35B38

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $b(x) > 0$ a smooth function defined on $\partial \Omega$. This paper deals with the analysis of solutions of the Robin boundary value problem

$$\begin{cases}
\Delta u + u^p = 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} + \lambda b(x) u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\nu$ denotes the exterior unit vector normal to $\partial \Omega$, $0 < \lambda < +\infty$ and $p > 1$ is a large exponent. Some solutions of (1.1) can be obtained as appropriately scaled extremals of

$$S_p^\lambda = \inf_{u \in H^1(\Omega)} P_p^\lambda(u),$$

where

$$P_p^\lambda(u) = \frac{\int_\Omega |\nabla u|^2 + \lambda \int_{\partial \Omega} b(x) u^2}{\left(\int_\Omega |u|^{p+1}\right)^{2/(p+1)}},$$

which are guaranteed to exist thanks to the compactness of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. They are known as least energy solutions of (1.1).

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Problem (1.1) has different names depending on the different values of the parameter \( \lambda \). It is called Dirichlet if \( \lambda = +\infty \), Neumann if \( \lambda = 0 \) and Robin if \( 0 < \lambda < +\infty \). From integration by parts it is trivial to observe that problem (1.1) has no solution when \( \lambda = 0 \). It is also worth pointing out that the Robin boundary condition in (1.1) is particularly interesting in various branches of biological models (see [10, 16]). Formally, as \( \lambda \to +\infty \), the Robin boundary condition in (1.1) tends to the Dirichlet boundary condition \( u|_{\partial \Omega} = 0 \), and problem (1.1) becomes

\[
\begin{aligned}
\begin{cases}
\Delta u + u^p = 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\tag{1.2}
\]

Standard variational methods show that problem (1.2) always has a least energy solution \( u_p \). In [20, 21] it is proved that as \( p \to +\infty \), the solution \( u_p \) develops one interior peak, namely \( u_p \) approaches zero except one interior point where it stays bounded and bounded away from zero. Furthermore, up to a subsequence, the renormalized energy \( p u_p^{p-1} \) concentrates as a Dirac mass around a critical point of the Robin function \( H_\infty(x, x) \), where \( H_\infty \) is the regular part of Green’s function \( G_\infty \) of the Dirichlet Laplacian in \( \Omega \). Namely, Green’s function \( G_\infty(x, y) \) is the solution of the problem

\[
\begin{aligned}
\begin{cases}
-\Delta G_\infty(x, y) = \delta_j(x) & \text{in } \Omega, \\
G_\infty(x, y) = 0 & \text{on } \partial \Omega, 
\end{cases}
\end{aligned}
\]

and \( H_\infty(x, y) \) is the regular part defined as

\[
H_\infty(x, y) = G_\infty(x, y) - \frac{1}{2\pi} \log \frac{1}{|x-y|}.
\]

In [1] and [11] Adimurthi and Grossi, and El Mehdi and Grossi, respectively, give a further description of the asymptotic behavior of \( u_p \) as \( p \to +\infty \), by identifying a limit profile problem of Liouville-type, that is,

\[
\begin{aligned}
\begin{cases}
\Delta u + e^u = 0 & \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^u < +\infty,
\end{cases}
\end{aligned}
\]

and show that \( \|u_p\|_{C^1} \to \sqrt{e} \) as \( p \to +\infty \). On the other hand, problem (1.2) may have solutions with an arbitrarily large number of interior peaks, as shown by Esposito, Musso and Pistoia in [13]. Indeed, they proved that if \( \Omega \) is not simply connected, then given any \( m \geq 1 \) and for \( p \) large enough, problem (1.2) has a family of solutions \( u_p \) which concentrates at \( m \) different points of \( \Omega \), namely

\[
p u_p^{p-1} \to 8\pi e \sum_{i=1}^m \delta_{\xi_i} \quad \text{weakly in the sense of measure in } \overline{\Omega} \tag{1.3}
\]

as \( p \to +\infty \), and the peaks of these solutions are located near the points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega^m \) corresponding to a critical point of the function

\[
\varphi_m^\infty(\xi_1, \ldots, \xi_m) = \sum_{j=1}^m H_\infty(\xi_j, \xi_j) + \sum_{j<k} G_\infty(\xi_j, \xi_k).
\]

More precisely, there is an \( m \)-tuple \( \xi^p = (\xi_{1,p}, \ldots, \xi_{m,p}) \in \Omega^m \) converging to \( \xi \) such that for any \( d > 0 \), as \( p \to +\infty \),

\[
u_p \to 0 \quad \text{uniformly in } \Omega \setminus \bigcup_{i=1}^m B_d(\xi_{i,p}) \quad \text{and} \quad \sup_{x \in \Omega \cap B_d(\xi_{i,p})} u_p(x) \to \sqrt{e}.
\]

Additionally, Grossi and Takahashi in [15] prove that when \( \Omega \) is convex, problem (1.2) has no solutions that have multiple interior peaks and satisfy property (1.3). Thus, the topological assumption on the domain in [13] is sharp for the construction of multi-peak solutions to problem (1.2).

In contrast, we will see that for any bounded smooth domain, when \( \lambda \) and \( p \) are large enough, the set of solutions of the so called Robin problem (1.1) is much richer.
Given $\lambda > 0$, let $G_\lambda(x, y)$ be Green’s function with Robin boundary condition, i.e.,

\[
\begin{align*}
-\Delta_x G_\lambda(x, y) &= \delta_y(x) & \text{in } \Omega, \\
\frac{\partial G_\lambda}{\partial \nu}(x, y) + \lambda b(x) G_\lambda(x, y) &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

and let $H_\lambda(x, y)$ be its regular part defined as

\[
H_\lambda(x, y) = G_\lambda(x, y) - \frac{1}{2\pi} \log \frac{1}{|x-y|}.
\]  

(1.4)

As in the Dirichlet problem (1.2), critical points of the Robin function $H_\lambda(x, x)$ may play a similar role in determining the location where solutions of problem (1.1) exhibit single-point concentration phenomena. From [7] the Robin function $H_\lambda(x, x)$ admits two precise asymptotic formulas, that is, if we set $d(x) := \text{dist}(x, \partial \Omega)$ and introduce the notation $\hat{x}$ as the point on $\partial \Omega$ closest to $x$, then for any $\alpha > 1$, there exists $\lambda_a$ such that for each $\lambda \geq \lambda_a$ and for each $x \in \Omega$ satisfying $a^{-1} < \lambda d(x) < a$, one has that if $b$ is not a constant function,

\[
H_\lambda(x, x) = \frac{1}{2\pi} h_\lambda(\lambda d(x), b(\hat{x})) + O\left(\frac{1}{\lambda}\right),
\]  

(1.5)

while if $b \equiv 1$,

\[
H_\lambda(x, x) = \frac{1}{2\pi} h_\lambda(\lambda d(x), 1) + \frac{1}{2\pi \lambda} \kappa(\hat{x}) \nu(\lambda d(x)) + O\left(\frac{1}{\lambda^{1+\alpha}}\right),
\]  

(1.6)

where $0 < a < 1$, $\kappa(\hat{x})$ is the mean curvature of $\partial \Omega$ at $\hat{x}$

\[
\begin{align*}
\lambda \kappa(\hat{x}) &= -\log \lambda - \log 2\theta + 2 \int_0^\infty e^{-t} \log \left(2\theta + \frac{1}{b}\right) \, dt
\end{align*}
\]  

(1.7)

and

\[
\begin{align*}
\nu(\theta) &= -\frac{\theta}{2} - \theta \int_0^\infty e^{-2bs} \frac{1}{(1+s)^2} \, ds.
\end{align*}
\]  

(1.8)

Moreover, for any $\hat{b} > 0$ the function $h_\lambda(\cdot, \hat{b})$ has a unique minimum $\theta_b \in (0, +\infty)$, which is non-degenerate and independent of $\lambda$. Thus, formulas (1.5)–(1.6) suggest that for $\lambda$ and $p$ large enough, problem (1.1) may have solutions which exhibit one or multiple points concentration phenomena in the intermediate region $\lambda d(x) = O(1)$. Our interest here is to find whether these solutions of problem (1.1) exist as $p \to +\infty$ and simultaneously as $\lambda \to +\infty$.

The first result we will establish is that when $p$ and $\lambda$ satisfy some restrained growth conditions, for any $m \geq 1$, problem (1.1) possesses at least two solutions which concentrate at $m$ different points with distance to the boundary uniformly approaching zero. Meanwhile, the location of such $m$ points is related to critical points of the function

\[
\varphi_m(\xi_1, \ldots, \xi_m) = \sum_{j=1}^m H_\lambda(\xi_j, \xi_j) + \sum_{j<k} G_\lambda(\xi_j, \xi_k).
\]  

(1.9)

**Theorem 1.1.** Let $m \geq 1$ be an integer. There exist $p_m > 0$ and $\lambda_m > 0$ such that for any $p > p_m$ and $\lambda > \lambda_m$ satisfying $\frac{\log \lambda}{p} = o(1)$ and $\frac{\lambda}{p \log \lambda} = O(1)$, there are at least two solutions $u_{p,\lambda}$ for problem (1.1) with $m$ different concentration points $\xi_{1,p,\lambda}, \ldots, \xi_{m,p,\lambda}$ located at distance $O(1/\lambda)$ from $\partial \Omega$. More precisely,

\[
u(\theta) = \frac{\theta}{2} - \theta \int_0^\infty e^{-2bs} \frac{1}{(1+s)^2} \, ds.
\]

\[
where the parameters $\gamma$, $\delta_j$ and $\mu_j$ satisfy
\[
\gamma = p^{p/(p-1)} \rho^{2/(p-1)}, \quad \delta_j = \mu_j \rho, \quad \mu_j = \frac{1}{\lambda} e^{-p/\rho}, \quad \frac{1}{\rho} < \mu_j < C
\]

for some $C > 0$, and $\xi_{j,p,\lambda} \in \partial \Omega$ satisfies

\[
|\lambda d(\xi_{j,p,\lambda}) - \theta_{b(\xi_{j,p,\lambda})}| = O(\lambda^{-1/2}) \quad \text{as } \lambda \to +\infty
\]
Theorem 1.3. is a non-degenerate critical point of the mean curvature function
and (1.10) has at least two solutions with a concentration point near the boundary. Additionally, when
to the boundary. More precisely, they prove that if
concentration phenomena as
\[
\lim_{\lambda \to +\infty} \nabla \varphi_m^\lambda (\xi_{1,p,\lambda}, \ldots, \xi_{m,p,\lambda}) = 0 \quad \text{uniformly for all large } \lambda,
\]
where \(\varphi_m^\lambda\) is defined in (1.9). In particular, for any \(\varepsilon \in (0, 1)\), as \(\lambda \to +\infty\),
\[
p \nu_{p,\lambda}^{\varepsilon+1} - 8\pi \varepsilon \sum_{j=1}^{m} \delta_{\xi_{j,p,\lambda}} \to 0 \quad \text{weakly in the sense of measure in } \Omega,
\]
\[
u_{p,\lambda} \to 0 \quad \text{uniformly in } \Omega \setminus \bigcup_{j=1}^{m} B_{\delta}(\xi_{j,p,\lambda})(\xi_{j,p,\lambda}),
\]
and
\[
\sup_{B_{\delta}(\xi_{j,p,\lambda})(\xi_{j,p,\lambda})} u_{p,\lambda} \to \sqrt{\varepsilon}.
\]
Note that as in the Dirichlet problem (1.2), when \(\Omega\) is not simple connected, one can find that given any \(m \geq 1\), for any fixed \(\lambda\) large, problem (1.1) has a third solution \(u_{p,\lambda}\) with \(m\) different peak points \(\xi_{1,p,\lambda}, \ldots, \xi_{m,p,\lambda}\) uniformly away from the boundary and each other as \(p \to +\infty\). We will not provide the construction of this solution because it can be easily realized using the same approach as in [13]. Besides, under a faster restrained growth condition of \(p\) and \(\lambda\), we also obtain the case \(m = 1\) of Theorem 1.1 and find solutions with a concentration point near the boundary.

Theorem 1.2. There exist \(p_0 > 0\) and \(\lambda_0 > 0\) such that for any \(p > p_0\) and \(\lambda > \lambda_0\) satisfying \(\frac{1}{p} \log \lambda = o(1)\), problem (1.1) has at least two solutions \(u_{p,\lambda}\) with a concentration point \(\xi_{p,\lambda}\) such that
\[
|\lambda d(\xi_{p,\lambda}) - \theta_{b(\xi_{p,\lambda})}| = O(\lambda^{-1/2}) \quad \text{as } \lambda \to +\infty.
\]
The following theorems deal with the case \(b = 1\) and the case where \(b\) is not a constant function, respectively.

Theorem 1.3. Assume \(b = 1\). If \(x_0 \in \partial \Omega\) is a non-degenerate critical point of \(\kappa\), then for each \(\beta \in (0, 1)\), there exist \(p_0 > 0\) and \(\lambda_0 > 0\) such that for any \(p > p_0\) and \(\lambda > \lambda_0\) satisfying \(\frac{1}{p} \lambda^{1+\beta} = o(1)\), problem (1.1) has a solution \(u_{p,\lambda}\) with a concentration point \(\xi_{p,\lambda}\) such that
\[
|\lambda d(\xi_{p,\lambda}) - \theta_{b(\xi_{p,\lambda})}| = O(\lambda^{-1/2}) \quad \text{and} \quad |\xi_{p,\lambda} - x_0| = O(\lambda^{\beta}) \quad \text{as } \lambda \to +\infty.
\]

Theorem 1.4. Let \(x_0 \in \partial \Omega\) be a non-degenerate critical point of \(b\). Then, for each \(\beta \in (0, 1)\), there exist \(p_0 > 0\) and \(\lambda_0 > 0\) such that for any \(p > p_0\) and \(\lambda > \lambda_0\) satisfying \(\frac{1}{p} (\lambda^{1+\beta}/3 + \lambda^{\beta}) = o(1)\), problem (1.1) has a solution \(u_{p,\lambda}\) with a concentration point \(\xi_{p,\lambda}\) such that
\[
|\lambda d(\xi_{p,\lambda}) - \theta_{b(\xi_{p,\lambda})}| = O(\lambda^{-1/2}) \quad \text{and} \quad |\xi_{p,\lambda} - x_0| = O(\lambda^{\beta}) \quad \text{as } \lambda \to +\infty.
\]
Let us remark the interesting analogy between all above results and those known for the Liouville equation with Robin boundary condition
\[
\begin{aligned}
\Delta u + \varepsilon^2 e^u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} + \lambda b(x)u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \(\Omega \subset \mathbb{R}^2\) is a bounded smooth domain, \(\varepsilon > 0\) is small but \(\lambda > 0\) is large, and \(b(x) > 0\) is a smooth function defined on \(\partial \Omega\). For \(b(x) \equiv 1\), Davila and Topp in [8] construct solutions of problem (1.10) which exhibit concentration phenomena as \(\varepsilon \to 0\) and simultaneously as \(\lambda \to +\infty\) at one or multiple points that get close to the boundary. More precisely, they prove that if \(\varepsilon\) and \(\lambda\) satisfy some restrained condition, then for any \(m \geq 1\), problem (1.10) has at least two solutions with \(m\) different concentration points located at distance \(O(1/\lambda)\) from \(\partial \Omega\), while if \(\varepsilon\) and \(\lambda\) satisfy a weaker restrained condition so that \(\lambda\) has a faster growth, problem (1.10) has at least two solutions with a concentration point near the boundary. Additionally, when \(x_0\) is a non-degenerate critical point of the mean curvature function \(\kappa\) on \(\partial \Omega\), they find that under a certain restrained condition of \(p\) and \(\lambda\), problem (1.10) has a solution with a concentration point located at distance \(O(1/\lambda)\) from \(x_0\).
Finally, it is necessary to compare Theorem 1.2 with the results given in [3], which is concerned with the following singularly perturbed elliptic problem with Robin boundary condition:

\[
\begin{aligned}
\epsilon^2 \Delta u - u + f(u) &= 0, \quad u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} + \lambda u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.11)

where \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is a bounded smooth domain, \( \epsilon > 0 \) is a small constant, \( 0 < \lambda < +\infty, f(u) = u^p - au^q \) with \( a \geq 0 \), and \( 1 < q < p < \frac{q(N-2)}{N-2} \) if \( N \geq 3 \) whereas \( 1 < q < p < +\infty \) if \( N = 2 \). The cases \( \lambda = 0 \) (Neumann problem) and \( \lambda = +\infty \) (Dirichlet problem) have been studied, respectively, in [17, 18] and [19], proving that the least energy solution of problem (1.11) with Dirichlet boundary condition concentrates a point that maximizes the distance to the boundary, while Neumann concentration takes place at a point on the boundary. In [3] it is proved that for fixed small \( \epsilon \), as \( \lambda \) moves from 0 to \( +\infty \), the asymptotic behavior of the least energy solution of problem (1.11) changes dramatically. It will be like in the case of Neumann boundary condition if \( \lambda < \lambda_0/\epsilon \) and like in the case of Dirichlet boundary condition if \( \lambda > \lambda_0/\epsilon \), where \( \lambda_0 \) is the borderline number associated to the corresponding problem on the half space. Thus, \( \lambda \sim 1/\epsilon \) represents a dramatic change in asymptotic behavior of the least energy solution of problem (1.11). Our results suggest that for least energy solutions of problem (1.1), the critical range of \( \lambda \) may be \( \log \lambda \sim p \).

The general strategy for proving our main results relies on the very well-known Lyapunov–Schmidt reduction, which has been used in many papers, for example, in [8, 9, 12, 13]. The paper is organized as follows. In Section 2 we exactly describe the ansatz for the solution of problem (1.1) and estimate the error. Then, we rewrite problem (1.1) in terms of a linearized operator for which a solvability theory, subject to suitable orthogonality conditions, is performed through solving a linearized problem in Section 3. In Section 4 we solve an auxiliary nonlinear problem. In Section 5 we reduce (1.1) to a finite system. In the last section, we give the proof of Theorems 1.1–1.4.

In this paper, the symbol \( C \) denotes always a generic positive constant independent of \( p \) and \( \lambda \), which could be changed from one line to another. The symbols \( O(t) \) (respectively \( o(t) \)) denote quantities for which \( \frac{O(t)}{t} \) stays bounded (respectively, \( \frac{o(t)}{t} \) tends to zero) as the parameter \( t \) goes to zero. In particular, we will often use the notations \( O(1) \) (respectively, \( o(1) \)), which represents a quantity that remains uniformly bounded (respectively, that tends to zero) as \( p \to +\infty \) and simultaneously as \( \lambda \to +\infty \).

## 2 A first approximation of the solution

In this section we will provide an ansatz for solutions of problem (1.1). A key ingredient to describe an approximate solution of problem (1.1) is given by the standard bubble:

\[
U_{\delta, \xi}(x) = \log \frac{8\delta^2}{(\delta^2 + |x - \xi|^2)^2}, \quad \delta > 0, \quad \xi \in \mathbb{R}^2.
\]  

(2.1)

It is well known (see [5]) that those are all the solutions of the following Liouville-type equation:

\[
\begin{aligned}
\Delta u + e^u &= 0 \quad \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^u &= +\infty.
\end{aligned}
\]

The configuration space we try to look for \( m \) concentration points \( \xi = (\xi_1, \ldots, \xi_m) \) is the following:

\[
\mathcal{O}_\varepsilon := \left\{ \xi = (\xi_1, \ldots, \xi_m) \in \Omega^m : |\xi_j - \xi_k| \geq \varepsilon, |\lambda d(\xi_j) - \theta_{\delta, \xi_j}| \leq \frac{1}{\varepsilon \lambda^{1/2}}, j, k = 1, \ldots, m, j \neq k \right\},
\]  

(2.2)

where \( \varepsilon > 0 \) is a sufficiently small but fixed number. Note that by the choice of \( \xi_j \), if \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_\varepsilon \), then we have

\[
d(\xi_j) = \text{dist}(\xi_j, \partial \Omega) = O\left( \frac{1}{\lambda} \right) \quad \text{and} \quad \theta_{\delta, \xi_j} = O(1) \quad \text{for all } j = 1, \ldots, m.
\]  

(2.3)
In what remains of this paper, we will always assume the following restrained growth condition:

\[
\frac{1}{p} \log \lambda \to 0 \quad \text{as } p \to +\infty \text{ and simultaneously as } \lambda \to +\infty. \tag{2.4}
\]

Also, for each \( j = 1, \ldots, m \), we set

\[
y(p) = p^{p/(p-1)} \rho^{2/(p-1)}, \quad \delta_j = \mu_j \rho, \quad \rho = \frac{1}{\lambda} e^{-p/4}, \tag{2.5}
\]

where the choice of \( \mu_j \) will be determined later. Define now

\[
U_j(x) = \frac{1}{y \mu_j^{2/(p-1)}} \left[ U_{\delta_j}(x) + \frac{1}{p} \omega_1 \left( \frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} \omega_2 \left( \frac{x - \xi_j}{\delta_j} \right) \right]. \tag{2.6}
\]

Here, \( \omega_i, i = 1, 2 \), are radial solutions of

\[
\Delta \omega_i + \frac{8}{(1 + |y|^2)^2} \omega_i = \frac{8}{(1 + |y|^2)^2} f_i(y) \quad \text{in } \mathbb{R}^2
\]

with

\[
f_1 = \frac{1}{2} U^2_{1,0}, \quad f_2 = \omega_1 U_{1,0} - \frac{1}{3} U^2_{1,0} - \frac{1}{2} \omega_2^2 - \frac{1}{8} U^4_{1,0} + \frac{1}{2} \omega_1 U^2_{1,0}, \tag{2.8}
\]

having the asymptotic properties

\[
\begin{align*}
\omega_i(y) &= C_i \log |y| + O \left( \frac{1}{|y|^\frac{1}{2}} \right) \quad \text{as } |y| \to +\infty, \\
\nabla \omega_i(y) &= C_i \cdot \frac{y}{|y|^2} + O \left( \frac{1}{1 + |y|^2} \right) \quad \text{for all } y \in \mathbb{R}^2,
\end{align*}
\]

where

\[
C_i = 8 \int_0^\infty \frac{t^2 - 1}{(t^2 + 1)^2} f_i(t) \, dt. \tag{2.9}
\]

In particular,

\[
\omega_1(y) = \frac{1}{2} U^2_{1,0}(y) + 6 \log(|y|^2 + 1) + \frac{2}{|y|^2 + 1} \log \left( \frac{|y|^2 - 1}{|y|^2 + 1} \right) + \frac{1}{|y|^2 + 1} \log \left( \frac{|y|^2 - 1}{|y|^2 + 1} \right)
\]

\[
\times \left[ \frac{1}{2} \log^2 8 \log(|y|^2 + 1) + 4 \int_0^\infty \frac{ds}{s+1} \log \frac{s+1}{s} - 8 \log |y| |\log(|y|^2 + 1)| \right]. \tag{2.10}
\]

and (see [4, 13])

\[
C_1 = 12 - 4 \log 8. \tag{2.11}
\]

Our ansatz for a solution of problem (1.1) is

\[
U_{\xi}(x) = \sum_{j=1}^m \left[ U_j(x) + H_j(x) \right], \tag{2.12}
\]

where \( H_j \) is a correction term defined as the solution of the following problem:

\[
\begin{align*}
-\Delta H_j &= 0 & \text{in } \Omega, \\
\frac{\partial H_j}{\partial \nu} + \lambda b(x) H_j &= - \frac{\partial U_j}{\partial \nu} - \lambda b(x) U_j & \text{on } \partial \Omega.
\end{align*} \tag{2.13}
\]

Lemma 2.1. For any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_\epsilon \),

\[
H_j(x) = \frac{1}{y \mu_j^{2/(p-1)}} \left[ \left( 1 - \frac{C_1}{4p} \right) \frac{C_2}{4p^2} \delta \log(8 \mu_j^2 \rho^2) + \left( \frac{C_1}{p} + \frac{C_2}{p^2} \right) \log(\mu_j \rho) + \frac{\mu \lambda \rho}{p} \right] \tag{2.14}
\]

uniformly in \( C(\bar{\Omega}) \) and in \( C^2_{\text{loc}}(\Omega) \) as \( p \) and \( \lambda \) go to \( +\infty \) and satisfy (2.4), where \( H_\lambda \) is the regular part of Green’s function defined in (1.4).
Proof. First, on the boundary, by (2.1) and (2.9), we have

\[
\frac{\partial H_j}{\partial \nu} + \lambda b(x)H_j = -\frac{1}{y\mu_j^{2/(p-1)}} \left[ \left( -4 + \frac{C_1}{p} + \frac{C_2}{p^2} \right) \frac{1}{\mu_j^{1/p^2} + |x - \xi_j|^2} \right] (x - \xi_j) \cdot \nu(x)
\]

\[
+ \lambda b(x) \log \frac{1}{\mu_j^{1/p^2} + |x - \xi_j|^2} + \left( \frac{1}{p} + \frac{C_2}{p^2} \right) \log |x - \xi_j|
\]

\[
+ H_j \left[ \log(\lambda b(x) + \lambda \rho) \right] + \frac{1}{p} O \left( \frac{\lambda b(x) + \lambda \rho}{|x - \xi_j|^2} \right)
\]

The regular part of Green’s function with Robin boundary condition \( H_{\lambda}(x, \xi) \) satisfies

\[
\begin{cases}
-\Delta H_{\lambda}(x, \xi) = 0 \\
\frac{\partial H_{\lambda}(x, \xi)}{\partial \nu} + \lambda b(x)H_{\lambda}(x, \xi) = \frac{1}{2\pi} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} - \frac{1}{2\pi} \lambda b(x) \log \frac{1}{|x - \xi_j|} \\
\end{cases}
\]

in \( \Omega \), on \( \partial \Omega \). (2.15)

So, if we set

\[
\bar{H}_j(x) = y\mu_j^{2/(p-1)} H_j(x) - \left[ \left( 1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi H_{\lambda}(x, \xi_j) \log(\lambda b(x) + \lambda \rho) + \left( \frac{1}{p} + \frac{C_2}{p^2} \right) \log |x - \xi_j| \right],
\]

then \( \bar{H}_j(x) \) satisfies \(-\Delta \bar{H}_j = 0 \) in \( \Omega \), and

\[
\frac{\partial \bar{H}_j}{\partial \nu} + \lambda b(x)\bar{H}_j = -\frac{4\mu_j^{1/p^2}}{\mu_j^{1/p^2} + |x - \xi_j|^2} (x - \xi_j) \cdot \nu(x) - \lambda b(x) \log \frac{|x - \xi_j|^2}{\mu_j^{1/p^2} + |x - \xi_j|^2} + \frac{1}{p} O \left( \frac{\lambda b(x) + \lambda \rho}{|x - \xi_j|^2} \right)
\]

on \( \partial \Omega \). From the maximum principle with Robin boundary condition (see [7, Lemma 2.6]), we deduce

\[
\max_{\bar{\Omega}} |\bar{H}_j(x)| + \max_{\bar{\Omega}} |d(x) \nabla \bar{H}_j(x)| \leq C \left( \frac{1}{\lambda} \left\| \frac{\partial \bar{H}_j}{\partial \nu} + \lambda b(x) \bar{H}_j \right\|_{L^\infty(\partial \Omega)} \right) = \frac{1}{p} O \left( \frac{\lambda b(x) + \lambda \rho}{|x - \xi_j|^2} \right).
\]

Thus, using the interior estimate of the derivative of a harmonic function (see [14, Theorem 2.10]), we get

\[
\max_K |D^a \bar{H}_j(x)| \leq \left( \frac{2|a|}{\lambda} \right)^{|a|} \max_{\bar{\Omega}} |\bar{H}_j(x)| = O \left( \frac{\lambda b(x) + \lambda \rho}{|x - \xi_j|^2} \right)
\]

for any compact subset \( K \) of \( \Omega \) and any multi-index \( a \) with \( |a| \leq 2 \). This derives estimate (2.14) uniformly in \( C(\Omega) \) and in \( C^2_{loc}(\Omega) \) for any sufficiently large \( p \) and \( \lambda \) satisfying (2.4). \( \square \)

From Lemma 2.1, away from the points \( \xi_j \), namely \( |x - \xi_j| \geq \epsilon d(\xi_j) \) for any \( j = 1, \ldots, m \), one has

\[
U_{\epsilon_j}(x) = \sum_{j=1}^m \frac{1}{y\mu_j^{2/(p-1)}} \left[ \left( 1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_{\lambda}(x, \xi_j) + \frac{1}{p} O(\lambda b(x)) \right]. \tag{2.16}
\]

While for \( |x - \xi_j| \leq \epsilon d(\xi_j) \) with some \( j \), if we write \( x = \xi_j + \delta_j y \), then, using (2.2), (2.5), (2.6), (2.12), (2.14), (2.16) and the fact that \( \nabla H_{\lambda}(x, \xi) = O(\lambda) \) uniformly holds in the intermediate region \( \lambda d(x) = O(1) \) (see [7]), we deduce

\[
U_{\epsilon_j}(x) = \frac{1}{y\mu_j^{2/(p-1)}} \left[ U_{\epsilon_j, \lambda}(x) + \frac{1}{p} \omega_1(x - \xi_j) + \frac{1}{p^2} \omega_2(x - \xi_j) + \left( 1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_{\lambda}(x, \xi_j) \right.
\]

\[
- \log(\lambda b(x) + \lambda \rho) + \left( \frac{1}{p} + \frac{C_2}{p^2} \right) \log(\lambda b(x) + \lambda \rho) + \frac{1}{p} O(\lambda b(x)) \right]
\]

\[
+ \sum_{k \neq j} \frac{1}{y\mu_k^{2/(p-1)}} \left[ \left( 1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_{\lambda}(x, \xi_k) + \frac{1}{p} O(\lambda b(x)) \right]
\]

\[
= \frac{1}{y\mu_j^{2/(p-1)}} \left[ U_{1,\lambda}(y) - \log(\lambda b(x) + \lambda \rho) + \frac{1}{p} \omega_1(y) + \frac{1}{p^2} \omega_2(y) + \log(\lambda b(x) + \lambda \rho) + O(\lambda b(x)) \right]
\]

\[
- \log(\lambda b(x) + \lambda \rho) + \left( \frac{1}{p} + \frac{C_2}{p^2} \right) \log(\lambda b(x) + \lambda \rho) + O(\lambda b(x)) \right]
\]

\[
+ \sum_{k \neq j} \frac{1}{y\mu_k^{2/(p-1)}} \left[ \left( 1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8\pi G_{\lambda}(x, \xi_k) + \frac{1}{p} O(\lambda b(x)) \right]
\]
Remark 2.2. Let us remark that $U_\xi$ is a positive uniformly bounded function. Observe that for $|y| \leq \varepsilon d(\xi)/\delta_j$,

$$p + U_{1,0}(y) + \frac{1}{p} \omega_1(y) + \frac{1}{p^2} \omega_2(y) \geq p + \log 8 - 2 \log \left( 1 + \frac{e^2 d(\xi)^2}{\delta_j^2} \right) + \frac{1}{4} + O\left( \frac{1}{p} \right)$$

$$= p + \log 8 - 4 \log \frac{\varepsilon d(\xi)}{\delta_j} + \frac{1}{4} + O\left( \frac{\delta_j^2}{e^2 d(\xi)^2} \right) + O\left( \frac{1}{p} \right)$$

$$= 4 \log \frac{1}{\varepsilon} + \log (8 \mu_j^0) - 4 \log (\lambda d(\xi_j)) + \frac{1}{4} + O\left( \frac{1}{p} \right).$$

By choosing $\varepsilon > 0$ small enough, it is easy to check that $U_\xi > 0$ in $B(\xi_j, \varepsilon d(\xi))$, and $\sup_{B(\xi_j, \varepsilon d(\xi))} U_\xi \to \sqrt{\varepsilon}$ as $p$ and $\lambda$ go to $+\infty$ and satisfy (2.4). Moreover, by the maximum principle, we see that $G_\lambda(x, \xi_j) > 0$ in $\overline{\Omega}$ and thus, by (2.16), $U_\xi$ is a positive uniformly bounded function in $\overline{\Omega}$. In conclusion, $0 < U_\xi \leq 2 \sqrt{\varepsilon}$ in $\overline{\Omega}$. 

We now choose the parameters $\mu_j$. We assume they are defined by the relation

$$\log(8 \mu_j^0) = 4 \log \lambda + \left( 1 - \frac{C_1}{4p} \right) 8 \pi H_\lambda(\xi_j, \xi_j) + \left( \frac{C_1}{p} + \frac{C_2}{p^2} \right) \log \left( \mu_j \cdot e^{-p/4} \right)$$

$$+ \sum_{k+j} \mu_k^{2/(p-1)} \left( 1 - \frac{C_1}{4p} - \frac{C_2}{4p^2} \right) 8 \pi G_\lambda(\xi_j, \xi_k).$$

Taking into account the explicit expression (2.11) of the constant $C_1$, we observe that as $p$ tends to $+\infty$, $\mu_j$ bifurcates from the value

$$\bar{\mu}_j = \lambda e^{-3/4} e^{2 \pi H_\lambda(\xi_j, \xi_j) + 2 \pi \sum_{k+j} G_\lambda(\xi_j, \xi_j)},$$

solution of the equation

$$\log(8 \mu_j^0) = 4 \log \lambda + 8 \pi H_\lambda(\xi_j, \xi_j) - \frac{1}{4} C_1 + 8 \pi \sum_{k+j} G_\lambda(\xi_j, \xi_k).$$

Thus, by (2.4), $\mu_j$ is a perturbation of order $\log \lambda/p$ of the value $\bar{\mu}_j$, namely

$$\mu_j = \lambda e^{-3/4} e^{2 \pi H_\lambda(\xi_j, \xi_j) + 2 \pi \sum_{k+j} G_\lambda(\xi_j, \xi_j)} \left[ 1 + O\left( \frac{\log \lambda}{p} \right) \right].$$

(2.17)

Moreover, from (1.5)–(1.8) and (2.2), there exists a constant $C > 0$ independent of $p, \lambda$ and $j$ such that for any sufficiently large $p$ and $\lambda$ satisfying (2.4),

$$\frac{1}{C} < \mu_j < C \tag{2.18}$$

and

$$|D_\xi \log \mu_j| < C \sqrt{\lambda}, \tag{2.19}$$

where the second inequality is due to the fact that if $x \in \Omega$ satisfies $|\lambda \text{dist}(x, \partial \Omega) - \theta_B(\xi)| = o(\lambda^{-1/2})$, then

$$|\nabla_\xi H_\lambda(x, x)| = O(\sqrt{\lambda}).$$

From this choice of the parameters $\mu_j$, we deduce that for $|x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi)_j,

$$U_\xi(x) = \frac{1}{\gamma \mu_j^{2/(p-1)}} \left[ p + U_{1,0}(y) + \frac{1}{p} \omega_1(y) + \frac{1}{p^2} \omega_2(y) + O(\lambda |y|) + \frac{1}{p} O(\lambda |\rho|) \right].$$

(2.20)
Let us define

\[ S_p(u) = \Delta u + u_p, \quad \text{where } u_+ = \max\{u, 0\}, \]

and introduce the following functional:

\[ J_p'(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega u^{p+1} + \frac{\lambda}{2} \int_\Omega b(x)u^2, \quad u \in H^1(\Omega), \]  \hspace{1cm} (2.21)

whose nontrivial critical points are solutions of problem (1.1). Obviously, by the maximum principle, problem (1.1) is equivalent to

\[ S_p(u) = 0, \quad u_+ \neq 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} + \lambda b(x)u = 0 \quad \text{on } \partial \Omega. \]

We will seek solutions of problem (1.1) in the form \( u = U_\xi + \phi \), where \( \phi \) will represent a higher order correction. Observe that

\[ S_p(U_\xi + \phi) = L(\phi) + R_\xi + N(\phi) = 0, \]

where

\[ L(\phi) = \Delta \phi + W_\xi \phi \quad \text{with} \quad W_\xi = pU_\xi^{p-1}, \]

\[ R_\xi = \Delta U_\xi + U_\xi^p \quad \text{and} \quad N(\phi) = (U_\xi + \phi)_+^p - U_\xi^p - pU_\xi^{p-1} \phi. \]

In terms of \( \phi \), problem (1.1) becomes

\[ \begin{cases} L(\phi) = -[R_\xi + N(\phi)] & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} + \lambda b(x)\phi = 0 & \text{on } \partial \Omega. \end{cases} \]  \hspace{1cm} (2.22)

The main step in solving problem (2.22) is a solvability theory for the linear operator \( L \) under a suitable choice of the points \( \xi_j \). In developing this theory, we will take into account the invariance, under translations and dilations, of the problem \( \Delta e^\varepsilon + e^\varepsilon = 0 \) in \( \mathbb{R}^2 \). We will apply the solvability theory for the linear operator \( L \) in a weighted \( L^\infty \)-norm space, following [9, 13]. For any \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_\varepsilon \) and \( h \in L^\infty(\Omega) \), define

\[ \|h\|_\star = \sup_{x \in \Omega} \left( \sum_{j=1}^m \left( \frac{\delta_j}{\delta_j + |x - \xi_j|^2} \right)^{3/2} h(x) \right). \]  \hspace{1cm} (2.23)

We conclude this section by showing an estimate of \( R_\xi \) in \( \| \cdot \|_\star \).

**Proposition 2.3.** Let \( \varepsilon > 0 \) be fixed and small. There exist \( C > 0, p_0 > 0 \) and \( \lambda_0 > 0 \) such that for any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_\varepsilon \), and any \( p \geq p_0 \) and \( \lambda \geq \lambda_0 \) satisfying (2.4),

\[ \|R_\xi\|_\star \leq \frac{C}{p^{4/3}}. \]  \hspace{1cm} (2.24)

**Proof.** Observe that by (2.6)–(2.7) and (2.12)–(2.13),

\[ \Delta U_\xi = \sum_{j=1}^m \frac{1}{y \mu_j^{p-1}} \left[ -e^{U_{\varepsilon_0}} + \frac{1}{p \delta_j^2} \Delta \omega_1 \left( \frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2 \delta_j^4} \Delta \omega_2 \left( \frac{x - \xi_j}{\delta_j} \right) \right] \]

\[ = \sum_{j=1}^m \frac{e^{U_{\varepsilon_0}}}{y \mu_j^{p-1}} \left[ -1 + \frac{1}{p} f_1 \left( \frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} f_2 \left( \frac{x - \xi_j}{\delta_j} \right) - \frac{1}{p} \omega_1 \left( \frac{x - \xi_j}{\delta_j} \right) - \frac{1}{p^2} \omega_2 \left( \frac{x - \xi_j}{\delta_j} \right) \right]. \]  \hspace{1cm} (2.25)

Far away from the points \( \xi_j \), namely \( |x - \xi_j| \geq \varepsilon d(\xi_j) \) for any \( j = 1, \ldots, m \), by (2.3)–(2.5), (2.16) and (2.25), we get

\[ \left( \sum_{k=1}^m \frac{\delta_k}{\delta_k + |x - \xi_k|^2} \right)^{-1} |\Delta U_\xi + U_\xi^p|(x) \leq \sum_{j=1}^m \frac{C}{\delta_j} \left( \frac{\delta_j^2 \log^4 \delta_j}{y \mu_j^2 d(\xi_j)^4} + \frac{1}{y_p} \right) = O(p \lambda^3 e^{-p/\lambda}). \]  \hspace{1cm} (2.26)
Let us now fix the index \(j \in \{1, \ldots, m\}\) and the region \(|x - \xi_j| \leq \varepsilon d(\xi_j)\). Taking into account (2.20) and the relation

\[
\left(\frac{p}{y \mu_j^{2/(p-1)}}\right)^{\frac{1}{p}} = \frac{1}{y^2 \mu_j^{2/(p-1)}},
\]  

(2.27)

for \(|x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j)\), we get

\[
U_\xi^p(x) = \frac{1}{y^{2 \delta_j^2 \mu_j^{2/(p-1)}}} \left[ 1 + \frac{1}{p} U_{1,0}(y) + \frac{1}{p^2} \omega_1(y) + \frac{1}{p^3} \left[ \omega_2(y) + O(p^2 \lambda_p |y|) + O(p \lambda_p) \right] \right]^p.
\]  

(2.28)

From the Taylor expansions of the exponential and logarithmic function, we have

\[
\left(1 + \frac{a}{p} + \frac{b}{p^2} + \frac{c}{p^3}\right)^p = e^a \left[ 1 + \frac{1}{p} \left( b - \frac{a^2}{2} \right) + \frac{1}{p^2} \left( c - ab + \frac{a^3}{3} + \frac{b^2}{2} - \frac{a^2b}{2} + \frac{a^4}{8} \right) + O\left( \frac{\log^2(|y| + 2)}{p^3} \right) \right],
\]  

(2.29)

which holds for \(|y| \leq C e^{p/8}\), provided that \(-4 \log(|y| + 2) \leq a(y) \leq C\) and \(|b(y)| + |c(y)| \leq C \log(|y| + 2)\). For \(|x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j) \sqrt{\delta_j}\), we deduce that

\[
U_\xi^p(x) = \frac{1}{y^{2 \delta_j^2 \mu_j^{2/(p-1)}}} e^{U_{1,0}(y)} \left[ 1 + \frac{1}{p} \left( \omega_1 - \frac{1}{2} U_{1,0}^2 \right) + \frac{1}{p^2} \left( \omega_2 - \omega_1 U_{1,0} + \frac{1}{3} U_{1,0}^3 + \frac{1}{2} \omega_1 U_{1,0}^2 + \frac{1}{8} U_{1,0}^4 \right) + O\left( \frac{\log^2(|y| + 2)}{p^3} \right) \right] + O\left( \lambda_p |y|\right) + O\left( \frac{\lambda_p}{p} \right) \left( \frac{2}{p^3} \right).
\]  

(2.30)

Combining (2.1), (2.8), (2.25) and (2.30), for \(|x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j) \sqrt{\delta_j}\), we have that

\[
(\Delta U_\xi + U_\xi^p)(x) = \frac{1}{y^{2 \delta_j^2 \mu_j^{2/(p-1)}}} \left( 1 + \frac{8}{(1 + |y|^2)^2} \right) O\left( \frac{\log^2(|y| + 2)}{p^3} \right),
\]  

and therefore

\[
\left( \sum_{k=1}^m \left( \delta_k \right)^{-1} \left( \frac{\xi_k - |x - \xi_k|^2}{2} \right)^{3/2} \right)^{-1} |(\Delta U_\xi + U_\xi^p)(x)| \leq \frac{\delta_j}{\delta_j} \left( \frac{\delta_j^2 + |x - \xi_j|^2}{2} \right)^{3/2} |(\Delta U_\xi + U_\xi^p)(x)| \leq \frac{C}{y} \left( \frac{1}{1 + |y|^2} \right)^{1/2} O\left( \frac{\log^2(|y| + 2)}{p^3} \right)
\]  

\[
\leq \frac{C}{p^2}.
\]  

(2.31)

On the other hand, if \(\varepsilon d(\xi_j) \sqrt{\delta_j} \leq |x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j)\), then by (2.28) we have

\[
U_\xi^p(x) = O\left( \frac{1}{y \delta_j^2 (1 + |y|^2)^2} \right),
\]  

since \((1 + \frac{a}{p})^p \leq e^a\). Thus, in this region,

\[
\left( \sum_{k=1}^m \left( \delta_k \right)^{-1} \left( \frac{\xi_k - |x - \xi_k|^2}{2} \right)^{3/2} \right)^{-1} |(\Delta U_\xi + U_\xi^p)(x)| \leq \frac{1}{y} \left( \frac{1}{1 + |y|^2} \right)^{1/2} O\left( \frac{1}{p^2} \log^2|y| \right) \leq p^{\sqrt{\lambda e}} e^{-p/8}.
\]  

(2.32)

Joining together (2.26), (2.31) and (2.32), we obtain estimate (2.24). \(\Box\)

### 3 Analysis of the linearized operator

In this section we prove the bounded invertibility of the operator \(L\), uniformly on \(\xi \in \mathcal{O}_\varepsilon\), by using the weighted \(L^\infty\)-norms introduced in (2.23). Let us recall that \(L(\phi) = \Delta \phi + W_\xi \phi\), where \(W_\xi(x) = p U_\xi^{-1}(x)\). As in Proposition 2.3, for the potential \(W_\xi(x)\) we have the following expansions.
Lemma 3.1. Let $\varepsilon > 0$ be fixed and small. There exist $D_0 > 0$, $p_0 > 0$ and $\lambda_0 > 0$ such that

$$W_\xi(x) \leq D_0 \sum_{j=1}^{m} e^{U_{j,\xi_j}(x)}$$

(3.1)

for any points $\xi = (\xi_1, \ldots, \xi_m) \in \Omega_\varepsilon$, and any $p \geq p_0$ and $\lambda \geq \lambda_0$ satisfying (2.4). Furthermore,

$$W_\xi(x) = \frac{8}{\delta_j^2(1 + |y|^2)^2} \left[ 1 + \frac{1}{p} \left( \omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right)(y) + O \left( \frac{\log^4(|y| + 2)}{p^2} \right) \right]$$

for any $|x - \xi| \leq \varepsilon d(\xi_j) \sqrt{\delta_j}$, where $y = \frac{x - \xi_j}{\delta_j}$.

Proof. If $|x - \xi| = \delta_j|y| \leq \varepsilon d(\xi_j)$ for some $j = 1, \ldots, m$, then by (2.20) and (2.27),

$$W_\xi(x) = p \left[ \frac{1}{p} U_{1,0}(y) + \frac{1}{p^2} \omega_1(y) + \frac{1}{p^3} \omega_2(y) + \frac{1}{p^4} O(\lambda p |y|) + \frac{1}{p^5} O(\lambda p) \right]^{p-1}$$

$$= \frac{1}{\delta_j^p} \left[ 1 + \frac{1}{p} U_{1,0}(y) + \frac{1}{p^2} \omega_1(y) + \frac{1}{p^3} \omega_2(y) + O(\lambda p |y|) + \frac{1}{p^5} O(\lambda p) \right]^{p-1},$$

where again we use the notation $y = \frac{x - \xi_j}{\delta_j}$. In this region, we have that

$$W_\xi(x) \leq \frac{C}{\delta_j^p} e^{U_{1,0}(y) + O(\lambda p |y|)} e^{-\frac{1}{p} \frac{1}{p} U_{1,0}(y) + O(\lambda p |y|)} = O(e^{U_{j,\xi_j}(x)}),$$

since $(1 + \frac{1}{p})^{p-1} \leq e^{(p-1)a/p}$ and $U_{1,0}(y) \geq -p + O(1)$. In particular, from a slight modification of formula (2.29), that is,

$$\left( 1 + a \frac{p}{p^2} + \frac{c}{p^3} \right)^{p-1} = e^a \left[ 1 + \frac{p}{p^2} \left( b - a - \frac{a^2}{2} \right) + O \left( \frac{\log^4(|y| + 2)}{p^2} \right) \right],$$

we obtain that if $|x - \xi_j| = \delta_j|y| \leq \varepsilon d(\xi_j) \sqrt{\delta_j}$, then

$$W_\xi(x) = \frac{8}{\delta_j^2(1 + |y|^2)^2} \left[ 1 + \frac{1}{p} \left( \omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right)(y) + O \left( \frac{\log^4(|y| + 2)}{p^2} \right) \right].$$

In addition, if $|x - \xi_j| \geq \varepsilon d(\xi_j)$ for all $j$, then by (2.16) we deduce

$$W_\xi(x) = O \left( p \left( \frac{C}{y} \right)^{p-1} \right),$$

which completes the proof. \qed

Remark 3.2. As for $W_\xi$, let us point out that if $|x - \xi_j| \leq \varepsilon d(\xi_j)$ for some $j = 1, \ldots, m$, then

$$p \left( U_{1,0}(y) + O \left( \frac{1}{p^3} \right) \right)^{p-2} \leq C p \left( \frac{p}{y y_j^2(p-2)} \right)^{p-2} e^{\frac{1}{p} U_{1,0} \left( \frac{c_0}{n} \right)} = O(e^{U_{j,\xi_j}(x)}).$$

Since this estimate is true if $|x - \xi_j| \geq \varepsilon d(\xi_j)$ for any $j = 1, \ldots, m$, we have

$$p \left( U_{1,0}(y) + O \left( \frac{1}{p^3} \right) \right)^{p-2} \leq C \sum_{j=1}^{m} e^{U_{j,\xi_j}(x)}.$$
is a linear combination of \( z_i, i = 0, 1, 2 \) (see [2]). Now we consider the following linear problem: given \( h \in C(\overline{\Omega}) \) and points \( \xi = (\xi_1, \ldots, \xi_m) \in \partial \epsilon \), we find a function \( \phi \in H^2(\Omega) \) such that

\[
\begin{cases}
L(\phi) = \Delta \phi + W_\xi \phi = h + \sum_{i=1}^{m} c_i e^{u_{i,\xi}} Z_{ij} & \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} + \lambda b(x) \phi = 0 & \text{on } \partial \Omega, \\
e^{u_{i,\xi}} Z_{ij} \phi = 0 & \text{for } i = 1, 2, j = 1, \ldots, m,
\end{cases}
\]

for some coefficients \( c_{ij}, i = 1, 2 \) and \( j = 1, \ldots, m \). Here and in the sequel, for any \( i = 0, 1, 2 \) and \( j = 1, \ldots, m \), we set

\[
Z_{ij}(x) := z_i \left( \frac{x - \xi_j}{\delta_j} \right) = \begin{cases}
|x - \xi|^2 - \delta_i^2 & \text{if } i = 0, \\
|x - \xi|^2 + \delta_i^2 & \text{if } i = 1, 2.
\end{cases}
\]

The main result of this section is the following proposition.

**Proposition 3.3.** Let \( \epsilon > 0 \) be fixed and small. There exist \( p_0 > 0, \lambda_0 > 0 \) and \( C > 0 \) such that for any \( h \in C(\overline{\Omega}) \), any points \( \xi = (\xi_1, \ldots, \xi_m) \in \partial \epsilon \), and any \( p > p_0 \) and \( \lambda > \lambda_0 \) satisfying (2.4), there exists a unique solution \( \phi \) to problem (3.2) for some scalars \( c_{ij}, i = 1, 2, j = 1, \ldots, m \), which satisfies

\[
\|\phi\|_{C^0} \leq C p \|h\|_C.
\]

**Proof.** The proof of this result will be divided into six steps.

**Step 1:** The operator \( L \) satisfies the maximum principle in \( \Omega := \Omega \setminus \bigcup_{j=1}^{m} B(\xi_j, R \delta_j) \) for \( R \) large, independent of \( p \) and \( \lambda \). Specifically, if \( \psi \) satisfies

\[
L(\psi) = \Delta \psi + W_\xi \psi \leq 0 \quad \text{in } \overline{\Omega}, \quad \psi \geq 0 \quad \text{on } \bigcup_{j=1}^{m} \partial B(\xi_j, R \delta_j) \quad \text{and} \quad \frac{\partial \psi}{\partial \nu} + \lambda b(x) \psi \geq 0 \quad \text{on } \partial \Omega,
\]

then \( \psi \geq 0 \) in \( \overline{\Omega} \). To prove this, it suffices to construct a positive function \( Z \) on \( \overline{\Omega} \) such that

\[
L(Z) = \Delta Z + W_\xi Z < 0 \quad \text{in } \Omega, \quad Z > 0 \quad \text{on } \bigcup_{j=1}^{m} \partial B(\xi_j, R \delta_j) \quad \text{and} \quad \frac{\partial Z}{\partial \nu} + \lambda b(x) Z > 0 \quad \text{on } \partial \Omega.
\]

Indeed, let

\[
Z(x) = \sum_{j=1}^{m} z_0 \left( \frac{a(x - \xi_j)}{\delta_j} \right), \quad a > 0.
\]

First, observe that if \( |x - \xi| \geq R \delta_j \) for \( R > \frac{1}{a} \), then \( Z(x) > 0 \). On the other hand, since \( Z(x) \leq m \),

\[
W_\xi(x) Z(x) \leq D_0 \left( \sum_{j=1}^{m} e^{u_{i,\xi}(x)} \right) Z(x) \leq D_0 Z \sum_{j=1}^{m} \frac{8 \delta_j^2}{|x - \xi_j|^4} \leq m D_0 \sum_{j=1}^{m} \frac{8 \delta_j^2}{|x - \xi_j|^4},
\]

where \( D_0 \) is the constant in Lemma 3.1. Further, by the definition of \( z_0 \),

\[
-\Delta Z(x) = \sum_{j=1}^{m} a^2 \frac{8 \delta_j^2 (a^2 |x - \xi_j|^2 - \delta_j^2)}{(a^2 |x - \xi_j|^2 + \delta_j^2)^3} \geq \frac{1}{3} \sum_{j=1}^{m} \frac{8 a^2 \delta_j^2}{(a^2 |x - \xi_j|^2 + \delta_j^2)^2} \geq \frac{4}{27} \sum_{j=1}^{m} \frac{8 \delta_j^2}{a^2 |x - \xi_j|^4},
\]

provided that \( R > \frac{\sqrt{3}}{a} \). Thus, if \( a \) is taken small and fixed but independent of \( p \) and \( \lambda \), and \( R \) is chosen sufficiently large depending on this \( a \), then we have that

\[
L(Z) = \Delta Z + W_\xi Z \leq \left( -\frac{4}{27} \frac{1}{a^2} + m D_0 \right) \sum_{j=1}^{m} \frac{8 \delta_j^2}{|x - \xi_j|^4} < 0.
\]
Moreover,
\[
\left| \frac{\partial}{\partial v} Z(x) \right| \leq \sum_{j=1}^{m} \frac{C \delta_j^2}{a^2 |x - \xi_j|^3} = O \left( \frac{\rho^2}{a^2} \sum_{j=1}^{m} \frac{1}{d(\xi_j)^3} \right)
\]
on $\partial \Omega$

and
\[
Z(x) \geq \frac{1}{2} \quad \text{on} \quad \partial \Omega \cup \left( \bigcup_{j=1}^{m} \partial B(\xi_j, R \delta_j) \right),
\]

which, together with (2.3)–(2.5), deduce that on $\partial \Omega$,
\[
\frac{\partial Z}{\partial v} + \lambda b(x) Z \geq O \left( \frac{1}{a^2} \Lambda^3 \rho^2 \right) + \frac{1}{2} \lambda b(x) \geq O(\lambda e^{-\rho/2}) + \frac{1}{2} \lambda \min_{x \in \partial \Omega} b(x) > 0,
\]

provided that $p$ and $\lambda$ are chosen sufficiently large. The function $Z(x)$ is what we are looking for.

**Step 2:** Let $R$ and $\varepsilon_0$ be as before. We define the “inner norm” of $\phi$ as
\[
\| \phi \|_i = \sup_{x \in (\cup_{j=1}^{m} B(\xi_j, R \delta_j))^{\circ}} |\phi|(x),
\]
and claim that there exists a constant $C > 0$ such that if $L(\phi) = h$ in $\Omega$ and $\frac{\partial \phi}{\partial v} + \lambda b(x) \phi = g$ on $\partial \Omega$, then
\[
\| \phi \|_{L^\infty(\partial \Omega)} \leq C \left( \| \phi \|_i + \| h \|_* + \frac{1}{\lambda} \| g \|_{L^\infty(\partial \Omega)} \right)
\]

for any $h \in C^{0,a}(\overline{\Omega})$ and $g \in C^{0,a}(\partial \Omega)$. We will establish this estimate with the use of suitable barriers. Let $M = 2 \text{diam} \Omega$. Consider the solution $\psi_j(x)$ of the problem
\[
\begin{aligned}
-\Delta \psi_j &= \frac{2 \delta_j}{|x - \xi_j|^3} \quad \text{in} \quad R \delta_j < |x - \xi_j| < M, \\
\psi_j(x) &= 0 \quad \text{on} \quad |x - \xi_j| = R \delta_j \quad \text{and} \quad |x - \xi_j| = M.
\end{aligned}
\]

The function $\psi_j(x)$ is the positive function given by
\[
\psi_j = -\frac{2 \delta_j}{|x - \xi_j|} + A + B \log|x - \xi_j|,
\]
where
\[
B = 2 \left( \frac{\delta_j}{M} - \frac{1}{R} \right) \frac{1}{\log \left( \frac{M}{R \delta_j} \right)} < 0 \quad \text{and} \quad A = \frac{2 \delta_j}{M} - B \log M.
\]

Observe that for $R \delta_j \leq |x - \xi_j| \leq M$,
\[
\psi_j(x) \leq A + B \log(R \delta_j) = \frac{2 \delta_j}{M} - B \log \frac{M}{R \delta_j} = \frac{2}{R}.
\]

Thus, $\psi_j(x)$ is uniformly bounded from above by a constant independent of $p$ and $\lambda$. In view of
\[
|\nabla \psi_j(x)| = O \left( \frac{\rho}{|x - \xi_j|^2} + \frac{1}{|\log \rho|} \left| \frac{1}{\rho} \right| \right),
\]
and by (2.3)–(2.5), we get
\[
\left| \frac{\partial}{\partial v} \psi_j(x) \right| = O \left( \lambda e^{-\rho/2} + \frac{A}{p} \right) \quad \text{on} \quad \partial \Omega.
\]

Now let
\[
\tilde{\phi}(x) = C_0 \left( Z(x) + \sum_{j=1}^{m} \psi_j(x) \right) \left( \| \phi \|_i + \| h \|_* + \frac{1}{\lambda} \| g \|_{L^\infty(\partial \Omega)} \right),
\]

where $Z$ was defined in the previous step, and $C_0 > 2$ is chosen larger if necessary. First of all, observe that for $x \in \bigcup_{j=1}^{m} \partial B(\xi_j, R \delta_j)$, by the definition of $Z$,
\[
\tilde{\phi}(x) \geq 2 C_0 \| \phi \|_i Z(x) + \| \phi \|_i \geq |\phi|(x).
\]
For $x \in \partial \Omega$, by (3.5), (3.7) and the positivity of $Z(x)$ and $\psi_j(x)$,
\[
\frac{\partial \bar{\phi}}{\partial v}(x) + Ab(x) \bar{\phi}(x) \geq \left[ O\left( \lambda e^{-p|1/2} + \lambda e^{-p|3/4} + \frac{1}{p} \right) + C_{0} \lambda \min_{x \in \Omega} b(x) \right]\left( \|\phi\|_{\ast} + \|h\|_{\ast} + \frac{1}{\lambda} \|g\|_{L^{\infty}(\partial \Omega)} \right) \\
\geq \frac{1}{2} C_{0} \lambda \left( \min_{x \in \Omega} b(x) \right) \left( \|\phi\|_{\ast} + \|h\|_{\ast} + \frac{1}{\lambda} \|g\|_{L^{\infty}(\partial \Omega)} \right) \\
\geq |g(x)|.
\]

Finally, for $x \in \Omega \setminus \bigcup_{j=1}^{m} B(\xi_j, R \delta_j)$, by (3.1), (3.6) and the definition of $\| \cdot \|_{\ast}$ in (2.23),
\[
L(\bar{\phi}) \leq C_{0} \left( \|\phi\|_{\ast} + \|h\|_{\ast} + \frac{1}{\lambda} \|g\|_{L^{\infty}(\partial \Omega)} \right) \sum_{j=1}^{m} L(\psi_j)(x) \\
= C_{0} \left( \|\phi\|_{\ast} + \|h\|_{\ast} + \frac{1}{\lambda} \|g\|_{L^{\infty}(\partial \Omega)} \right) \sum_{j=1}^{m} \left( -\frac{2 \delta_j}{|x - \xi_j|^3} \right) W(\xi_j)(\psi_j(x)) \\
\leq C_{0} \left( \|\phi\|_{\ast} + \|h\|_{\ast} + \frac{1}{\lambda} \|g\|_{L^{\infty}(\partial \Omega)} \right) \sum_{j=1}^{m} \left( -\frac{2 \delta_j}{|x - \xi_j|^3} \right) \left( \frac{2 m D_0}{R} e^{U_{\xi_j}}(x) \right) \\
\leq -C_{0} \|h\|_{\ast} \sum_{j=1}^{m} \frac{\delta_j}{(\delta_j^3 + |x - \xi_j|^2)^{3/2}} \\
\leq -|h(x)| \\
\leq -|L(\phi)|\end{align}\]
provided that $R > 16 m D_0$, and $p$ and $\lambda$ are large enough. Hence, by the maximum principle in Step 1, we obtain
\[
|\phi(x)| \leq \bar{\phi}(x) \quad \text{for } x \in \overline{\Omega},
\]
and therefore, since $Z(x) \leq m$ and $\psi_j(x) \leq \frac{2}{R}$,
\[
\|\phi\|_{L^{\infty}(\Omega)} \leq C \left( \|\phi\|_{\ast} + \frac{1}{\lambda} \|g\|_{L^{\infty}(\partial \Omega)} \right).
\]

**Step 3:** We prove uniform a priori estimates for the solutions $\phi$ of the problem
\[
\begin{cases}
L(\phi) = h & \text{in } \Omega, \\
\frac{\partial \phi}{\partial v} + \lambda b(x) \phi = g & \text{on } \partial \Omega,
\end{cases}
\]
where $h \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{0,\alpha}(\partial \Omega)$, and in addition $\phi$ satisfy the orthogonality conditions
\[
\int_{\Omega} e^{U_{\xi_j}} Z_{ij} \phi = 0 \quad \text{for } i = 0, 1, 2, j = 1, \ldots, m.
\]
Namely, we prove that there exists a positive constant $C$ such that for any points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_e$, $h \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{0,\alpha}(\partial \Omega)$,
\[
\|\phi\|_{L^{\infty}(\Omega)} \leq C \left( \|h\|_{\ast} + \frac{1}{\lambda} \|g\|_{L^{\infty}(\partial \Omega)} \right)
\]
for $p$ and $\lambda$ sufficiently large. By contradiction, assume the existence of sequences $p_n \to +\infty$, $\lambda_n \to +\infty$, of points $\xi^n = (\xi^n_1, \ldots, \xi^n_m) \in \mathcal{O}_e$, and of functions $h_n$, $g_n$ and associated solutions $\phi_n$ such that $\|h_n\|_{\ast} \to 0$, $\frac{1}{\lambda_n} \|g_n\|_{L^{\infty}(\partial \Omega)} \to 0$ and $\|\phi_n\|_{L^{\infty}(\Omega)} = 1$.

Since $\|\phi_n\|_{L^{\infty}(\Omega)} = 1$, Step 2 shows that $\liminf_{n \to +\infty} \|\phi_n\|_{\ast} > 0$. Set $\tilde{\phi}_j^n(y) = \phi_n(\delta_j^n y + \xi^n_j)$ for $j = 1, \ldots, m$.

Observe that by (2.3), (2.5) and (2.18),
\[
\min_{j=1, \ldots, m} \frac{\text{dist} (\xi^n_j, \partial \Omega)}{\delta^n_j} \to +\infty.
\]
which means that the domain of definition of $\tilde{\phi}_j^n$ approaches $\mathbb{R}^2$ as $n \to +\infty$. By Lemma 3.1, elliptic estimates readily imply that $\tilde{\phi}_j^n$ converges uniformly over compact sets to a bounded solution $\tilde{\phi}_j^{\infty}$ of

$$
\Delta \phi + \frac{8}{(1 + |y|^2)^2} \phi = 0 \quad \text{in} \quad \mathbb{R}^2.
$$

This implies that $\tilde{\phi}_j^{\infty}$ is a linear combination of the functions $z_i$, $i = 0, 1, 2$. Since $\|\tilde{\phi}_j^n\|_{L^\infty(\Omega)} \leq 1$, by Lebesgue's theorem, the orthogonality conditions on $\tilde{\phi}_j^n$ pass to the limit and give

$$
\int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} z_i(y) \tilde{\phi}_j^{\infty} \, dy = 0 \quad \text{for} \quad i = 0, 1, 2.
$$

Hence, $\tilde{\phi}_j^{\infty} \equiv 0$ for any $j = 1, \ldots, m$ contradicting $\liminf_{n \to +\infty} \|\phi_n\| > 0$.

**Step 4:** We prove that for any solution $\phi$ of the problem

$$
\begin{align*}
L(\phi) &= h & \text{in} \quad \Omega, \\
\frac{\partial \phi}{\partial \nu} + \lambda b(x) \phi &= 0 & \text{on} \quad \partial \Omega,
\end{align*}
$$

which in addition satisfies the orthogonality conditions

$$
\int_{\Omega} e^{uh_i} z_i \phi = 0 \quad \text{for} \quad i = 1, 2, j = 1, \ldots, m,
$$

there exists a positive constant $C > 0$ such that

$$
\|\phi\|_{L^\infty(\Omega)} \leq C\|h\|_*
$$

for $h \in C^{0,\alpha}(\overline{\Omega})$. Proceeding by contradiction, as in Step 3, we can suppose further that

$$
\|\phi_n\|_{L^\infty(\Omega)} = 1 \quad \text{and} \quad \|p_n\|_{H^1_0(\Omega)} \to 0 \quad \text{as} \quad n \to +\infty \quad \text{(3.8)}
$$

but we lose the condition

$$
\int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} z_0(y) \tilde{\phi}_j^{\infty} = 0
$$

in the limit. Hence, we have that

$$
\tilde{\phi}_j^n \to \tilde{\phi}_j^{\infty} = C_j \frac{|y|^2 - 1}{|y|^2 + 1} \quad \text{in} \quad C^{0,\alpha}_0(\mathbb{R}^2) \quad \text{(3.9)}
$$

for some constants $C_j$. To reach a contradiction, we have to show that $C_j = 0$ for any $j = 1, \ldots, m$. We will obtain it from the stronger condition (3.8) on $h_n$.

To this end, we perform the following construction. According to [4, 13], there exist radial solutions $\omega$ and $\zeta$, respectively, of the equations

$$
\Delta \omega + \frac{8}{(1 + |y|^2)^2} \omega = \frac{8}{(1 + |y|^2)^2} z_0(y) \quad \text{and} \quad \Delta \zeta + \frac{8}{(1 + |y|^2)^2} \zeta = \frac{8}{(1 + |y|^2)^2} \quad \text{in} \quad \mathbb{R}^2,
$$

such that

$$
\omega(y) = \frac{4}{3} \log |y| + O\left(\frac{1}{|y|}\right) \quad \text{and} \quad \zeta(y) = O\left(\frac{1}{|y|}\right) \quad \text{as} \quad |y| \to +\infty,
$$

and

$$
\nabla \omega(y) = \frac{4}{3} \frac{y}{1 + |y|^2} + O\left(\frac{1}{1 + |y|^2}\right) \quad \text{and} \quad \nabla \zeta(y) = O\left(\frac{1}{1 + |y|^2}\right) \quad \text{for all} \quad y \in \mathbb{R}^2,
$$

since

$$
8 \int_0^{+\infty} \frac{r^2 - 1)^2}{(r^2 + 1)^2} \, dr = \frac{4}{3} \quad \text{and} \quad 8 \int_0^{+\infty} \frac{r^2 - 1}{(r^2 + 1)^2} \, dr = 0.
$$
For simplicity, from now on we define the dependence on $n$. For $j = 1, \ldots, m$, define now
\[
u_j(x) = \omega \left( \frac{x - \xi_j}{\delta_j} \right) + \frac{a_j}{3} (\log \delta_j) Z_0(x) + \frac{8\pi}{3} H_\lambda(\xi_j, \xi_j) \left( \frac{x - \xi_j}{\delta_j} \right),
\]
and denote its projection $P\nu_j = \nu_j + \tilde{H}_j$, where $\tilde{H}_j$ is a correction term defined as the solution of
\[
\begin{align*}
-\Delta \tilde{H}_j &= 0 \\ \frac{\partial \tilde{H}_j}{\partial \nu} + \lambda b(x) \tilde{H}_j &= -\frac{\partial \nu_j}{\partial \nu} - \lambda b(x) \nu_j
\end{align*}
\]
on $\partial \Omega$.

Observe that on $\partial \Omega$,
\[
\left( \frac{\partial}{\partial \nu} + \lambda b(x) \left( \tilde{H}_j + \frac{8\pi}{3} H_\lambda(x, \xi_j) \right) \right) = O(\lambda^2 \rho) + (\log \delta_j) O(\lambda^3 \rho^2) + H_\lambda(\xi_j, \xi_j) O(\lambda^2 \rho).
\]

By (1.5)--(1.8) and the maximum principle with Robin boundary condition, we get
\[
\begin{align*}
Pu_j &= u_j - \frac{8\pi}{3} H_\lambda(x, \xi_j) + O(\lambda \rho \log \lambda) \quad \text{in } C(\overline{\Omega}), \\
Pu_j &= -\frac{8\pi}{3} G_\lambda(x, \xi_j) + O\left( \lambda \rho \log \lambda + \frac{\rho \log \lambda}{|x - \xi_j|} + \frac{\rho^2 \log \rho}{|x - \xi_j|^2} \right) \quad \text{in } C_{\text{loc}}(\overline{\Omega} \setminus \{\xi_j\})
\end{align*}
\]
(3.10)
The function $Pu_j$ solves
\[
\begin{align*}
\Delta Pu_j + W_\xi Pu_j &= e^{U_{\xi_j, \xi_j}} Z_0 + (W_\xi - e^{U_{\xi_j, \xi_j}}) Pu_j + R_j(u) \quad \text{in } \Omega, \\
\frac{\partial}{\partial \nu} Pu_j + \lambda b(x) Pu_j &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]
(3.11)
where
\[
R_j(x) = \left( Pu_j - u_j - \frac{8\pi}{3} H_\lambda(x, \xi_j) \right) e^{U_{\xi_j, \xi_j}}.
\]
(3.12)
Multiply (3.11) by $\phi$ and integrate by parts to obtain
\[
\int_{\Omega} e^{U_{\xi_j, \xi_j}} Z_0 \phi + \int_{\Omega} (W_\xi - e^{U_{\xi_j, \xi_j}}) Pu_j \phi = \int_{\Omega} Pu_j h - \int_{\Omega} R_j \phi.
\]
(3.13)
We estimate each term of (3.13). First of all, by Lebesgue’s theorem and (3.9) we get
\[
\int_{\Omega} e^{U_{\xi_j, \xi_j}} Z_0 \phi \rightarrow C_j \int_{\mathbb{R}^2} \frac{8(|y|^2 - 1)^2}{(|y|^2 + 1)^{4}} dy = \frac{8\pi}{3} C_j.
\]
(3.14)
Note that for fixed small $\delta > 0$, there exists a constant $C > 0$ such that for all $|x - \xi_j| = \delta$ and $\lambda d(x) = O(1)$,
\[
G_\lambda(x, \xi_j) \geq \frac{C}{\delta \lambda},
\]
(3.15)
provided that $\lambda$ is sufficiently large (see [8, p. 2685]). Thus, by Lemma 3.1, (2.4) and (3.10), we have
\[
\begin{align*}
\int_{\Omega} (W_\xi - e^{U_{\xi_j, \xi_j}}) Pu_j \phi &= \int_{B(\xi_j, \varepsilon \delta_j \cap \overline{\Omega})} (W_\xi - e^{U_{\xi_j, \xi_j}}) Pu_j \phi - \frac{8\pi}{3} \sum_{k \neq j} G_\lambda(\xi_k, \xi_j) \int_{B(\xi_j, \varepsilon \delta_j \cap \overline{\Omega})} W_\xi \phi + O(\lambda^2 \sqrt{\rho} + \lambda \rho \log \lambda) \\
&= \frac{8}{(1 + |y|^2)^2} \left( \omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right) \left( \frac{1}{p} \right) \\
&= \frac{8}{(1 + |y|^2)^2} \left( \omega_1 - U_{1,0} - \frac{1}{2} U_{1,0}^2 \right) \left( \frac{1}{p} \right) + o(1)
\end{align*}
\]
provided that $\lambda$ is sufficiently large.
As before, arguing by contradiction of (3.19), we can proceed, as in Step 3, and suppose further that

\[ \text{(see [13, p.50]). So we obtain} \]

we find that

\[ \text{Using (2.10), in a straightforward but tedious way, we can compute} \]

and

\[ \text{Using (2.15) and the maximum principle with Robin boundary condition,} \]

Finally, using (2.15) and the maximum principle with Robin boundary condition, we find that \( |\nabla \delta H| = (|\nabla j|/\text{radical.alt}) \lambda \), a contradiction and the claim is proved.

(3.18)

Hence, inserting (3.14), (3.16), (3.17) and (3.18) in (3.13), and taking into account (3.8), we conclude that

\[ \frac{16\pi}{3} C_j = o(1) \quad \text{for any} \ j = 1, \ldots, m. \]

Necessarily, \( C_j = 0 \), a contradiction and the claim is proved.

**Step 5:** We establish the validity of the a priori estimate

\[ \| \phi \|_\infty \leq C p \| h \|_* \]  

for solutions of problem (3.2) and \( h \in C^{0,\alpha}(\overline{\Omega}) \). Step 4 gives

\[ \| \phi \|_{L^\infty(\Omega)} \leq C p \left( \| h \|_* + \sum_{i=1}^{2} \sum_{j=1}^{m} |c_{ij}| \cdot |e^{U_{ij}/4} Z_{ij}|_* \right) \leq C p \left( \| h \|_* + \sum_{i=1}^{2} \sum_{j=1}^{m} |c_{ij}| \right). \]

As before, arguing by contradiction of (3.19), we can proceed, as in Step 3, and suppose further that

\[ \| \phi_n \|_{L^\infty(\Omega)} = 1, \quad p_n \| h_n \|_* \to 0 \quad \text{and} \quad p_n \sum_{i=1}^{2} \sum_{j=1}^{m} |c_{ij}| \geq \delta > 0 \quad \text{as} \ n \to +\infty. \]
We omit the dependence on \( n \). It suffices to estimate the values of the constants \( c_{ij} \). To this end, we define \( PZ_{ij} \) as the projection of \( Z_{ij} \) under homogeneous Robin boundary condition, namely

\[
\begin{cases}
\Delta PZ_{ij} = \Delta Z_{ij} = -e^{U_{i,j}} Z_{ij} & \text{in } \Omega, \\
\frac{\partial PZ_{ij}}{\partial \nu} + \lambda b(x) PZ_{ij} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

As in Lemma 2.1, for \( i = 1, 2 \) and \( j = 1, \ldots, m \), we have the following expansions:

\[ PZ_{ij} = Z_{ij} + 8 \pi \delta_j \frac{\partial H_A(x, \xi_j)}{\partial \xi_j} + O(\lambda^3 \rho^3) \quad \text{and} \quad PZ_{ij} = Z_{ij} - 1 + O(\lambda^2 \rho^2) \]  

(3.21)

in \( C(\overline{\Omega}) \) and in \( C^1_{\text{loc}}(\Omega) \), and

\[ PZ_{ij} = 8 \pi \delta_j \frac{\partial H_A(x, \xi_j)}{\partial \xi_j} + O\left(\lambda^3 \rho^3 + \frac{\rho^3}{|x - \xi_j|^3}\right) \quad \text{and} \quad PZ_{ij} = O\left(\lambda^2 \rho^2 + \frac{\rho^2}{|x - \xi_j|^2}\right) \]  

(3.22)

in \( C(\overline{\Omega} \setminus \{\xi_j\}) \) and in \( C^1_{\text{loc}}(\Omega \setminus \{\xi_j\}) \). By (3.21) and (3.22), we deduce the following “orthogonality” relations: for each \( i, l = 1, 2 \) and \( j, k = 1, \ldots, m \),

\[
\int_{\Omega} e^{U_{i,j}} Z_{il} PZ_{jk} = \left( 64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1 + |y|^2)^4} \right) \delta_{lj} + O(\lambda \rho),
\]

(3.23)

and

\[
\int_{\Omega} e^{U_{i,j}} Z_{ij} PZ_{lk} = O(\lambda \rho)
\]

(3.24)

uniformly for any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_e \), where \( \delta_{lk} \) and \( \delta_{lj} \) denote Kronecker’s symbols. Indeed, we have

\[
\int_{\Omega} e^{U_{i,j}} Z_{ij} PZ_{lk} = \int_{B(\xi, \epsilon d(\xi_j))} e^{U_{i,j}} Z_{ij} \left[ Z_{ij} + 8 \pi \delta_j \frac{\partial H_A(x, \xi_j)}{\partial \xi_j} + O(\lambda^3 \rho^3) \right] \ dx + O(\lambda^4 \rho^4)
\]

\[
= \left( 64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1 + |y|^2)^4} \right) \delta_{lj} + O(\lambda \rho),
\]

and for \( j \neq k \),

\[
\int_{\Omega} e^{U_{i,j}} Z_{ij} PZ_{lk} = \int_{B(\xi, \epsilon d(\xi_j))} e^{U_{i,j}} Z_{ij} \left[ 8 \pi \delta_k \frac{\partial G_k(x, \xi_k)}{\partial \xi_k} + O\left(\lambda^3 \rho^3 + \frac{\rho^3}{|x - \xi_k|^3}\right) \right] \ dx + O(\lambda^4 \rho^4)
\]

\[
= O\left(\lambda \rho \int_{B(\xi, \epsilon d(\xi_j))} e^{U_{i,j}} Z_{ij} \ dx \right) + O(\lambda^5 \rho^4)
\]

\[
= O(\lambda \rho),
\]

and

\[
\int_{\Omega} e^{U_{i,j}} Z_{ij} PZ_{lk} = \int_{B(\xi, \epsilon d(\xi_j))} e^{U_{i,j}} Z_{ij} \left[ 8 \pi \delta_k \frac{\partial G_k(x, \xi_k)}{\partial \xi_k} + O\left(\lambda^3 \rho^3 + \frac{\rho^3}{|x - \xi_k|^3}\right) \right] \ dx + O(\lambda^3 \rho^3)
\]

\[
= O\left(\lambda \rho \int_{B(\xi, \epsilon d(\xi_j))} e^{U_{i,j}} Z_{ij} \ dx \right) + O(\lambda^3 \rho^3)
\]

\[
= O(\lambda \rho),
\]
because $|\partial_1(\xi_0) H_s(x, \zeta)| = O(\lambda)$ holds uniformly in $\Omega$, which can be directly proved by (2.15) and the maximum principle with Robin boundary condition.

Now, for $i = 1, 2$ and $j = 1, \ldots, m$, multiplying (3.2) by $PZ_{ij}$ and integrating by parts we find

$$
\sum_{l=1}^m \sum_{k=1}^m c_{lk} \int_\Omega e^{U_{ij} \delta l_0} Z_{lk} PZ_{ij} + \int_\Omega hPZ_{ij} = \int_\Omega W_\xi \phi PZ_{ij} - \int_\Omega e^{U_{ij} \delta l_0} Z_{ij} \phi.
$$

(3.25)

Since $\int_\Omega hPZ_{ij} = O(\|h\|_*)$, by (3.23)–(3.24),

$$
\sum_{l=1}^m \sum_{k=1}^m c_{lk} \int_\Omega e^{U_{ij} \delta l_0} Z_{lk} PZ_{ij} + \int_\Omega hPZ_{ij} = DC_{ij} + O\left( e^{-p/4} \sum_{l=1}^m \sum_{k=1}^m |c_{lk}| \right) + O(\|h\|_*),
$$

where $D = 64 \int_{\mathbb{R}^2} \frac{y^2}{(1 + |y|^2)^3 \delta j}$. Moreover, by Lemma 3.1,

$$
DC_{ij} + O\left( e^{-p/4} \sum_{l=1}^m \sum_{k=1}^m |c_{lk}| \right) + O(\|h\|_*) = \int_{B(\xi_j, \delta j) \setminus \overline{\mathbb{B}_0}} W_\xi \phi PZ_{ij} - \int_{\Omega} e^{U_{ij} \delta l_0} Z_{ij} \phi + O(\lambda \sqrt{p} \|\phi\|_{\infty})
$$

$$
= \int_{B(\xi_j, \delta j) \setminus \overline{\mathbb{B}_0}} (W_\xi - e^{U_{ij} \delta l_0}) \phi PZ_{ij} + \int_{\Omega} e^{U_{ij} \delta l_0} (PZ_{ij} - Z_{ij}) \phi + O(\lambda \sqrt{p} \|\phi\|_{\infty})
$$

$$
= \frac{1}{p} \int_{B(0, \delta j) \setminus \overline{\mathbb{B}_0}} \frac{32 y^2}{1 + |y|^2} \left( \omega_1 - U_{1,0} - \frac{1}{2} U_{2,0}^2 \right) \phi(y) + O\left( \frac{1}{p} \|\phi\|_{\infty} \right),
$$

in view of (3.21)–(3.22), where $\phi_j(y) = \phi(\delta_j y + \xi_j)$. Accordingly, (3.25) can be reduced to

$$
DC_{ij} + O\left( e^{-p/4} \sum_{l=1}^m \sum_{k=1}^m |c_{lk}| \right) = O\left( \|h\|_* + \frac{1}{p} \|\phi\|_{\infty} \right)
$$

for each $i = 1, 2$ and $j = 1, \ldots, m$. Therefore, we obtain

$$
\sum_{l=1}^m \sum_{k=1}^m |c_{lk}| = O\left( \|h\|_* + \frac{1}{p} \|\phi\|_{\infty} \right) = o(1).
$$

As in Step 4, we conclude that for each $j = 1, \ldots, m$,

$$
\phi_j \to C_j \frac{|y|^2 - 1}{|y|^2 + 1} \quad \text{in } C_0^0(\mathbb{R}^2)
$$

for some constant $C_j \in \mathbb{R}$, and thus

$$
\int_{B(0, \delta j) \setminus \overline{\mathbb{B}_0}} \frac{32 y^2}{1 + |y|^2} \left( \omega_1 - U_{1,0} - \frac{1}{2} U_{2,0}^2 \right) \phi_j(y) + C_j \int_{\mathbb{R}^2} \frac{32 y^2 (|y|^2 - 1)}{1 + |y|^2} \left( \omega_1 - U_{1,0} - \frac{1}{2} U_{2,0}^2 \right) \phi_j(y) = 0.
$$

Therefore,

$$
\sum_{l=1}^m \sum_{k=1}^m |c_{lk}| = o\left( \frac{1}{p} \right) + O(\|h\|_*),
$$

which is impossible because of (3.20).

**Step 6:** We prove the solvability of problem (3.2). To this purpose, we consider the spaces

$$
K_\xi = \left\{ \sum_{l=1}^m \sum_{j=1}^m c_{lj} PZ_{lj} : c_{lj} \in \mathbb{R} \text{ for } i = 1, 2, j = 1, \ldots, m \right\}
$$

and

$$
K_\xi^* = \left\{ \phi \in L^2(\Omega) : \int_\Omega e^{U_{ij} \delta l_0} Z_{ij} \phi = 0 \text{ for } i = 1, 2, j = 1, \ldots, m \right\}.
$$
Define $\Pi_\xi : L^2(\Omega) \to K_\xi$ by

$$\Pi_\xi \phi = \sum_{i=1}^{2} \sum_{j=1}^{m} c_{ij} P Z_{ij},$$

where the coefficients $c_{ij}$ are uniquely determined (as it follows by (3.23)–(3.24)) by the system

$$\int_{\Omega} e^{U_{\xi} + \phi} Z_{ik} \left( \phi - \sum_{i=1}^{2} \sum_{j=1}^{m} c_{ij} P Z_{ij} \right) = 0 \quad \text{for any } l = 1, 2, k = 1, \ldots, m.$$ 

Let $\Pi_\xi = \operatorname{Id} - \Pi_\xi : L^2(\Omega) \to K_\xi^\perp$. Moreover, the Hilbert space $K_\xi^+ \cap H^1(\Omega)$ is endowed with the inner product

$$\langle \phi, \psi \rangle_H = \int_{\Omega} \nabla \phi \nabla \psi + \lambda \int_{\partial \Omega} b(x) \phi \psi.$$ 

Problem (3.2), expressed in a weak form, is equivalent to finding $\phi \in K_\xi^+ \cap H^1(\Omega)$ such that

$$\langle \phi, \psi \rangle_H = \int_{\Omega} (W_{\xi} \phi - h) \psi \quad \text{for all } \psi \in K_\xi^+ \cap H^1(\Omega).$$

With the aid of Riesz’s theorem, this equation gets rewritten in $K_\xi^+ \cap H^1(\Omega)$ in the operator form

$$\phi = K(\phi) + \bar{h},$$

where

$$\bar{h} = -\Pi_\xi^+ \left[ (-\Delta)_{\Omega} + \left( \frac{\partial}{\partial v} + \lambda b(x) \right) \right]^{-1} h \quad \text{and} \quad K(\phi) = \Pi_\xi^+ \left[ (-\Delta)_{\Omega} + \left( \frac{\partial}{\partial v} + \lambda b(x) \right) \right]^{-1} (W_{\xi} \phi).$$

is a linear compact operator in $K_\xi^+ \cap H^1(\Omega)$. By Fredholm’s alternative with Robin boundary condition (see [6, 14] and references therein), we obtain the unique solvability of this problem for any $h \in K_\xi^\perp$ provided that the homogeneous equation $\phi = K(\phi)$ has only the trivial solution in $K_\xi^+ \cap H^1(\Omega)$, which in turn follows from the a priori estimate (3.19) in Step 5. Moreover, by the elliptic regularity theory the solution constructed in this way belongs to $H^2(\Omega)$. For $p > p_0$ and $\lambda > \lambda_0$ fixed and under the restrained growth condition (2.4), and by the density of $C^{0,\alpha}(\Omega)$ in $(C(\Omega), \| \cdot \|_{C^{0,\alpha}})$, we can approximate $h \in C(\Omega)$ by smooth functions and, by (3.19) and the elliptic regularity theory, we can show that (3.4) holds for any $h \in C(\Omega)$. This completes the proof. 

**Remark 3.4.** Given $h \in C(\Omega)$, let $\phi$ be the solution of problem (3.2) given by Proposition 3.3. Multiplying (3.2) against $\phi$ and integrating by parts, we get

$$\| \phi \|^2 = \int_{\Omega} |\nabla \phi|^2 + \lambda \int_{\partial \Omega} b(x) \phi^2 = \int_{\Omega} W_{\xi} \phi^2 - \int_{\Omega} h \phi.$$

By Lemma 3.1, we can prove that $\left| \int_{\Omega} W_{\xi} \phi^2 \right| \leq C \| \phi \|_{C^{0,\alpha}}^2$, and therefore $\| \phi \|_H \leq C (\| h \|_+ + \| \phi \|_{C^{0,\alpha}}).$

### 4 The nonlinear problem

In what follows we want to solve the nonlinear auxiliary problem: for any points $\xi = (\xi_1, \ldots, \xi_m) \in \Omega$, we look for a function $\phi \in H^2(\Omega)$ such that

$$\begin{aligned}
\Delta (U_{\xi} + \phi) + (U_{\xi} + \phi)^p &= \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} e^{U_{\xi} + \phi} Z_{ij} \quad \text{in } \Omega, \\
U_{\xi} + \phi &> 0 \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial v} + \lambda b(x) \phi &= 0 \quad \text{on } \partial \Omega, \\
e^{U_{\xi} + \phi} Z_{ij} \phi &= 0 \quad \text{for } i = 1, 2, j = 1, \ldots, m,
\end{aligned}$$

(4.1)
for some coefficients $c_{ij}$, $i = 1, 2, j = 1, \ldots, m$, which depend on $\xi$. Recalling that
\[
N(\phi) = (U_\xi + \phi)^p - U_\xi^p - pU_\xi^{p-1}\phi \quad \text{and} \quad R_\xi = \Delta U_\xi + U_\xi^p,
\]
we can rewrite the first equation in (4.1) in the form
\[
L(\phi) = -[R_\xi + N(\phi)] + \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} e^{U_\xi + \xi} Z_{ij}. \tag{5.1}
\]
Using the theory developed in the previous section for the linear operator $L$, we have the following result.

**Proposition 4.1.** Let $\varepsilon > 0$ be fixed and small. There exist $C > 0$, $p_0 > 0$ and $\lambda_0 > 0$ such that for any points $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{O}_\varepsilon$, and any $p > p_0$ and $\lambda > \lambda_0$ satisfying (2.4), problem (4.1) has a unique solution $\phi_\xi$ for some scalars $c_{ij}(\xi)$, $i = 1, 2, j = 1, \ldots, m$, such that
\[
\|\phi_\xi\|_{L^\infty} \leq \frac{C}{p^3}, \quad \sum_{i=1}^{m} \sum_{j=1}^{m} |c_{ij}(\xi)| \leq \frac{C}{p^4} \quad \text{and} \quad \|\phi_\xi\|_{H^1} \leq \frac{C}{p^3}. \tag{4.2}
\]
Furthermore, the map $\xi \to \phi_\xi$ is a $C^1$-function in $C(\overline{\Omega})$ and $H^1(\Omega)$.

**Proof.** The proof of this proposition can be done along the lines of the proof of [13, Lemma 4.1]. We omit the details. \hfill \square

## 5 Variational reduction

After problem (4.1) has been solved, we find a solution of problem (2.22) and hence to the original problem (1.1) if $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{O}_\varepsilon$ satisfies
\[
c_{ij}(\xi) = 0 \quad \text{for all } i = 1, 2, j = 1, \ldots, m. \tag{5.1}
\]
Equation (1.1) is the Euler–Lagrange equation of the functional $F_p^1 : H^1(\Omega) \to \mathbb{R}$, defined by
\[
F_p^1(u) = \frac{1}{2} \int_{\Omega} \nabla u^2 - \frac{1}{p+1} \int_{\Omega} u^{p+1} + \frac{\lambda}{2} \int_{\Omega} b(x) u^2. \tag{5.2}
\]
We define
\[
F_p^1(\xi) = F_p^1(U_\xi + \phi_\xi) \quad \text{for all } \xi \in \mathbb{O}_\varepsilon, \tag{5.3}
\]
where $\phi_\xi$ is the unique solution to problem (4.1) given by Proposition 4.1. The critical points of $F_p^1$ correspond to solutions of (5.1) for all sufficiently large $p$ and $\lambda$ satisfying (2.4), as the following result states.

**Proposition 5.1.** The function $F_p^1 : \mathbb{O}_\varepsilon \to \mathbb{R}$ is of class $C^1$. Moreover, for all sufficiently large $p$ and $\lambda$ satisfying (2.4), if $D_\xi F_p^1(\xi) = 0$, then $\xi$ satisfies (5.1).

**Proof.** Since the map $\xi \to \phi_\xi$ is a $C^1$-map into $H^1(\Omega)$, $F_p^1$ is a $C^1$-function of $\xi$. Suppose now that $\xi \in \mathbb{O}_\varepsilon$ is a point such that $D_\xi F_p^1(\xi) = 0$. Then, we have that
\[
0 = \int_{\Omega} \nabla(U_\xi + \phi_\xi) \nabla[D_\xi(U_\xi + \phi_\xi)] - \int_{\Omega} (U_\xi + \phi_\xi)^p D_\xi(U_\xi + \phi_\xi) + \lambda \int_{\Omega} b(x)(U_\xi + \phi_\xi) D_\xi(U_\xi + \phi_\xi)
\]
\[
= \int_{\Omega} \left[ \Delta(U_\xi + \phi_\xi) + (U_\xi + \phi_\xi)^p \right] D_\xi(U_\xi + \phi_\xi)
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(\xi) \int_{\Omega} e^{U_\xi + \xi} Z_{ij} D_\xi(U_\xi + \phi_\xi)
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(\xi) \int_{\Omega} e^{U_\xi + \xi} Z_{ij} D_\xi(U_\xi + \phi_\xi) + \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(\xi) \int_{\Omega} D_\xi(e^{U_\xi + \xi} Z_{ij}) \phi_\xi. \tag{5.4}
\]
because \( \int_{\Omega} e^{U_{k, \xi}} Z_{ij} \phi_{\xi} = 0 \). From the definition of \( U_{\xi} \) in (2.12), we obtain

\[
\delta_{(\xi)} U_{\xi} = \sum_{j=1}^{m} \frac{1}{y_{j}^{2(p-1)}} \left\{ \delta_{(\xi)} U_{\delta_{i}, \xi}(x) + \frac{1}{p} \omega_{1}' \left( \frac{x - \xi_{k}}{\delta_{i}} \right) + \frac{1}{p^{2}} \omega_{2}' \left( \frac{x - \xi_{k}}{\delta_{i}} \right) + y_{j}^{2(p-1)} H_{j}(x) \right\} - \frac{2 \delta_{(\xi)} \log y_{j}}{p-1} \left[ U_{\delta_{i}, \xi}(x) + \frac{1}{p} \omega_{1}' \left( \frac{x - \xi_{k}}{\delta_{i}} \right) + \frac{1}{p^{2}} \omega_{2}' \left( \frac{x - \xi_{k}}{\delta_{i}} \right) + y_{j}^{2(p-1)} H_{j}(x) \right]
\]

\[
= \sum_{j=1}^{m} \frac{1}{y_{j}^{2(p-1)}} \left\{ \delta_{(\xi)} U_{\delta_{i}, \xi}(x) + \frac{1}{p} \omega_{1}' \left( \frac{x - \xi_{k}}{\delta_{i}} \right) + \frac{1}{p^{2}} \omega_{2}' \left( \frac{x - \xi_{k}}{\delta_{i}} \right) + y_{j}^{2(p-1)} H_{j}(x) \right\} - \frac{2 \delta_{(\xi)} \log y_{j}}{p-1} \left[ p + O(1) \right] .
\]

Notice that by (2.1) and (3.3),

\[
\delta_{(\xi)} U_{\delta_{i}, \xi}(x) = \frac{1}{\delta_{j}} \delta_{k, j} Z_{ij} + 2 Z_{ij} \log y_{j} = \frac{1}{\delta_{j}} \delta_{k, j} Z_{ij} + 2 \left( 1 - \frac{2 \delta_{j}^{2}}{|x_{j} - \xi_{j}|^{2} + \delta_{j}^{2}} \right) \delta_{(\xi)} \log y_{j},
\]

while for \( i = 1, 2 \), we have that

\[
\delta_{(\xi)} U_{\delta_{i}, \xi}(x) = \frac{1}{\delta_{j}} \delta_{k, j} Z_{ij} + 2 Z_{ij} \log y_{j} = \frac{1}{\delta_{j}} \delta_{k, j} Z_{ij} + 2 \left( 1 - \frac{2 \delta_{j}^{2}}{|x_{j} - \xi_{j}|^{2} + \delta_{j}^{2}} \right) \delta_{(\xi)} \log y_{j}.
\]

Moreover, as in the proof of Lemma 2.1, we can prove that

\[
\delta_{(\xi)}[y_{j}^{2(p-1)} H_{j}(x)] = \delta_{k, j} \left( 1 - \frac{C_{2}}{p^{2}} \right) 8 \pi \delta_{(\xi)} H_{j}(x, \xi_{j}) + \left( 2 - \frac{C_{1}}{p} + \frac{C_{2}}{p^{2}} \right) \delta_{(\xi)} \log y_{j} + \frac{1}{p} O \left( \frac{\rho}{d(\xi_{j})^{3}} + \frac{\rho}{d(\xi_{j})^{2} \delta_{j}^{2}} \right) \log y_{j} .
\]

Furthermore, by (2.3)–(2.5), (2.19) and (3.21), we get

\[
\delta_{(\xi)} U_{\xi} = \frac{1}{\delta_{k, j} y_{j}^{2(p-1)}} \left[ Z_{ik} - \frac{1}{p} \omega_{1}' \left( \frac{|x - \xi_{k}|}{\delta_{k}} \right) \left( \frac{|x - \xi_{k}|}{|x - \xi_{k}|} \right) + \frac{1}{p^{2}} \omega_{2}' \left( \frac{|x - \xi_{k}|}{\delta_{k}} \right) \left( \frac{|x - \xi_{k}|}{|x - \xi_{k}|} \right) \right] + \left( 1 - \frac{C_{1}}{4 p} - \frac{C_{2}}{4 p^{2}} \right) \delta_{(\xi)} H_{j}(x, \xi_{j})
\]

\[
+ \frac{1}{\delta_{k, j} y_{j}^{2(p-1)}} \left[ - 2 + O \left( \frac{1}{p} \frac{\delta_{j}^{2}}{|x_{j} - \xi_{j}|^{2} + \delta_{j}^{2}} \right) \right] \delta_{(\xi)} \log y_{j} + O \left( \frac{\rho^{3}}{y_{j}} \right)
\]

\[
= \frac{1}{\delta_{k, y_{j}^{2(p-1)}}} \left[ \left( 1 - \frac{C_{1}}{4 p} - \frac{C_{2}}{4 p^{2}} \right) P Z_{jk} + O \left( \frac{\rho^{3}}{y_{j}} \right) \right] + O \left( \frac{\sqrt{\lambda}}{y_{j}} \right). \tag{5.5}
\]

On the other hand, by (2.1) and (3.3), for \( i = 1, 2 \), we have that

\[
\delta_{(\xi)} \left( e^{U_{k, \xi}} Z_{ij} \right) = -4 \delta_{ij} e^{U_{k, \xi}} \left[ \frac{\delta_{ij}}{|x_{j} - \xi_{j}|^{2} + \delta_{j}^{2}} - \frac{6}{|x_{j} - \xi_{j}| \delta_{k, j}} \left( \frac{|x_{j} - \xi_{j}|}{|x_{j} - \xi_{j}| + \delta_{j}^{2}} \right) \delta_{k, j} Z_{ij} + 3 e^{U_{k, \xi}} Z_{ij} \right] \delta_{(\xi)} \log y_{j} . \tag{5.6}
\]

Consequently, for each \( l = 1, 2 \) and \( k = 1, \ldots, m \), (5.4) can be written as

\[
- \sum_{l,j} c_{lj}(\xi) \left[ 1 + O \left( \frac{\sqrt{\lambda}}{y_{j}} \right) \right] e^{U_{k, \xi}} Z_{lj} \left[ \frac{\delta_{ij}}{|x_{j} - \xi_{j}|^{2} + \delta_{j}^{2}} - \frac{6}{|x_{j} - \xi_{j}| \delta_{k, j}} \left( \frac{|x_{j} - \xi_{j}|}{|x_{j} - \xi_{j}| + \delta_{j}^{2}} \right) \delta_{k, j} Z_{lj} + 3 e^{U_{k, \xi}} Z_{lj} \right] \delta_{(\xi)} \log y_{j} = 0,
\]

so that, using (2.19), (3.23), (4.2) and (5.6),

\[
- 6 \lambda c_{lj}(\xi) \left( \frac{\delta_{ij} y_{j}^{2(p-1)}}{\delta_{k, y_{j}^{2(p-1)}}} \right) \left( \frac{|y_{j}|^{2}}{1 + |y_{j}|^{2}} \right) dy + O \left( \frac{\rho}{\delta_{k, y_{j}^{2(p-1)}}} \right) + \frac{1}{p^{3} \delta_{k}^{2}} \frac{\sqrt{\lambda}}{y_{j}} \sum_{l=1}^{m} c_{lj}(\xi) = 0,
\]

which, together with (2.5), implies \( c_{lk}(\xi) = 0 \) for each \( l = 1, 2 \) and \( k = 1, \ldots, m \).
Next, we need to write the expansion of $F_p^A(\xi)$ in terms of $\varphi_m^A(\xi)$ as $p$ and $\lambda$ go to $+\infty$ and satisfy (2.4).

**Proposition 5.2.** Let $\varepsilon > 0$ be fixed and small. With the choice (2.17) for the parameters $\mu_j$, there exist $p_0 > 0$ and $\lambda_0 > 0$ such that for any points $\xi = (\xi_1, \ldots, \xi_m) \in \Omega$, and any $p > p_0$ and $\lambda > \lambda_0$ satisfying (2.4), the following expansion holds uniformly:

$$F_p^A(\xi) = \frac{4\pi m p}{y^2} - \frac{16\pi m \log \lambda}{y^2} - \frac{32\pi^2}{y^2} \varphi_m^A(\xi) + \frac{4\pi m}{2y^2} \int_{\mathbb{R}^2} \left( \frac{8}{(1 + |y|^2)^2} U_{1,0} - \Delta \omega_1 \right) + O\left( \frac{\log \lambda}{p^3} \right). \quad (5.7)$$

**Proof.** Multiplying the first equation in (4.1) by $U_\xi + \varphi_\xi$ and integrating by parts, we obtain

$$\int_{\Omega} (U_\xi + \varphi_\xi)^{p+1} = \int_{\Omega} |\nabla (U_\xi + \varphi_\xi)|^2 + \lambda \int_{\partial \Omega} b(x)(U_\xi + \varphi_\xi)^2 + \sum_{i=1}^{m} c_{ij}(\xi) \int_{\Omega} e^{U_{ij} + \xi} U_{ij} \varphi_\xi.$$ 

Since $U_\xi$ is a bounded function, by (4.2) we get

$$\int_{\Omega} (U_\xi + \varphi_\xi)^{p+1} = \int_{\Omega} |\nabla (U_\xi + \varphi_\xi)|^2 + \lambda \int_{\partial \Omega} b(x)(U_\xi + \varphi_\xi)^2 + O\left( \frac{1}{p^4} \right)$$

uniformly for any points $\xi = (\xi_1, \ldots, \xi_m) \in \Omega$. Hence, by (5.2)–(5.3) we have

$$F_p^A(\xi) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left[ \int_{\Omega} |\nabla (U_\xi + \varphi_\xi)|^2 + \lambda \int_{\partial \Omega} b(x)(U_\xi + \varphi_\xi)^2 \right] + O\left( \frac{1}{p^4} \right),$$

$$= \left( \frac{1}{2} - \frac{1}{p+1} \right) \left[ \left( \int_{\Omega} |\nabla U_\xi|^2 + \lambda \int_{\partial \Omega} b(x)U_\xi^2 \right) + 2 \left( \int_{\Omega} \nabla U_\xi \nabla \varphi_\xi + \lambda \int_{\partial \Omega} b(x)U_\xi \varphi_\xi \right) \right] + O\left( \frac{1}{p^4} \right). \quad (5.8)$$

In view of (2.16), (2.20) and (2.25) we have

$$\int_{\Omega} |\nabla U_\xi|^2 + \lambda \int_{\partial \Omega} b(x)U_\xi^2 = \left( -\Delta U_\xi \right)U_\xi$$

$$= \sum_{j=1}^{m} \frac{1}{\mu_j^{2/3}(p-1)} \int_{B(\xi_j, \delta_j)} \left[ \frac{8}{(1 + |y|^2)^2} - \frac{1}{p \delta_j^2} \Delta \omega_1 \left( \frac{x - \xi_j}{\delta_j} \right) - \frac{\lambda}{p \delta_j^2} \Delta \omega_2 \left( \frac{x - \xi_j}{\delta_j} \right) \right] U_\xi$$

$$+ O(p^2 \lambda^3 \log \lambda)$$

$$= \sum_{j=1}^{m} \frac{1}{\mu_j^{2/3}(p-1)} \int_{B(\xi_j, \delta_j)} \left[ \frac{8}{(1 + |y|^2)^2} - \frac{1}{p} \Delta \omega_1(y) - \frac{1}{p^2} \Delta \omega_2(y) \right]$$

$$\times \left[ p U_{1,0}(y) + U_1(y) + \frac{1}{p} \omega_1(y) + \frac{1}{p^2} \omega_2(y) + O(\lambda \rho |y|) + \frac{1}{p} O(\lambda \rho) \right] dy + O(p^2 \lambda^2 \log \lambda)$$

$$= \sum_{j=1}^{m} \frac{1}{\mu_j^{2/3}(p-1)} \left[ 8 \pi p + \int_{\mathbb{R}^2} \left( \frac{8}{(1 + |y|^2)^2} U_{1,0} - \Delta \omega_1 \right) + O\left( \frac{1}{p^3} \right) \right]$$

$$= \frac{8 \pi p}{y^2} - \frac{32 \pi m \log \mu_j}{y^2} + \frac{m}{y^2} \int_{\mathbb{R}^2} \left( \frac{8}{(1 + |y|^2)^2} U_{1,0} - \Delta \omega_1 \right) + O\left( \frac{\log \lambda}{p^3} \right).$$

since $\mu_j^{2/3}(p-1) = 1 - \frac{2}{p} \log \mu_j + O\left( \frac{1}{p^2} \right)$. Recalling the expansion (2.17) of $\mu_j$ and the definition (1.9) of $\varphi_m^A(\xi)$, we get

$$\int_{\Omega} |\nabla U_\xi|^2 + \lambda \int_{\partial \Omega} b(x)U_\xi^2 = \frac{8 \pi p}{y^2} - \frac{32 \pi m \log \lambda}{y^2} - \frac{64 \pi^2}{y^2} \varphi_m^A(\xi)$$

$$+ \frac{24 \pi m}{y^2} + \frac{m}{y^2} \int_{\mathbb{R}^2} \left( \frac{8}{(1 + |y|^2)^2} U_{1,0} - \Delta \omega_1 \right) + O\left( \frac{\log \lambda}{p^3} \right). \quad (5.9)$$
On the other hand, by virtue of (4.2), we have
\[
2\left( \int_{\Omega} \nabla U_{\xi} \nabla \phi_{\xi} + \lambda \int_{\Omega} b(x) U_{\xi} \phi_{\xi} \right) + \left( \int_{\Omega} |\nabla \phi_{\xi}|^2 + \lambda \int_{\Omega} b(x) \phi_{\xi}^2 \right) = O\left( \frac{1}{p^{1/2}} \right).
\] (5.10)

Consequently, inserting (5.9) and (5.10) in (5.8), we obtain (5.7).

Finally, we want to show that the expansion of $F^l_p(\xi)$ in terms of $\varphi^l_p(\xi)$ holds in a $C^1$-sense.

**Proposition 5.3.** Let $\varepsilon > 0$ be fixed and small. Then, there exist $p_0 > 0$ and $\lambda_0 > 0$ such that for any points $\xi = (\xi_1, \ldots, \xi_m) \in \Omega_\varepsilon$, and any $p > p_0$ and $\lambda > \lambda_0$ satisfying (2.4), the following expansion holds uniformly:

\[
\nabla \varphi^l_p(\xi) \left[ \frac{32\pi^2}{\nu^2} \nabla \varphi^l_p(\xi_1, \ldots, \xi_m) + O\left( \frac{\nu}{p^2} \right) + O\left( \frac{1}{p^3} \right) \right],
\]

where $l = 1, 2$ and $k = 1, \ldots, m$.

**Proof.** Observe that for any $l = 1, 2$ and $k = 1, \ldots, m$,

\[
\partial_{(\xi_k)} F^l_p(\xi) = -\int_{\Omega} (\Delta u_{\xi} + u_{\xi}^p) \partial_{(\xi_k)} U_{\xi} - \int_{\Omega} (\Delta u_{\xi} + u_{\xi}^p) \partial_{(\xi_k)} \phi_{\xi},
\] (5.11)

where $u_{\xi} = U_{\xi} + \phi_{\xi}$. For each $k = 1, \ldots, m$, let $\eta_k$ be a radial smooth cut-off function such that $0 \leq \eta_k \leq 1$, $\eta_k = 1$ if $|x| \leq \varepsilon d(\xi_k)$, and $\eta_k = 0$ if $|x| \geq 2\varepsilon d(\xi_k)$. In view of (4.1), we deduce

\[
\int_{\Omega} (\Delta u_{\xi} + u_{\xi}^p) \partial_{(\xi_k)} \phi_{\xi} = -\sum_{i=1}^m \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} e^{U_{\xi_i} Z_{ij}} \partial_{(\xi_k)} \phi_{\xi}
\]

\[
= -\sum_{i=1}^m \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} \partial_{(\xi_i)} (e^{U_{\xi_i} Z_{ij}}) \phi_{\xi}
\]

\[
+ \sum_{i=1}^m \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} \partial_{x_i} (e^{U_{\xi_i} Z_{ij}}) \eta_k(x - \xi_k) \phi_{\xi}
\]

\[
- \sum_{i=1}^m \sum_{j=1}^m c_{ij}(\xi) \int_{\Omega} \partial_{(\xi_i)} (e^{U_{\xi_i} Z_{ij}}) \eta_k(x - \xi_k) \partial_{x_i} (e^{U_{\xi_i} Z_{ij}}) \phi_{\xi} \phi_{\xi}.
\]

Observe that $p^2 \phi_{\xi}(x) \to 0$ in $C(\bar{\Omega})$ and thus, by the elliptic regularity theory,

\[
p^2 \phi_{\xi}(x) \to 0 \quad \text{uniformly in } C^1_{\text{loc}}\left( \Omega \setminus \bigcup_{h=1}^m B(\xi_h, \varepsilon d(\xi_h)) \right).
\] (5.12)

Since $\eta_k(x - \xi_k) = 0$ on $\partial \Omega$, using an integration by parts of the derivative in $x_i$ and (4.2), we get

\[
\int_{\Omega} \partial_{x_i} (e^{U_{\xi_i} Z_{ij}}) \eta_k(x - \xi_k) \phi_{\xi} = \int_{\Omega} \partial_{x_i} (e^{U_{\xi_i} Z_{ij}}) \eta_k(x - \xi_k) \phi_{\xi} - \int_{\Omega} e^{U_{\xi_i} Z_{ij}} \partial_{x_i} (\eta_k(x - \xi_k)) \phi_{\xi}
\]

\[
= -\int_{\Omega} e^{U_{\xi_i} Z_{ij}} \partial_{x_i} (\eta_k(x - \xi_k)) \phi_{\xi}
\]

\[
- \int_{B(\xi_i, \varepsilon d(\xi_i))} e^{U_{\xi_i} Z_{ij}} \partial_{x_i} \phi_{\xi} + O\left( \frac{\lambda^4}{p^3} \right),
\]

in view of $|\partial_{x_i} \eta_k(x - \xi_k)| = O(\lambda)$. On the other hand, by (5.6) we have

\[
\partial_{(\xi_i)} (e^{U_{\xi_i} Z_{ij}}) + \eta_k(x - \xi_k) \partial_{x_i} (e^{U_{\xi_i} Z_{ij}})
\]

\[
= 4 \delta_i e^{U_{\xi_i} Z_{ij}} \eta_k(x - \xi_k) - \delta_{jk} \delta_{ij} + 24 \delta_i e^{U_{\xi_i} Z_{ij}} (x - \xi_k) - \delta_{jk} \delta_{ij} (x - \xi_k) 2 \delta_{jk} (x - \xi_k) + 3 e^{U_{\xi_i} Z_{ij}} \partial_{x_i} \log \mu_{ij} + O(\lambda^3 p^{-3}),
\]

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which, together with (2.19), (4.2), and the fact that \(|Z_{ij}Z_{ij}| \leq 2\), implies
\[
\left| \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(\xi) \int_{\Omega} \left( \partial_i (e^{U_j \xi} Z_{ij}) + \eta_{k}(x - \xi k) \partial_{x_i} (e^{U_j \xi} Z_{ij}) \right) \phi_{\xi} \right| = O\left( \frac{\sqrt{\lambda}}{p^7} \right).
\]

Therefore,
\[
\int_{\Omega} \left( \Delta u_{\xi} + u_{\xi}^{p} \right) \partial_{i}(\xi_{i}) \phi_{\xi} = - \int_{\Omega} \left( \Delta u_{\xi} + u_{\xi}^{p} \right) \partial_{i} \phi_{\xi} + O\left( \frac{\sqrt{\lambda}}{p^7} \right).
\]

Now, by the expression of \(U_{\xi}\) in (2.12) we obtain
\[
\partial_{x_{i}} U_{\xi} = \sum_{j=1}^{m} \frac{1}{\delta_{j} y_{jk}} \partial_{x_{i}} \left[ U_{\delta_{i}, \xi}(x) + \frac{1}{p} \omega_{i} \left( \frac{x - \xi_{i}}{\delta_{j}} \right) + \frac{1}{p^2} \omega_{2} \left( \frac{x - \xi_{i}}{\delta_{j}} \right)^{2} + \gamma p^{2}(1 - \eta_{k}) H_{j}(x) \right].
\]

Obviously, by (2.1) and (3.3),
\[
\partial_{x_{i}} U_{\delta_{i}, \xi}(x) = - \frac{1}{\delta_{j}} Z_{ij},
\]
while for \(i = 1, 2\), by (2.9) and (3.3),
\[
\partial_{x_{i}} \omega_{i} \left( \frac{x - \xi_{i}}{\delta_{j}} \right) = \omega_{i} \left( \frac{x - \xi_{i}}{\delta_{j}} \right) \frac{1}{\delta_{j}} \left[ \frac{C_{1}}{4} Z_{ij} + O\left( \frac{\delta_{i}^{2}}{|x - \xi_{i}|^{2} + \delta_{i}^{2}} \right) \right].
\]

Moreover, as in the proof of Lemma 2.1, for \(|x - \xi_{k}| \leq 2p d(\xi_{k})\), we have that
\[
\partial_{x_{i}} [\gamma p^{2}(1 - \eta_{k}) H_{j}(x)] = \left( 1 - \frac{C_{1}}{4p} - \frac{C_{2}}{4p^{2}} \right) 8 \pi \delta_{i} \partial_{x_{i}} H_{j}(x, \xi_{j}) + O\left( \frac{p}{d(\xi_{j}) d(\xi_{k})} \right).
\]

Then, for \(|x - \xi_{k}| \leq 2p d(\xi_{k})\),
\[
\partial_{x_{i}} U_{\xi} = \sum_{j=1}^{m} \frac{1}{\delta_{j} y_{jk}^{2}(p - 1)} \left[ - Z_{ij} + \frac{1}{p} \omega_{i} \left( \frac{x - \xi_{i}}{\delta_{j}} \right) \left( \frac{x - \xi_{i}}{\delta_{j}} \right) \frac{1}{\delta_{j}} \left[ \frac{C_{1}}{4} Z_{ij} + O\left( \frac{\delta_{i}^{2}}{|x - \xi_{i}|^{2} + \delta_{i}^{2}} \right) \right] \right] + \left( 1 - \frac{C_{1}}{4p} - \frac{C_{2}}{4p^{2}} \right) 8 \pi \delta_{i} \partial_{x_{i}} H_{j}(x, \xi_{j}) + O\left( \frac{1}{\gamma} \right).
\]

Combing (5.5) and (5.14), we find
\[
\partial_{i}(\xi_{i}) U_{\xi} + \eta_{k}(x - \xi_{k}) \partial_{x_{i}} U_{\xi} = \frac{8 \pi [1 + O\left( \frac{1}{p} \right)]}{\gamma p^{2}(p - 1)} \left[ \partial_{i}(\xi_{i}) H_{j}(x, \xi_{j}) + \eta_{k}(x - \xi_{k}) \partial_{x_{i}} H_{j}(x, \xi_{j}) \right] + O\left( \frac{\sqrt{\lambda}}{\gamma} \right) = O\left( \frac{\lambda}{\gamma} \right).
\]

Furthermore,
\[
\int_{\Omega} \left( \Delta u_{\xi} + u_{\xi}^{p} \right) \partial_{i}(\xi_{i}) U_{\xi} = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(\xi) \int_{\Omega} \left( e^{U_{j} \xi} Z_{ij} \partial_{i}(\xi_{i}) U_{\xi} + \eta_{k}(x - \xi_{k}) \partial_{x_{i}} U_{\xi} \right)
\]
\[
= - \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(\xi) \int_{\Omega} \left( e^{U_{j} \xi} Z_{ij} \eta_{k}(x - \xi_{k}) \partial_{x_{i}} U_{\xi} \right)
\]
\[
= - \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij}(\xi) \int_{B(\xi_{i}, d(\xi_{j}))} \left( e^{U_{j} \xi} Z_{ij} \partial_{x_{i}} U_{\xi} + O\left( \frac{\lambda}{p^{7}} \right) \right)
\]
\[
= \int_{B(\xi_{i}, d(\xi_{j}))} \left( \Delta u_{\xi} + u_{\xi}^{p} \right) \partial_{x_{i}} U_{\xi} + O\left( \frac{\lambda}{p^{7}} \right).
\]
Inserting (5.13) and (5.15) in (5.11), we get
\[
\partial_{(\xi)}, \mathcal{F}_p^\lambda(\xi) = \int_{B(\xi, \bar{d}(\xi))} (\Delta u_\xi + u_\xi^p) \partial_{\xi} u_\xi + O\left(\frac{\lambda}{p^5}\right) - \frac{26}{\bar{d}(\xi)} \beta(\xi, \bar{d}(\xi)) + O(1).
\]
(5.16)

Finally, by integrating by parts we have the following Pohozaev-type identities. For any ball \(B \subset \Omega\) and for any function \(u\),
\[
\int_B \Delta u \partial_{\xi} u = \int_{\partial B} \left(\partial_{\nu} u \partial_{\xi} u - \frac{1}{2} |\nabla u|^2 \nu_1\right) \quad \text{and} \quad \int_B \partial^p u \partial_{\xi} u = \frac{1}{p^3 + 1} \int_{\partial B} u^{p+1} \nu_1,
\]
where \(\nu_1(x)\) denotes the \(l\)-component of the exterior unit normal vector \(\nu(x)\) of \(\partial B\) at \(x \in \partial B\). Let us define
\[
\varphi_k(x) = H_k(x, \xi) + \sum_{j \neq k} G_k(x, \xi).
\]
Observe that for sufficiently large \(p\) and \(\lambda\) satisfying (2.4), by (2.16) and (3.15), we have
\[
yU_k(x) = 8\pi \sum_{j=1}^m G_k(x, \xi_j) + O\left(\frac{1}{p}\right) = 8\pi \left(\frac{1}{2\pi} \log \frac{1}{|x - \xi_k|} + \varphi_k(x)\right) + O\left(\frac{1}{p}\right)
\]
(5.18)
uniformly in \(C^1_{\text{loc}}(\Omega \setminus \bigcup_{h=1}^m B(\xi_h, \bar{d}(\xi_h)))\). Using (5.12), (5.17), (5.18) and the fact that \(u_\xi = U_\xi + \varphi_\xi\), we obtain
\[
\int_{B(\xi, \bar{d}(\xi))} (\Delta u_\xi + u_\xi^p) \partial_{\xi} u_\xi = \int_{\partial B(\xi, \bar{d}(\xi))} \left(\frac{1}{2\pi} \frac{1}{|x - \xi_k|} + \varphi_\xi\right) + O\left(\frac{1}{p}\right)
\]
\[
= \frac{64\pi^2}{\bar{d}(\xi)} \int_{\partial B(\xi, \bar{d}(\xi))} \left[ -\frac{1}{2\pi} \frac{1}{|x - \xi_k|} \varphi_\xi + \frac{1}{2} \nabla \varphi_\xi^2 \right] + O\left(\frac{1}{p}\right)
\]
\[
= \frac{64\pi^2}{\bar{d}(\xi)} \int_{\partial B(\xi, \bar{d}(\xi))} \left[ -\frac{1}{2\pi} \frac{1}{|x - \xi_k|} \varphi_\xi + \frac{1}{2} \nabla \varphi_\xi^2 \right] + O\left(\frac{1}{p}\right).
\]
where the last equality is due to the harmonicity of \(\varphi_\xi\) near \(\xi_k\), more precisely,
\[
\frac{1}{2\pi} \frac{1}{|x - \xi_k|} \int_{\partial B(\xi, \bar{d}(\xi))} \partial_{\xi} \varphi_\xi = \partial_{\xi} \varphi_\xi(\xi_k),
\]
and by (5.17),
\[
\int_{\partial B(\xi, \bar{d}(\xi))} \left(\varphi_\xi \partial_{\xi} \varphi_\xi - \frac{1}{2} \nabla \varphi_\xi^2 \nu_1\right) = \int_{B(\xi, \bar{d}(\xi))} \Delta \varphi_\xi \partial_{\xi} \varphi_\xi = 0.
\]
Accordingly, by (5.16) we get
\[
\partial_{(\xi)}, \mathcal{F}_p^\lambda(\xi) = -\frac{64\pi^2}{\bar{d}(\xi)} \partial_{\xi} \varphi_\xi(\xi_k) + O\left(\frac{\lambda}{p^5}\right) = -\frac{32\pi^2}{\bar{d}(\xi)} \partial_{(\xi)}, \varphi_m^\lambda(\xi) + O\left(\frac{\lambda}{p^5}\right),
\]
since
\[
\partial_{\xi} \varphi_\xi(\xi_k) = \frac{1}{2} \partial_{(\xi)}, \varphi_m^\lambda(\xi).
\]
This completes the proof.
6 Proof of theorems

Proof of Theorem 1.1. According to Proposition 5.1, the function $u_{p,\lambda} = U_{\lambda} + \phi_{\lambda}$, where $U_{\lambda}$ is defined in (2.12) and $\phi_{\lambda}$ is the unique solution to problem (4.1) given by Proposition 4.1, is a solution to problem (1.1) if we adjust $\xi$ so that it is a critical point of $F_p(\xi)$, defined by (5.3). This is equivalent to finding a critical point of

$$F_p(\xi) = -\frac{y^2}{2\pi^2} \left[ \frac{4\pi mp}{y^2} - \frac{4\pi m}{y^2} - \frac{m}{2\pi} \left( \frac{8}{\pi^2} \frac{U_{1,0}}{U_{1,1}} \right) \right].$$

On the other hand, from Proposition 5.2 and 5.3, for $\xi = (\xi_1, \ldots, \xi_m) \in \Omega_\epsilon$, we have that

$$F_p(\xi) = \varphi^1_m(\xi) + \frac{m}{2\pi} \log \lambda + \frac{\lambda}{p} \Theta_{p,\lambda}(\xi),$$

(6.1)

where $\varphi^1_m$ is given by (1.9), and $\Theta_{p,\lambda}$ and $\nabla \Theta_{p,\lambda}$ are uniformly bounded in the considered region as $p$ and $\lambda$ go to $\infty$ such that $\frac{\log \lambda}{p} = O(1)$ and $\frac{\lambda}{p^2 \log \lambda} = O(1)$.

Let us first claim that for any small $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\inf_{\xi \in \Omega_\epsilon} F_p(\xi) \geq \frac{m}{2\pi} \inf_{\xi \in \Omega_\epsilon} h_\lambda(\beta_{b(\xi)}, b(x)) + \frac{m}{2\pi} \log \lambda + \frac{C}{\epsilon \lambda} - C,$$

uniformly for any large $p$ and $\lambda$ satisfying $\frac{\log \lambda}{p} = O(1)$ and $\lambda p^2 \log \lambda = O(1)$. To prove this, we recall that

$$\Omega_\epsilon = \left\{ \xi = (\xi_1, \ldots, \xi_m) \in \Omega^m : \xi_j - \xi_k \geq \epsilon, |\lambda d(\xi_j) - \theta_{b(\xi_j)}| \leq \frac{1}{\epsilon \lambda^{1/2}}, j, k = 1, \ldots, m, j \neq k \right\},$$

where $\epsilon > 0$ is a sufficiently small but fixed number. Obviously, if $\xi = (\xi_1, \ldots, \xi_m) \in \partial \Omega_\epsilon$, then either we have $|\lambda d(\xi_j) - \theta_{b(\xi_j)}| = \frac{1}{\epsilon \lambda^{1/2}}$ for some $j_0$, or $|\xi_j - \xi_k| = \epsilon$ for some $j_0 \neq k_0$. Using (1.5)–(1.8) and the fact that $h_\lambda(\theta, \beta)$ has a unique non-degenerate minimum $\theta_{b} \in (0, \infty)$ independent of $\lambda$, we have, for $|\lambda d(\xi_j) - \theta_{b(\xi_j)}| = \frac{1}{\epsilon \lambda^{1/2}}$, that

$$F_p(\xi) \geq \sum_{j=1}^{m} H_\lambda(\xi_j, \xi_j) + \sum_{j \neq k} G_\lambda(\xi_j, \xi_k) + \frac{m}{2\pi} \log \lambda + \frac{\lambda}{p} \Theta_{p,\lambda}(\xi) \geq \sum_{j=1}^{m} h_\lambda(\theta_{b(\xi_j)}, b(\xi_j)) + \frac{m}{2\pi} \log \lambda + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{p} \log \lambda\right) \geq \frac{m}{2\pi} \inf_{\xi \in \Omega_\epsilon} h_\lambda(\theta_{b(\xi)}, b(x)) + \frac{m}{2\pi} \log \lambda + \frac{C}{\epsilon \lambda} - C,$$

while for $|\xi_j - \xi_k| = \epsilon$, by (3.15),

$$F_p(\xi) \geq \sum_{j=1}^{m} H_\lambda(\xi_j, \xi_j) + \sum_{j \neq k} G_\lambda(\xi_j, \xi_k) + \frac{m}{2\pi} \log \lambda + \frac{\lambda}{p} \Theta_{p,\lambda}(\xi) \geq \sum_{j=1}^{m} h_\lambda(\theta_{b(\xi_j)}, b(\xi_j)) + \frac{m}{2\pi} \log \lambda + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{p} \log \lambda\right) \geq \frac{m}{2\pi} \inf_{\xi \in \Omega_\epsilon} h_\lambda(\theta_{b(\xi)}, b(x)) + \frac{m}{2\pi} \log \lambda + \frac{C}{\epsilon \lambda} - C.$$

We shall show that $F_p(\xi)$ has at least two distinct critical points in the region $\Omega_\epsilon$, a fact that will prove our result. Observe that $F_p(\xi)$ is $C^1$ and uniformly bounded from below in $\Omega_\epsilon$, and such that

$$\inf_{\xi \in \Omega_\epsilon} F_p(\xi) \to +\infty \quad \text{as} \quad \epsilon \to 0.$$

Hence, since $\epsilon$ is sufficiently small, $F_p(\xi)$ has an absolute minimum $M_{p,\lambda}$ in $\Omega_\epsilon$ whenever $p$ and $\lambda$ are sufficiently large such that $\frac{\log \lambda}{p} = O(1)$ and $\frac{\lambda}{p^2 \log \lambda} = O(1)$. On the other hand, the Lusternik–Schnirelmann theory is applicable in our setting, so that the number of critical points of $F_p(\xi)$ in $\Omega_\epsilon$ can be estimated from
below by \( \text{cat}_{O_\varepsilon}(O_\varepsilon) \), the Lusternik–Schnirelmann category of \( O_\varepsilon \) relative to \( O_\varepsilon \). Let us recall that \( \text{cat}_{O_\varepsilon}(A) \), the Lusternik–Schnirelmann category of a closed subset \( A \) of \( O_\varepsilon \) relative to \( O_\varepsilon \), is the minimal integer \( I \) such that \( A \) can be covered by \( I \) closed and contractible subsets of \( O_\varepsilon \).

We claim that \( \text{cat}_{O_\varepsilon}(O_\varepsilon) > 1 \). Indeed, by contradiction, assume that \( \text{cat}_{O_\varepsilon}(O_\varepsilon) = 1 \). This means that \( O_\varepsilon \) is contractible in itself, namely there exist a point \( \xi^0 \in O_\varepsilon \) and a continuous function \( \Gamma: [0, 1] \times O_\varepsilon \to O_\varepsilon \) such that for all \( \xi \in O_\varepsilon \),

\[
\Gamma(0, \xi) = \xi \quad \text{and} \quad \Gamma(1, \xi) = \xi^0.
\]

Given one connected component \( C_0 \) of \( \partial \Omega \), let \( \Lambda: S^1 \to C_0 \) be a continuous bijective function that parametrizes \( C_0 \). Observe that for any \( x \in \Omega \) close to \( \partial \Omega \), there exists a unique point \( \bar{x} \) on \( \partial \Omega \) closest to \( x \). Then, we can define a continuous map \( \tilde{P}: O_\varepsilon \to \partial \Omega \) such that \( \tilde{P}_\xi = \tilde{\xi}_1 \) for any \( \xi = (\xi_1, \ldots, \xi_m) \in O_\varepsilon \). Set

\[
g_\lambda(x) = x - \frac{\theta(x)}{\lambda}v(x), \quad x \in C_0,
\]

where \( v \) is the exterior unit normal vector of \( \partial \Omega \). Obviously, for \( \lambda \) sufficiently large, \( g_\lambda \) is a continuous invertible function. We represent \( S^1 = \{ z \in C : |z| = 1 \} \). Let \( f: S^1 \to O_\varepsilon \) be the continuous function defined by

\[
f_\lambda(z) = \left( g_\lambda(\lambda(z)), g_\lambda(\lambda(z)e^{i2\pi \frac{1}{m}}), \ldots, g_\lambda(\lambda(z)e^{i(2\pi - 1)\frac{1}{m}}) \right), \quad z \in S^1.
\]

Let \( \eta: [0, 1] \times S^1 \to S^1 \) be the well-defined continuous map given by

\[
\eta_\lambda(t, z) = \Lambda^{-1} \circ \tilde{P} \circ \Gamma(t, f_\lambda(z)).
\]

The function \( \eta_\lambda \) is a contraction of \( S^1 \) to a point and this gives a contradiction. Thus, we conclude that \( \text{cat}_{O_\varepsilon}(O_\varepsilon) \geq 2 \).

Let us now define

\[
M^{p, A}_2 = \inf_{A \in \mathcal{E}} \sup_{\xi \in A} \bar{F}^A_p(\xi),
\]

where

\[
\mathcal{E} = \{ A \subset O_\varepsilon : A \text{ closed and } \text{cat}_{O_\varepsilon}(A) \geq 2 \}.
\]

By the above construction of \( f_\lambda \), we have that

\[
\Lambda^{-1} \circ \tilde{P} \circ f_\lambda = z \quad \text{for any } z \in S^1,
\]

so \( \Lambda^{-1} \circ \tilde{P} \circ f_\lambda \) has a nonzero winding number, which implies that \( \text{cat}_{O_\varepsilon}(A_0) \geq 2 \) with \( A_0 = \{ f_\lambda(z) : z \in S^1 \} \). Note that for any \( \xi = (\xi_1, \ldots, \xi_m) \in A_0 \), the \( m \) coordinates of \( \xi \) are uniformly separated and independent of \( \varepsilon \) and \( \lambda \). Then, for any \( \xi \in A_0 \),

\[
\bar{F}^A_p(\xi) = \sum_{j=1}^m H_\lambda(\xi_j, \xi_j) + \sum_{j \neq k} G_\lambda(\xi_j, \xi_k) + \frac{m}{2\pi} \log \lambda + \frac{\log \lambda}{p} \Theta_{p, A}(\xi)
\]

\[
= \frac{1}{2\pi} \sum_{j=1}^m h_\lambda(\theta(b(\xi_j)), b(\xi_j)) + \frac{m}{2\pi} \log \lambda + O(1) + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{p} \log \lambda\right),
\]

so that, using (1.7) and (2.3),

\[
M^{p, A}_2 = \inf_{A \in \mathcal{E}} \sup_{\xi \in A} \bar{F}^A_p(\xi) \leq \sup_{\xi \in A_0} \bar{F}^A_p(\xi) \leq C < +\infty
\]

uniformly for any large \( p \) and \( \lambda \) such that \( \frac{\log \lambda}{p} = o(1) \) and \( \frac{\lambda}{p \log \lambda} = O(1) \). Thus, Lusternik–Schnirelmann theory gives that \( M^{p, A}_2 \) is a critical level for \( F^A_p \).

Obviously, \( M^{p, A}_1 \leq M^{p, A}_2 \). If \( M^{p, A}_1 < M^{p, A}_2 \), then we conclude that there are at least two distinct critical points for \( F^A_p \) (distinct up to permutations) in \( O_\varepsilon \) corresponding to two distinct solutions of problem (1.1). If \( M^{p, A}_1 = M^{p, A}_2 \), then we get that there is at least one set \( A \), with \( \text{cat}_{O_\varepsilon}(A) \geq 2 \), where the function \( F^A_p \) reaches its absolute minimum. In this case, we conclude that there are infinitely many critical points for \( F^A_p \) in \( O_\varepsilon \). Since permutations are only finite in number, the result is thus proven.

\[\square\]
Proof of Theorem 1.2. For the case \( m = 1 \), it suffices to prove the existence of critical points of \( \tilde{F}_p^A \) given by (6.1), whose proof is based on the asymptotic formulas (1.5)-(1.6) for \( H_1(\xi, \cdot) \) combined with a linking argument.

In fact, for \( m = 1 \), by (6.1) for any large \( p \) and \( \lambda \) satisfying \( \frac{1}{p} \log \lambda = o(1) \), we get that

\[
\tilde{F}_p^A(\xi) = H_1(\xi, \cdot) + \frac{1}{2\pi} \log \lambda + \frac{\log \lambda}{p} \Theta_{p, \lambda}(\xi), \quad \xi \in \mathcal{O}_\varepsilon, \tag{6.2}
\]

where

\[
\frac{\log \lambda}{p} |\Theta_{p, \lambda}(\xi)| = O\left( \frac{\log \lambda}{p} \right) = o(1), \quad \frac{\log \lambda}{p} |\nabla_\xi \Theta_{p, \lambda}(\xi)| = O\left( \frac{\lambda}{p^3} \right) + O\left( \frac{1}{p} \right) \tag{6.3}
\]

and

\[
\mathcal{O}_\varepsilon = \left\{ \xi \in \Omega : |\lambda d(\xi) - \theta_{b(\xi)}| \leq \frac{1}{\varepsilon \lambda^{1/2}} \right\}. \tag{6.4}
\]

Using the asymptotic formulas of \( H_1(\xi, \cdot) \) in (1.5)-(1.6), for \( \xi \in \mathcal{O}_\varepsilon \), we have that

\[
\tilde{F}_p^A(\xi) = \frac{1}{2\pi} h_1(\lambda d(\xi), b(\xi)) + \frac{1}{2\pi} \log \lambda + O\left( \frac{1}{\lambda} \right) + O\left( \frac{1}{p} \log \lambda \right). \tag{6.5}
\]

Note that for any \( \tilde{b} > 0 \) the function \( h_1(\cdot, \tilde{b}) \) has a unique non-degenerate minimum \( \theta_{\tilde{b}} \in (0, +\infty) \), which is a continuous function with respect to the variable \( \tilde{b} \). We define

\[
S^* := \{ \xi \in \Omega : \lambda d(\xi) = \theta_{\tilde{b}(\xi)} \} \subset \mathcal{O}_\varepsilon.
\]

Then, for \( \varepsilon > 0 \) sufficiently small, it follows that

\[
\inf_{\xi \in \mathcal{O}_\varepsilon} h_1(\lambda d(\xi), b(\xi)) > \sup_{\xi \in S^*} h_1(\lambda d(\xi), b(\xi)),
\]

and therefore, by (6.5),

\[
\inf_{\xi \in \mathcal{O}_\varepsilon} \tilde{F}_p^A(\xi) > \sup_{\xi \in S^*} \tilde{F}_p^A(\xi) \tag{6.6}
\]

uniformly for any large \( p \) and \( \lambda \) satisfying \( \frac{1}{p} \log \lambda = o(1) \). Accordingly, there exists \( \xi_{1, p, \lambda} \in \mathcal{O}_\varepsilon \) such that

\[
\inf_{\xi \in \mathcal{O}_\varepsilon} \tilde{F}_p^A(\xi) = \tilde{F}_p^A(\xi_{1, p, \lambda}).
\]

We now seek another critical point of \( \tilde{F}_p^A \) in \( \mathcal{O}_\varepsilon \). For each \( x \in \partial \Omega \), we consider the set

\[
Q_x := \{ \xi \in \mathcal{O}_\varepsilon : \xi = x \}.
\]

Let us assume that there exists \( x_0 \in \partial \Omega \) such that

\[
\inf_{\xi \in Q_{x_0}} \tilde{F}_p^A(\xi) > \tilde{F}_p^A(\xi_{1, p, \lambda}). \tag{6.7}
\]

Otherwise, if such a point does not exist then the theorem is proved. Let \( \partial Q_{x_0} \) denote the boundary of \( Q_{x_0} \). Obviously, the sets \( \partial Q_{x_0} \) and \( S^* \) link in \( \mathcal{O}_\varepsilon \). Moreover, by (6.6) we have

\[
\inf_{\xi \in \partial Q_{x_0}} \tilde{F}_p^A(\xi) > \sup_{\xi \in S^*} \tilde{F}_p^A(\xi).
\]

Let

\[
P = \{ f \in C^0(\overline{Q_{x_0}}, \overline{\mathcal{O}_\varepsilon}) : f|_{\partial Q_{x_0}} = \text{Id}|_{\partial Q_{x_0}} \}.
\]

Then,

\[
\beta = \sup_{f \in P} \inf_{\xi \in \mathcal{O}_\varepsilon} \tilde{F}_p^A(\xi)
\]

is a critical value of \( \tilde{F}_p^A \), which is different from \( \tilde{F}_p^A(\xi_{1, p, \lambda}) \) by (6.7). Hence, we obtain the existence of the second critical point \( \xi_{2, p, \lambda} \) for \( \tilde{F}_p^A \) in \( \mathcal{O}_\varepsilon \), which is different from \( \xi_{1, p, \lambda} \). \qed
Proof of Theorem 1.3. Let \( x_0 \in \partial \Omega \) be a non-degenerate critical point of the mean curvature \( k \) and \( b \equiv 1 \). As in the proof of Theorem 1.2, it suffices to prove the existence of critical points of \( \tilde{F}_p^\lambda \) near \( x_0 \), where \( \tilde{F}_p^\lambda \) is given by (6.2)–(6.4). For this purpose, we will take advantage of the asymptotic formula (1.6) of \( H_{\lambda}(\xi, \xi) \) to relate the topological degree of the \( \nabla \tilde{F}_p^\lambda \) in a suitable small set close to \( x_0 \) with that of the \( \nabla F \).

Let \( R_\lambda(\xi) = H_{\lambda}(\xi, \xi) \) and \( \nabla F \) denote the tangential gradient which is defined in a neighborhood of \( \partial \Omega \). For any \( \alpha \in (0, 1) \), there exist the following expansions for the gradient of \( R_\lambda(\xi) \) (see [7]):

\[
\nabla R_\lambda(\xi) = \frac{1}{2\pi \lambda} \nabla \kappa(\xi) \nabla (\lambda d(\xi)) + O\left(\frac{1}{\lambda^{1+\alpha}}\right),
\]

\[
\langle \nabla R_\lambda(\xi), \nabla \cdot (\lambda d(\xi), 1) - \frac{1}{2\pi \lambda} \kappa(\xi) \nu(\lambda d(\xi)) + O\left(\frac{1}{\lambda^2}\right).
\]

uniformly for \( |\lambda d(\xi) - \theta_b| \leq \frac{1}{\alpha^{1/2}} \) with \( b \equiv 1 \).

Since \( x_0 \in \partial \Omega \) is a non-degenerate critical point of \( k \), there exist \( C > 0 \) and \( \sigma > 0 \) such that

\[
|\nabla \kappa(\xi)| \geq C|\xi - x_0| \quad \text{for all} \quad \xi \in \partial \Omega \quad \text{such that} \quad |\xi - x_0| \leq \sigma.
\]

On the other hand, we know that the function \( h_{\lambda}(\cdot, 1) \) has a unique minimum \( \theta_b > 0 \), which is non-degenerate, and hence by taking \( C > 0 \) and \( \sigma > 0 \) smaller if necessary, we have

\[
\left| \frac{\partial}{\partial \theta} h_{\lambda}(\theta, 1) \right| \geq C(\theta - \theta_b) \quad \text{for all} \quad |\theta - \theta_b| \leq \sigma.
\]

Using the fact that the function \( v \) is continuous and negative in \( \mathbb{R} \), we can choose \( \sigma > 0 \) smaller such that

\[
\inf_{\theta \in [\theta_b - \sigma, \theta_b + \sigma]} |v(\theta)| > 0.
\]

Let \( 0 < \beta < \alpha \). For \( \lambda \) large and \( \epsilon \) small but fixed, we can also assume \( \max\{\frac{1}{\alpha^{1/2}}, \frac{1}{\lambda^{1/2}}\} < \sigma < \theta_b \), since \( \theta_b > 0 \). Consider the compact set

\[
\mathcal{K}_\lambda := \left\{ \xi \in \Omega : |\lambda d(\xi) - \theta_b| \leq \frac{1}{\epsilon \lambda^{1/2}}, \ |\xi - x_0| \leq \frac{1}{\lambda^{1/2}} \right\}.
\]

Then, \( \mathcal{K}_\lambda \subset O_\epsilon \) with \( O_\epsilon \) defined in (6.4). Let

\[
\tilde{F}_p^\lambda(\xi) = \frac{1}{2\pi} h_{\lambda}(\lambda d(\xi), 1) + \frac{1}{2\pi \lambda} \kappa(\xi) v(\lambda d(\xi)).
\]

Obviously, \( \tilde{F}_p^\lambda \) has a unique non-degenerate critical point in the interior of \( \mathcal{K}_\lambda \). Now we define the homotopy between \( \tilde{F}_p^\lambda \) and \( \tilde{F}_0^\lambda \) as follows:

\[
\tilde{F}_p^{t, \lambda}(\xi) = t \tilde{F}_p^\lambda(\xi) + (1 - t) \tilde{F}_0^\lambda(\xi), \quad t \in [0, 1].
\]

We observe that for any large \( p \) and \( \lambda \) satisfying \( \frac{1}{p} \log \lambda = o(1) \), there exists \( C' > 0 \) such that if \( \xi \in \partial \mathcal{K}_\lambda \) then:

(i) if \( |\lambda d(\xi) - \theta_b| = \frac{1}{\epsilon \lambda^{1/2}} \), then by (6.2), (6.3), (6.9) and (6.11), we have

\[
|\nabla \tilde{F}_p^{t, \lambda}(\xi)| \geq |\langle \nabla \tilde{F}_p^{t, \lambda}(\xi), \nu(\lambda d(\xi)) \rangle| \geq C' \sqrt{\lambda} + O\left(\frac{\lambda}{p^{1/2}}\right) + O\left(\frac{1}{p}\right);
\]

(ii) if \( |\xi - x_0| = \frac{1}{\lambda^{1/2}} \), then by (6.2), (6.3), (6.8), (6.10) and (6.12), we have

\[
|\nabla \tilde{F}_p^{t, \lambda}(\xi)| \geq |\nabla \tilde{F}_p^{t, \lambda}(\xi)| \geq \frac{C'}{\lambda^{1+\beta}} + O\left(\frac{\lambda}{p^{1/2}}\right) + O\left(\frac{1}{p}\right).
\]

Then, it is easily checked that by choosing sufficiently large \( p \) and \( \lambda \) such that \( \frac{1}{p} \lambda^{1+\beta} = o(1) \), \( |\nabla \tilde{F}_p^{t, \lambda}(\xi)| \) remains uniformly positive on \( \partial \mathcal{K}_\lambda \). By the degree theory \( \tilde{F}_p^\lambda(\xi) = \tilde{F}_p^\lambda(\xi) \) has a critical point in the interior of \( \mathcal{K}_\lambda \), and hence it is located at distance \( O(\lambda^{-\beta}) \) from \( x_0 \).
Proof of Theorem 1.4. Let $x_0 \in \partial \Omega$ be a non-degenerate critical point of the function $b$. The proof is similar to that of Theorem 1.3, that is, to prove the existence of critical points of $\tilde{F}_A^1$ defined in (6.2)–(6.4), we will take advantage of the asymptotic formula (1.5) of $H_A(\xi, \tilde{\xi})$ to relate the topological degree of the $\nabla \tilde{F}_A^1$ in a suitable small set close to $x_0$ with that of the $\nabla b$.

Note that from [7], the gradient of $R_\lambda(\xi)$ has the following expansions:

$$\nabla R_\lambda(\xi) = \frac{1}{2\pi} \frac{\partial h_\lambda}{\partial \theta}(\lambda \theta(\xi), b(\tilde{\xi})) \nabla b(\tilde{\xi}) + O\left(\frac{1}{\lambda}\right),$$

$$\nabla (\theta R_\lambda(\xi), \varphi(\tilde{\xi})) = -\frac{\lambda}{2\pi} \frac{\partial h_\lambda}{\partial \theta}(\lambda \theta(\xi), b(\tilde{\xi})) + O(1)$$

uniformly for $| \lambda \theta(\xi) - \theta_{b(\tilde{\xi})} | \leq \frac{1}{2\lambda^2}$.

Since $x_0 \in \partial \Omega$ is a non-degenerate critical point of the function $b$, there exist $C > 0$ and $\sigma > 0$ such that

$$|\nabla b(\tilde{\xi})| \geq C|\tilde{\xi} - x_0| \quad \text{for all } \tilde{\xi} \in \partial \Omega \text{ such that } |\tilde{\xi} - x_0| \leq \sigma.$$ (6.15)

On the other hand, we know that for any $\tilde{b} > 0$ the function $h_\lambda(\cdot, \tilde{b})$ has a unique non-degenerate minimum $\theta_{b(\tilde{\xi})} \in (0, \infty)$, which is a continuous function with respect to the variable $\tilde{b}$, and hence taking $\tilde{\xi} > 0$ and $\sigma > 0$ smaller if necessary, we have

$$\left| \frac{\partial}{\partial \theta} h_\lambda(\theta, \tilde{b}(\tilde{\xi})) \right| \geq C|\theta - \theta_{b(\tilde{\xi})}| \quad \text{for all } |\theta - \theta_{b(\tilde{\xi})}| \leq \sigma \text{ and } \tilde{\xi} \in \partial \Omega.$$ (6.16)

Using that the function $\frac{\partial h_\lambda}{\partial \theta}(\theta, \tilde{b})$ is continuous and negative in $\mathbb{R} \times \mathbb{R}_+$, we can choose $\sigma > 0$ smaller such that

$$\inf_{\tilde{\xi} \in \partial \Omega, \theta \in [\theta_{b(\tilde{\xi})} - \sigma, \theta_{b(\tilde{\xi})} + \sigma]} \left| \frac{\partial}{\partial \theta} h_\lambda(\theta, \tilde{b}(\tilde{\xi})) \right| > 0.$$ (6.17)

Let $0 < \beta < 1$. For large $\lambda$ and small but fixed $\varepsilon$, we can also assume $\max\{\frac{1}{\varepsilon^{1/2^\beta}}, \frac{1}{\varepsilon^{1/2^{1-\beta}}}\} < \sigma < \min_{\tilde{\xi} \in \partial \Omega} \theta_{b(\tilde{\xi})}$, since the function $\theta_{b(\cdot)}$ is continuous and positive on $\partial \Omega$. Consider the compact set

$$\mathcal{K}_\lambda := \left\{ \xi \in \Omega : |\lambda \theta(\xi) - \theta_{b(\tilde{\xi})}| \leq \frac{1}{\varepsilon^{1/2^\beta}}, |\tilde{\xi} - x_0| \leq \frac{1}{\lambda^\beta} \right\}.$$ (6.18)

Then, $\mathcal{K}_\lambda \subset \mathcal{O}_\varepsilon$ with $\mathcal{O}_\varepsilon$ defined in (6.4). Let

$$\tilde{F}_A^1(\xi) = \frac{1}{2\pi} h_\lambda(\lambda \theta(\xi), b(\tilde{\xi})).$$

Obviously, $\tilde{F}_A^0$ has a unique non-degenerate critical point $\xi_\lambda$ located in the interior of $\mathcal{K}_\lambda$ such that $\xi_\lambda = x_0$ and $\lambda \theta(\xi_\lambda) = \theta_{b(\tilde{\xi}_\lambda)}$. Now we define the homotopy between $\tilde{F}_A^p$ and $\tilde{F}_A^1$ as follows:

$$\tilde{F}_A^{p,t}(\xi) = t \tilde{F}_A^p(\xi) + (1 - t) \tilde{F}_A^0(\xi), \quad t \in [0, 1].$$

We observe that for any large $p$ and $\lambda$ satisfying $\frac{1}{p} \log \lambda = o(1)$, there exists $C' > 0$ such that if $\xi \in \partial \mathcal{O}_\lambda$ then:

(i) if $|\lambda \theta(\xi) - \theta_{b(\tilde{\xi})}| = \frac{1}{\lambda^{1/2}}$, then by (6.2), (6.3), (6.14) and (6.16), we have

$$|\nabla \tilde{F}_A^{p,t}(\xi)| \geq |\nabla \tilde{F}_A^p(\xi), \varphi(\tilde{\xi})| \geq C' \sqrt{\lambda} + O\left(\frac{1}{\lambda^{1/2}}\right) + O\left(\frac{1}{p}\right);$$

(ii) if $|\xi - x_0| = \frac{1}{\lambda^\beta}$, then by (6.2), (6.3), (6.13) and (6.17), we have

$$|\nabla \tilde{F}_A^{p,t}(\xi)| \geq |\nabla \tilde{F}_A^{p,t}(\xi)| \geq C' \frac{1}{\lambda^\beta} + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{p}\right).$$

Then, it is easily checked that, choosing sufficiently large $p$ and $\lambda$ such that $\frac{1}{p} (\lambda^{1/3(1+\beta)} + \lambda^\beta) = o(1)$, $|\nabla \tilde{F}_A^{p,t}(\xi)|$ remains uniformly positive on $\partial \mathcal{K}_\lambda$. By the degree theory, $\tilde{F}_A^p(\xi) = \tilde{F}_A^{p,1}(\xi)$ has a critical point in the interior of $\mathcal{K}_\lambda$, and hence it is located at distance $O(\lambda^{-\beta})$ from $x_0$. 

\hfill \Box
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**References**


