L3 DYNAMICS AND POINCARÉ MAPS IN THE RESTRICTED FULL THREE BODY PROBLEM

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Abstract. Poincaré maps are a basic dynamical systems tool yielding information about the geometric structure of the phase space of the system. Poincaré maps are however time consuming to compute. In this paper we have analysed and compared two different schemes to compute Poincaré maps in the context of accuracy versus computation time: a Runge-Kutta method of 7th and 8th order and a time transformed geometric method of 6th order. The dynamical system used is the Restricted Full Three Body Problem, with the primaries, an elongated body and a sphere, in a short axis relative equilibrium configuration. Using these Poincaré maps we have studied the dynamics near the collinear Lagrange point L3, located on the outer side of the elongated body. We present evidence that the L3 point in this system can have saddle-center, stable or complex unstable behaviour depending on system parameters. We further show that when a low accuracy regime that still captures the correct structure of the Poincaré map is considered, the geometric method clearly outperforms the Runge-Kutta method being up to 4 times faster to compute and free from accumulating local errors that smear the structure of the Poincaré maps.

Key words: methods: numerical – celestial mechanics – minor planets, asteroids

1. INTRODUCTION

Nowadays in the space community, there is an increasing interest in asteroids and comets, which, due to their very irregular shapes, leads to the study of mathematical models that take shape into account. Moreover, since the discovery of binary asteroids, various mathematical models have been developed, such as the Full Two Body Problem (F2BP) to model a binary asteroid (Bellerose & Scheeres 2007; Bellerose & Scheeres 2008a; Scheeres 2002, 2004), or the Restricted Full Three Body Problem (RF3BP) to model a spacecraft in the vicinity of the binary (Bellerose & Scheeres 2007, 2008c, 2005, 2008b; Bellerose 2008; Gabern et al. 2006).
A useful way to study dynamical systems like the F2BP or the RF3BP is to use Poincaré maps, as they are a basic dynamical systems tool to reduce the dimension of the system but still conserve many of their properties. However, Poincaré maps are very time consuming to compute, to the point of taking several hours on a typical personal computer. In mathematical models with many parameters where several Poincaré maps have to be computed, it is important to find a suitable way to compute them fast without compromising the structure of the map.

In this paper, two different methods of computing Poincaré maps have been studied and compared in the context of accuracy versus computation time, when applied to the Hamiltonian system of the RF3BP. We have investigated the possibility of obtaining a computational advantage by using a geometric propagator with the secant method to find the cut, instead of a traditional high order Runge-Kutta propagator combined with the Newton method to find the cut. The goal was to develop a method that is simple to implement and confers computational advantage in such accuracy regimes that the resulting maps have correct qualitative features.

A defining characteristic of geometric integration methods is conservation of phase space structure during integration. This allows a longer time step to be used while still retaining correct long-term behaviour of the solution. The drawback is a loss of local accuracy of the solution. These advantages are best realised in integrations spanning a long time interval. As calculating Poincaré maps requires long propagation of each set of initial values, the a priori expectation was that geometric methods should do well.

2. THEORY: THE MATHEMATICAL MODELS

2.1. The planar spherical restricted full two body problem

The planar spherical restricted F2BP is a mathematical model of two gravitationally attracting rigid bodies where it has been assumed that one of the bodies is a constant density perfect sphere and the other a non-spherical or elongated body. In Newtonian gravity, the spherical body can be reduced to a point mass and its rotational dynamics decoupled from the system. The non-spherical body is assumed to rotate about the shortest principal axis and it must have a symmetry about its equatorial plane, the same plane where the point mass is located at, allowing us to consider a planar problem.

Two frames of reference centered at the center of mass of the non-spherical body are used throughout the paper. One is fixed in space, and the other is rotating with the non-spherical body. All coordinates refer to the fixed frame unless the rotating frame is explicitly specified. In both frames, the \( x \) and \( y \) axis are located on the plane of symmetry of the non-spherical body and the \( z \) axis is in the direction of the shortest principal axis. Using these frames of reference, it is possible to express the system using the following coordinates: \( \vec{r} = (x, y) \), the relative position between the centre of the elongated body and the centre of the sphere, and \( \theta \), the angle of rotation of the elongated body about the \( z \) axis. Naturally in the rotating frame we have \( \theta \equiv 0 \). We further note that \( \vec{v} \) is the relative velocity and \( \omega = \dot{\theta} \) is the angular velocity.

To express the potential energy of the elongated body, McGullagh’s formula
Restricted full three body problem

(Scheeres 2009), an expansion up to second order in terms of the mass and inertia moments of the non-spherical body, is used. As the spherical body is treated as a point mass, the rotation of the sphere decouples and the expression for the potential reduces to

\[
V(\vec{r}, \theta) = -GM_1M_2 \left\{ \frac{1}{||\vec{r}||} + \frac{Tr(I)}{2 ||\vec{r}||^3} - \frac{3}{2 ||\vec{r}||^5} (x^2I_{xx} + y^2I_{yy} + z^2I_{zz}) - \frac{3}{2 ||\vec{r}||^5} (I_{xx} - I_{yy}) (y^2 - x^2 \sin^2 \theta + xy \sin(2\theta)) \right\}.
\]

(1)

where \( I \) is the inertia matrix per unit mass of the elongated body, with diagonal elements \( I_{xx}, I_{yy} \) and \( I_{zz} \) in the direction of the principal longest, middle and shortest axis of the body, respectively. This expression for the gravitational potential is only valid in the region outside a sphere circumscribing the body, therefore, only the dynamics outside this sphere are considered. In this paper we assume a planar problem, so that the \( z \)-coordinate and the respective velocity component are suppressed.

When considering the total kinetic energy we can leave out the rotational energy of the sphere, because the rotational motion of the sphere decouples from the rest of the system. The expression for the kinetic energy can then be written as

\[
T = \frac{1}{2} I_{zz} \omega^2 + \frac{1}{2} \frac{M_1M_2}{M_1 + M_2} (\dot{x}^2 + \dot{y}^2).
\]

(2)

In order to have dimensionless equations normalization is necessary. Therefore, a fundamental unit of length and a fundamental unit of time are defined. Assuming that the moments of inertia of the non-spherical body can be expressed in the form

\[
I_{zz} = M_2R^2, \quad I_{yy} = M_2R^2\alpha_y, \quad I_{xx} = M_2R^2\alpha_x
\]

\[
\text{with } 1 \geq \alpha_y \geq \alpha_x \geq 0,
\]

we can define the following quantities:

\[
x = \frac{x}{R}, \quad y = \frac{y}{R}, \quad n = \sqrt{\frac{G(M_1 + M_2)}{R^3}},
\]

\[
t' = nt, \quad \omega = \frac{\omega}{n}, \quad \nu = \frac{M_1}{M_1 + M_2}.
\]

Using these new variables (without the tildes) the normalised Euler-Lagrange
equations of motion have the following form:

\[ \ddot{x} = \frac{\partial V}{\partial x} \]
\[ \ddot{y} = \frac{\partial V}{\partial y} \]
\[ \dot{\omega} = 3\nu \left( \frac{\alpha_y - \alpha_x}{(x^2 + y^2)^{3/2}} \right) \left( xy \cos(2\theta) + \sin \theta \cos \theta (y^2 - x^2) \right) . \]

(3)  
(4)  
(5)

Here the normalised potential is

\[ V(\vec{r}, \theta) = -\left\{ \frac{1}{\|\vec{r}\|} + \frac{1}{2} \frac{\alpha_x + \alpha_y}{\|\vec{r}\|^3} \right. \]
\[ - \frac{3}{2} \frac{\alpha_x + \alpha_y}{\|\vec{r}\|^2} \left( x^2 \alpha_x + y^2 \alpha_y \right) \]
\[ - \frac{3}{2} \frac{\alpha_x - \alpha_y}{\|\vec{r}\|^2} \left[ (y^2 - x^2) \sin^2 \theta + xy \sin(2\theta) \right] \}

(6)

the normalised kinetic energy is

\[ T = \frac{1}{2\nu} \omega^2 + \frac{1}{2} (\dot{x}^2 + \dot{y}^2) , \]

(7)

and we have three parameters: the mass parameter \( \nu \in [0, 1] \) and the shape parameters \( \alpha_x, \alpha_y \in [0, 1] \).

If \( \nu \neq 0 \), the system allows for two independent integrals of motion, the total energy \( E \) and the angular momentum \( K \), with

\[ E = \frac{1}{2} \omega^2 + \frac{1}{2} \nu (\dot{x}^2 + \dot{y}^2) + \nu V(\vec{r}, \theta) , \]

(8)

and

\[ K = \omega^2 + \nu (\dot{y}x - \dot{x}y) . \]

(9)

When \( \nu = 0 \) we recover the problem of a massless particle orbiting a non-spherical body with constant angular velocity, problem that has been studied in several references such as Augenstein & Scheeres (2003); Scheeres (1994); Chauvineau et al. (1993, 1994).

In a frame of reference always aligned with the principal axes of the elongated body, the constants of motions have the following expressions

\[ E = \frac{1}{2} \omega^2 + \frac{1}{2} \left[ (\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 \right] + V(x, y) , \]

(10)

\[ K = \omega \left[ 1 + \nu(x^2 + y^2) \right] + \nu (\dot{y}x - \dot{x}y) , \]

(11)

where

\[ V(x, y) = -\frac{1}{\sqrt{x^2 + y^2}} - \frac{1}{2} \frac{\alpha_x + \alpha_y}{(x^2 + y^2)^{3/2}} + \frac{3(\alpha_x x^2 + \alpha_y y^2)}{2(x^2 + y^2)^{5/2}} . \]

(12)
Then, the equations of motion in rotating frame can be expressed as

\[ \ddot{x} - 2y\omega - \omega^2 x = \frac{\partial V}{\partial x}, \] (13)

\[ \ddot{y} + 2x\omega + \omega^2 y = \frac{\partial V}{\partial y}, \] (14)

\[ \dot{\omega} = \frac{3\nu xy(\alpha_y - \alpha_x)}{(x^2 + y^2)^{3/2}}. \] (15)

From Equation (15) it is possible to see that there are two relative equilibrium points (actually four as the system is symmetric with respect to the \(x\) and \(y\) axes). Two when \(x = 0\) and \(y \neq 0\), and two when \(y = 0\) and \(x \neq 0\). Using Equations (13) and (14), one can compute the angular velocity in each equilibrium configuration yielding

\[ \omega^2 = \frac{1}{|x|^3} \left[ 1 + \frac{3}{2x^2} (1 + \alpha_y - 2\alpha_x) \right] \] (16)

and

\[ \omega^2 = \frac{1}{|y|^3} \left[ 1 + \frac{3}{2y^2} (1 + \alpha_x - 2\alpha_y) \right] \] (17)

for the long and short axis equilibrium configurations respectively. These are illustrated in Figure 1. These two relative equilibrium solutions of each axis are symmetric, so, from now on, only the solutions where \(x > 0\) and \(y > 0\) are considered. The spectral stability as well as the energetic stability of these relative equilibrium configurations have been studied in the literature, see Bellerose & Scheeres (2007); Bellerose & Scheeres (2008a); Scheeres (2002, 2004) and references therein.
2.2. The restricted full three body problem

Assume that we have a spherical body (point mass) and an elongated body following the dynamics of the Planar Spherical Restricted Full Two Body Problem, defined in the previous section. And suppose now, that we have a massless particle near the two bodies (that we call primaries) that does not affect their dynamics but is affected by them.

Using the same frame of reference as before and the non-dimensional coordinates $\vec{r}_p = (x_p, y_p, z_p)$ for the position of the particle, $\vec{r} = (x, y)$ for the relative position of the binary and $\theta$ for the rotation angle of the non-spherical body, the potential that the massless particle experiences has the following expression

$$U(\vec{r}_p, \theta) = \frac{\nu}{\|\vec{r} - \vec{r}_p\|} + (1 - \nu)V(\vec{r}_p, \theta) - \nu\vec{r}_p \cdot \nabla V(\vec{r}, \theta),$$

where $V(\vec{r}_p, \theta)$ is the potential due to the elongated body. The last term in the potential arises from the non-inertial choice of coordinates. It offsets the acceleration of the ellipsoidal body and the coordinate frame.

In these coordinates the kinetic energy of the particle is

$$T = \frac{1}{2}(\dot{x}_p^2 + \dot{y}_p^2 + \dot{z}_p^2).$$

Then, the Euler-Lagrange equations of motion for the massless particle have the following expressions.

$$\ddot{x}_p = \frac{\partial U}{\partial x_p} + \nu \frac{\partial V}{\partial x},$$
$$\ddot{y}_p = \frac{\partial U}{\partial y_p} + \nu \frac{\partial V}{\partial y},$$
$$\ddot{z}_p = \frac{\partial U}{\partial z_p}.$$ 

Expressing the system using Hamiltonian formulation, we have the following Hamiltonian

$$H_p(\vec{r}_p, \vec{v}_p) = \frac{1}{2} \|\vec{v}_p\|^2 - (1 - \nu)V(\vec{r}_p, \theta) - \nu \left( \frac{1}{\|\vec{r}_p - \vec{r}\|} - \vec{r}_p \cdot \nabla V(\vec{r}, \theta) \right).$$

The Hamiltonian (22) is non-autonomous due to the implicit time dependence of $\vec{r}$, $\vec{v}$ and $\theta$. To obtain an autonomous system we introduce time $t$ and the moment of time $p_0$ as new canonical conjugates and write the Hamiltonian in the form

$$\tilde{H}_p(\vec{r}_p, \vec{v}_p) = \frac{1}{2} \|\vec{v}_p\|^2 + (1 - \nu)V(\vec{r}_p, \theta) - \nu \left( \frac{1}{\|\vec{r}_p - \vec{r}\|} - \vec{r}_p \cdot \nabla V(\vec{r}, \theta) \right) + p_0.$$ 

This Hamiltonian can be seen to give the same equations of motion as (22) and in addition we obtain

$$\frac{dt}{dt} = \frac{\partial H_p}{\partial p_0} = 1$$
$$\frac{dp_0}{dt} = -\frac{\partial H_p}{\partial t}.$$ 

Thus we see that the new Hamiltonian (23) is constant in time.
If the primaries are in one of the relative equilibrium configurations, the problem simplifies: the angular velocity of the non-spherical body is constant. Furthermore, expressed in the rotating frame the system becomes autonomous, and we have a conserved quantity, the Jacobi constant, which in rotating frame can be written as

\[ C = \frac{1}{2} (\dot{x}_p^2 + \dot{y}_p^2) - \tilde{V}(\vec{r}_p) - \frac{1}{2} \nu \omega^2 \| \vec{r}_p \|^2 + \nu \omega^2 \| \vec{r} \| f(\vec{r}), \]  

(26)

with

\[ \tilde{V}(\vec{r}_p) = \frac{\nu}{\| \vec{r}_p - \vec{r} \|} + (1 - \nu) V(\vec{r}_p, 0), \]  

(27)

and \( f(x, y) = x \) in long axis equilibrium configuration and \( f(x, y) = y \) in short axis equilibrium configuration. Moreover, when the primaries are in relative equilibrium there are relative equilibrium solutions for the massless particle in the rotating frame, called the Lagrange points.

2.2.1. Primaries in long axis relative equilibrium

In this case, the system resembles the ordinary Restricted Three Body Problem (R3BP), and, in the same way, there are five Lagrange equilibrium points: three collinear, \( L_1, L_2, L_3 \), which are aligned with the primaries, and two that form triangles with the masses, \( L_4 \) and \( L_5 \), which are symmetric with respect to the \( x \) axis. The location and stability of these equilibrium points differs from the location and stability of the R3BP ones due to the non-spherical shape of the elongated body. The three collinear points are always unstable, with a saddle-centre behaviour, and the triangular ones undergo a Hopf bifurcation, depending on the three parameters of the system.

2.2.2. Primaries in short axis relative equilibria

In this case, the Lagrange points appear as well, but this time, not only the
location and stability are different compared to the R3BP, but also the number of equilibrium points. There are three collinear points with the masses, and two or four triangular ones, depending on the stability of the collinear points. The behaviour of the collinear points L1 and L2, which are the ones on both sides of the spherical body, is the same as in the long axis case, saddle-centre behaviour. On the other hand, the behaviour of L3, which is the collinear point on the outer side of the elongated body, depends on the system parameters. When L3 has a saddle-centre behaviour, the problem is similar to the long axis case, so there are only five Lagrange points, the three collinear and the symmetric with respect to the y axis, L4 and L5. When L3 is stable or complex unstable two other unstable Lagrange points appear, L6 and L7. In both cases, the stabilities of L4 and L5 depend on the system parameters, experiencing a Hopf bifurcation as well.

3. NUMERICAL STUDY OF RF3BP DYNAMICS NEAR L3 WITH POINCARÉ MAPS

The RF3BP when the primaries are in short axis equilibrium configuration is more interesting in terms of dynamics than the long axis equilibrium case, as it differs from the R3BP in the behaviour of L3. For the R3BP or the RF3BP in long axis, only the Lagrange points L4 and L5 can be stable, though only for very small or very large values of the mass parameter. This constraint of the mass parameter is satisfied for the planets and their moons in the Solar System, but not for binary asteroids as their masses tend to be of comparable magnitude.

Analysing the linearised system around L3, it is observed that this Lagrange point is a saddle-centre when both bodies are spherical, but becomes stable for relatively short distances between the bodies and small mass parameter and as we increase the elongation of the non-spherical body. Increasing the elongation even further, the region of stability increases and a region with complex instability behaviour appears.
Fig. 4. Behaviour of Lagrange point $L_3$ in the short axis equilibrium configuration for different shape parameters $\alpha_x$ and $\alpha_y$. going from a very spherical body to a very elongated body, from left to right and top to bottom. The white region corresponds to the case where $L_3$ is inside the sphere circumscribing the elongated body, for which our equations are not valid.

3.1. Computing the Poincaré map

For the following, we assume that all coordinates refer to the rotating frame. Let $\dot{x} = f(x)$ be a dynamical system, where $f : \mathbb{R}^N \to \mathbb{R}^N$, and let $\Sigma : \mathbb{R}^N \to \mathbb{R}$ be a real valued function of the phase space $\mathbb{R}^N$ of the dynamical system, so that $\Sigma = 0$ defines a Poincaré section of the phase space. Now given the dynamical system and two Poincaré sections defined by $\Sigma_1$ and $\Sigma_2$, the Poincaré map $P$ is defined as

$$P : \Sigma_1 \rightarrow \Sigma_2$$

$$x_1 \mapsto x_2 = P(x) = \varphi(t(x_1), x_1)$$

where $\varphi(t, x)$ is the solution of the dynamical system and $t(x)$ is the time it takes for the solution $\varphi(t, x)$ to propagate from $x_1 \in \Sigma_1$ to $x_2 \in \Sigma_2$, (Meyer & Hall 1992). Often, as in this paper, it is useful to have $\Sigma_1 = \Sigma_2$.

Let now $\Sigma : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$, define a Poincaré section of the phase space of $H_p$. Let $\Psi : \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4 \times \mathbb{R}^4$ be the numerical flow of the propagator. We wish
to find all such points, called returns, of a trajectory $\Psi(t, \vec{x}_0)$ starting from the phase space point $\vec{x}_0 \in \mathbb{R}^4 \times \mathbb{R}^4$ that cut the section $\Sigma$, so that

$$\Sigma(\Psi(t, \vec{x}_0)) = 0.$$  \hspace{1cm} (29)

Furthermore, we wish to find these points with as little computational effort as possible for a given level of accuracy.

In order to analyse the dynamics around $L_3$ in the different stability regimes only one Poincaré section is used, i.e. $\Sigma_1 = \Sigma_2$, with the section given by the equation $x_p = 0$ in rotating coordinates. First, the parameters of the problem are fixed, i.e., the shape parameters $\alpha_x$ and $\alpha_y$, the mass parameter $\nu$, and the distance $\|\vec{r}\|$ between the massive bodies. Secondly, a set of initial conditions on this section is selected.

These initial values are then propagated and Poincaré cuts with the surface $\Sigma = 0$ computed until the particle escapes beyond the limit $\|\vec{r}_p\| > 9$ or strays within $r_{ell}$ of the ellipsoidal body or within $r_{sph}$ of the spherical body, with

$$r_{ell} = (1 - \nu) \frac{\sqrt{5}}{2} \sqrt{1 + |\alpha_x - \alpha_y|} \hspace{1cm} (30)$$

and

$$r_{sph} = \nu \sqrt{\frac{5}{2}}, \hspace{1cm} (31)$$

in which case a collision is recorded.

The integration of the map is done using two different methods of integration in order to do a numerical comparison. The first is a Runge-Kutta method combined with a Newton-Raphson iteration to compute the cut – i.e. the root of Equation (29) – and the second one is a geometric integrator with a secant method to compute the cut.

4. THE RUNGE-KUTTA INTEGRATOR

In this section, a well known Runge-Kutta-Fehlberg (RKF) method of order 7 and 8 is used to integrate the equations of motion (3)–(5) in the rotating frame. After the Poincaré surface is cut, a Newton iteration scheme is used to find the precise (up to tolerance) point of the cut.

The RKF methods are characterised by the use of two embedded Runge-Kutta methods of different order to reduce the local error and estimate the necessary time step to stay within given error tolerance (Grau & Noguera 1993). In our case, one method is of order 7 and the other one of order 8. Due to the fact that both methods share the majority of the force function evaluations at each step of integration, computing both doesn’t cause a large overhead cost. During the integration, the method with higher order is used to estimate the error in the method of lower order. If the error is found to be greater than the given tolerance, the time step is scaled down and the step repeated. Details of the procedure can be found in reference Burden & Faires (2005). For the 7th and 8th order algorithm a total of 13 evaluations of the force function have to be done at each time step, multiplied by the number of repeated steps.
4.1. Numerical results using the RKF method

In order to have a glimpse of the dynamics around $L_3$, in the regime where this Lagrange point has a completely different behaviour than $L_3$ in the R3BP, we have selected a binary system that has $L_3$ as a stable Lagrange point. Then, we have selected a set of initial conditions on the Poincaré section $x = 0$ with $y < 0$ with a particular horizontal velocity. The initial values have the form $\vec{r}_p = (0, y, 0, \dot{x}, 0, 0)$ where $\dot{x}$ is such that all the different $\vec{r}_p$ have the same value of the Jacobi constant $C$ given by the equation (26). We have integrated these values forward in time and we have plotted the returns on the Poincaré section with $y < 0$.

We have computed two different Poincaré maps using two very similar binary systems. In both systems, the relative distance between primaries is $\vec{r} = (0, 5, 0)$ in the rotating frame, the mass parameter is $\nu = 0.001$, the shape parameter $\alpha_y = 1$ and the value of the Jacobi constant is $C = -0.30011$. In the first one the shape parameter $\alpha_x = 0.93$, and in the second $\alpha_x = 0.91$. This small difference in the shape parameter makes the zero velocity curve around $L_3$, having a horseshoe shape in the first case ($\alpha_x = 0.93$), break at $L_6$ and $L_7$ when $\alpha_x = 0.91$, allowing the possibility of having stable orbits around $L_3$.

In both Poincaré maps we have selected our initial conditions on the outer side of the zero velocity curve and we have plotted the first 2000 returns of the trajectories, with the exception of the ones that escape from the system or crash with one of the components of the binaries.

Observing Figure 6, which corresponds to the case where the zero velocity curves have a horseshoe shape, only two structures appear in the middle of the chaotic sea. They correspond to stable orbits around the whole system, see Figure 7.

On the other hand, when reducing the shape parameter $\alpha_x$ to 0.91, we observe...
Fig. 6. Returns of the Poincaré map with starting values on the $y$ axis below the Lagrange point. $L_3$ is a stable equilibrium point and the zero velocity curve has a horseshoe shape around $L_4$, $L_3$, and $L_5$.

Fig. 7. Two stable symmetric periodic orbits with $C = -0.30011$ for the shape parameters $\alpha_x = 0.93$ and $\alpha_y = 1$.

in Figure 8 that some additional structure appears. This structure corresponds to periodic orbits about $L_3$, as the horseshoe shape of the zero velocity curve breaks forming two gaps around the points $L_6$ and $L_7$.

In Figure 8 we can observe that there are symmetric periodic orbits that cross the Poincaré map once, twice, thrice and up to four times before returning to the same point. Some examples of these orbits can be found in Figure 9.
Fig. 8. Returns of the Poincaré map with starting values on the $y$ axis below the Lagrange point. $L_3$ is a stable equilibrium point and the zero velocity curve has opened on $L_6$ and $L_7$.

5. THE GEOMETRIC INTEGRATOR

The system can be integrated with a symplectic leapfrog propagator, also known as a splitting method, since the Hamiltonian (23) separates into a kinetic part

$$T(\vec{v}_p, p_0) = \frac{1}{2} \|\vec{v}_p\|^2 + p_0,$$

and a potential part

$$V(\vec{r}, \theta) = (1 - \nu)V(\vec{r}_p, \theta) - \nu \left( \frac{1}{\|\vec{r}_p - \vec{r}\|} - \vec{r}_p \cdot \nabla_{\vec{r}} V(\vec{r}, \theta) \right),$$

(33)
(a) Periodic orbit with 1 cut of the Poincaré section \( x = 0 \) before returning to the same initial point.

(b) Periodic orbit with 2 cuts of the Poincaré section \( x = 0 \) before returning to the same initial point.

Fig. 9. Some stable symmetric periodic orbits with \( C = -0.30011 \) for the shape parameters \( \alpha_x = 0.91, \alpha_y = 1 \).

so that
\[
\tilde{H}_p(\vec{r}, \theta, \vec{v}_p, p_0) = T(\vec{v}_p, p_0) + V(\vec{r}, \theta), \tag{34}
\]

with the kinetic part depending only on momenta (velocities) and the potential part depending only on the coordinates.

However, instead of a standard symplectic leapfrog, we chose to use the Time Transformed Leapfrog (TTL) scheme as detailed in Mikkola & Aarseth (2002) combined with a sixth order symplectic leapfrog. The TTL scheme requires little extra implementation effort, but offers computational and accuracy benefits. In TTL method the system is modified so that it is not precisely Hamiltonian. Thus when integrated with a leapfrog the composite method is not symplectic except in special cases, and belongs to the broader category of geometric integrators instead. The behaviour of this composite method with respect to energy, and phase space structure conservation is however similar to symplectic methods, so that there is no secular growth in the energy error, and the phase space structure is well conserved. Geometric and symplectic integration methods are well documented in the literature (see e.g. references Leimkuhler & Reich 2004; Hairer et al. 2006).

In the TTL scheme we start with the equations of motion we obtain from Hamilton’s equations
\[
\frac{dq}{dt} = \nabla_p H \tag{35}
\]
\[
\frac{dq}{dt} = -\nabla_q H. \tag{36}
\]

using the Hamiltonian of the massless particle given by (23). Computing this and
solving for coordinates and velocities we get
\[
\frac{d\vec{r}_p}{dt} = \vec{v}_p \tag{37}
\]
\[
\frac{d\vec{v}_p}{dt} = -\nabla_{\vec{r}_p} V. \tag{38}
\]

The equations of motions for the massive binary are not solved separately as the binary is assumed to be in a relative equilibrium state and rotating with a constant angular velocity.

Next we introduce a time transformation function
\[
\Omega(\vec{r}_p, \vec{r}) = \frac{1}{||\vec{r}||} + \frac{1}{||\vec{r}_p||} + \frac{1}{||\vec{r} - \vec{r}_p||}. \tag{39}
\]

This function relates the physical time \( t \) and a new fictitious time \( s \) so that
\[
\frac{ds}{dt} = \Omega(\vec{r}_p, \vec{r}). \tag{40}
\]

With respect to this fictitious time, our equations of motion become
\[
\frac{dt}{ds} = \frac{1}{\Omega}, \tag{41}
\]
\[
\frac{d\vec{r}}{ds} = \frac{1}{\Omega} \vec{v}_p \tag{42}
\]
\[
\frac{dp_0}{ds} = -\frac{1}{\Omega} \frac{\partial V}{\partial t} \tag{43}
\]
\[
\frac{d\vec{v}}{ds} = -\frac{1}{\Omega} \nabla_{\vec{r}} V. \tag{44}
\]

Now instead of using the function \( \Omega \) when updating the positions, we instead introduce an auxiliary variable, the time transformation variable \( W \). Initially we set \( W = \Omega(\vec{r}_p, \vec{r}_0) \) and subsequently update the value using
\[
\frac{dW}{dt} = \nabla_{\vec{r}_p} \Omega \cdot \vec{v}_p + \nabla_{\vec{r}} \Omega \cdot \vec{v}. \tag{45}
\]
from which we immediately get
\[
\frac{dW}{ds} = \frac{1}{\Omega} \left( \nabla_{\vec{r}_p} \Omega \cdot \vec{v}_p + \nabla_{\vec{r}} \Omega \cdot \vec{v} \right). \tag{46}
\]

Our equations of motion are then in a separable form
\[
\frac{dt}{ds} = \frac{1}{W}, \tag{47}
\]
\[
\frac{d\vec{r}}{ds} = \frac{1}{W} \vec{v}, \tag{48}
\]
\[
\frac{dp_0}{ds} = -\frac{1}{\Omega} \frac{\partial V}{\partial t} \tag{49}
\]
\[
\frac{d\vec{v}}{ds} = -\frac{1}{\Omega} \nabla_{\vec{r}} V \tag{50}
\]
\[
\frac{dW}{ds} = \frac{1}{\Omega} \left( \nabla_{\vec{r}_p} \Omega \cdot \vec{v}_p + \nabla_{\vec{r}} \Omega \cdot \vec{v} \right). \tag{51}
\]
and can be integrated using a leapfrog method.

To integrate the equations of motion, we chose to use the 6th order leapfrog by Yoshida (Yoshida 1990). We use a common notation derived from the exponential map of a Lie algebra by designating the propagation of coordinates and time for a step of length $h$ of the fictitious time with the exponential operator $\exp \left( \frac{1}{2} hT \right)$ and the propagation of velocities and the time transformation variable similarly with $\exp (hV)$. With this notation one step of length $h$ of the usual second order leapfrog can be written as the operator composition

$$LF2 (h) = \exp \left( \frac{1}{2} hT \right) \exp (hV) \exp \left( \frac{1}{2} hT \right).$$

(52)

Yoshida's 6th order method can then be written as the symmetric composition

$$Y6 (h) = LF2 (a_1 h) LF2 (a_2 h) \cdots LF2 (a_7 h),$$

(53)

where the approximate values of coefficients $a_i$ are

$$a_1 = a_7 = 0.78451361047755726381949763$$

$$a_2 = a_6 = 0.23557321335935813368479318$$

$$a_3 = a_5 = -1.17767998417887100694641568$$

$$a_4 = 1.31518632068391121888424973.$$

(54)

In line with our goal of simple implementation, we chose to find the Poincaré section crossing without calculating the time derivative of the numerical flow. To this purpose we chose to use the secant method to find the root of equation (29). The secant method finds the root of the equation $f(x) = 0$, $f : \mathbb{R} \rightarrow \mathbb{R}$ by iterating the equation

$$x_{n+1} = x_n - \frac{f(x_n) x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

(55)

In the implementation the iteration was continued until the value of the surface function $\Sigma$ was within the given tolerance $\epsilon_{cut}$.

Considering that the cut is detected by a change in sign of $\Sigma$, we have a good localisation of the root at the start of the iteration and convergence is typically not a problem. Furthermore, when the computational cost and rate of convergence are considered together, the secant method is seen to be more efficient that Newton’s method (Deuflhard & Bornemann 2002).

6. NUMERICAL COMPARISONS

We compared the combination of the geometric propagator and secant method, referred to as SYMP (alluding to the symplectic characteristics of the method), against the reference implementation described in section , which we denote RK78.

The test suite consisted of initial values near the $L_3$ point of the binary system. We computed a total of 250 sets of initial values of the form $\vec{r}_p = (0, y, 0)$ and $\vec{v}_p = (\dot{x}, 0, 0)$ in the rotating frame. The initial values were computed by setting the Jacobi constant to $C = -3.0011$ and $y = -5.279 - 1.6 \cdot 10^{-3} (k - 1)$ with $k = 1 \ldots 250$. Finally $\dot{x}$ was solved from equation (26). The parameters of the system were set to $\alpha_x = 0.91$, $\alpha_y = 1.0$, $\nu = 10^{-3}$ and the relative position of the
massive binary was set to $\vec{r} = (0, 5, 0)$ in the rotating frame. Barring collisions and escapes as defined in section , the orbits were propagated until a total of 4000 intersections with the Poincaré surface were recorded.

The calculations were done with several choices of step sizes and tolerance parameters for both methods to investigate the computational cost of the methods in different tolerance regimes. For the SYMP method, the only adjustable parameters are the (constant) time step $h$ and the root finder tolerance $\epsilon_{\text{cut}}$. For the RK78 method we can adjust the maximum and minimum time steps $h_{\text{max}}$ and $h_{\text{min}}$, the local error tolerance $\epsilon_{\text{int}}$ and the root finder tolerance $\epsilon_{\text{cut}}$.

Several runs were done with both methods, and the parameter sets used are listed in Table 1. In addition a single run with the RK78 method was done with strict tolerance limits to serve as the truth model. In line with our goal of investigating quick computation with limited accuracy we chose to use a constant value of $\epsilon_{\text{cut}} = 10^{-10}$ throughout all runs, except for the truth model for which we used $\epsilon_{\text{cut}} = 10^{-14}$.

All runs were computed on the same desktop computer, featuring an AMD Sempron LE-1200 processor and 3 GiB of RAM, running a GNU/Linux operating system. The system was considered to have computing power representative of a typical lower end desktop system for a contemporary researcher, a quality useful for the objectives of the comparison.

To assess the computational costs the propagation runs were timed with the GNU/Linux built-in `time` utility. This results in a only proxy of the amount of computations done, but is a direct measurement of the physical time to compute the Poincaré map. The observed run times were long, from approximately 2.5 minutes to 170 minutes, depending on parameters, with little variance in the run times between runs observed in initial testing. Thus it was considered to be adequate to time each run only once. The run times for each run are detailed in Table 2.

### Table 1. Choices of integration parameters for each run.

<table>
<thead>
<tr>
<th>Run</th>
<th>SYMP</th>
<th>RK78</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$h$</td>
<td>$\epsilon_{\text{cut}}$</td>
</tr>
<tr>
<td>1</td>
<td>0.020</td>
<td>$3 \cdot 10^{-14}$</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>$9 \cdot 10^{-14}$</td>
</tr>
<tr>
<td>3</td>
<td>0.100</td>
<td>$3 \cdot 10^{-13}$</td>
</tr>
<tr>
<td>4</td>
<td>0.125</td>
<td>$9 \cdot 10^{-13}$</td>
</tr>
<tr>
<td>5</td>
<td>0.160</td>
<td>$3 \cdot 10^{-12}$</td>
</tr>
<tr>
<td>6</td>
<td>0.200</td>
<td>$9 \cdot 10^{-12}$</td>
</tr>
<tr>
<td>7</td>
<td>0.300</td>
<td>$3 \cdot 10^{-11}$</td>
</tr>
<tr>
<td>8</td>
<td>0.500</td>
<td>$9 \cdot 10^{-11}$</td>
</tr>
<tr>
<td>9</td>
<td>0.700</td>
<td>$3 \cdot 10^{-10}$</td>
</tr>
<tr>
<td>10</td>
<td>0.900</td>
<td>$9 \cdot 10^{-10}$</td>
</tr>
<tr>
<td>11</td>
<td>1.00</td>
<td>$3 \cdot 10^{-9}$</td>
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<tr>
<td>12</td>
<td>1.25</td>
<td>$9 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>13</td>
<td>1.60</td>
<td>$1 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>Truth model</td>
<td>$1 \cdot 10^{-14}$</td>
<td>$2 \cdot 10^3$</td>
</tr>
</tbody>
</table>

After the propagation, only the trajectories that had exactly 4000 intersections with the Poincaré surface were kept. Of these trajectories, only those 20 that were
Table 2. Runtimes for each run.

<table>
<thead>
<tr>
<th>Run</th>
<th>SYMP</th>
<th>RK78</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10012.20</td>
<td>2959.97</td>
</tr>
<tr>
<td>2</td>
<td>4055.95</td>
<td>2759.35</td>
</tr>
<tr>
<td>3</td>
<td>1974.98</td>
<td>2221.51</td>
</tr>
<tr>
<td>4</td>
<td>1581.80</td>
<td>1986.26</td>
</tr>
<tr>
<td>5</td>
<td>1240.19</td>
<td>1765.62</td>
</tr>
<tr>
<td>6</td>
<td>981.58</td>
<td>1468.16</td>
</tr>
<tr>
<td>7</td>
<td>657.04</td>
<td>1294.66</td>
</tr>
<tr>
<td>8</td>
<td>401.21</td>
<td>1175.32</td>
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<td>9</td>
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<td>998.16</td>
</tr>
<tr>
<td>10</td>
<td>239.91</td>
<td>880.97</td>
</tr>
<tr>
<td>11</td>
<td>215.64</td>
<td>771.98</td>
</tr>
<tr>
<td>12</td>
<td>181.72</td>
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</tr>
<tr>
<td>13</td>
<td>149.39</td>
<td>668.17</td>
</tr>
<tr>
<td>Truth model</td>
<td>3770.56</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Number of trajectories $n$ with 4000 intersections and the number of trajectories excluded as not common to all runs.

<table>
<thead>
<tr>
<th>Run number</th>
<th>SYMP</th>
<th>RK78</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n$</td>
<td>Excluded</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>0</td>
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<tr>
<td>8</td>
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<tr>
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<tr>
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<td>20</td>
<td>0</td>
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<td>11</td>
<td>21</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>22</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>23</td>
<td>3</td>
</tr>
<tr>
<td>Truth model</td>
<td></td>
<td>23</td>
</tr>
</tbody>
</table>

common to all runs for both methods were kept. The numbers of trajectories kept in the first step and excluded in the second step for each set of parameters are listed in Table 3. The number of trajectories that make 4000 intersections is relatively stable over the range of parameters, with a maximum deviation of three from the median value of 20 trajectories. More importantly, there is no significant trend as a function of the strictness of the used tolerance parameters. Figure 10 shows the resulting Poincaré map produced by the truth model parameters, along with closeups of the structure.

For the sets of 20 trajectories we computed maximum and root mean square (RMS) values of the error in Jacobi constant, value of $x$-coordinate and the difference in $y$-coordinate compared to the truth model, point by point. The averaging
Fig. 10. Poincaré map produced with the truth model parameters (top) with closeups of the structure for \( y < 0 \) (bottom left) and \( y > 0 \) (bottom right).

was done sequentially over all trajectories of a single run at once. The results are shown in Figures 11, 12 and 13. The resulting maximum and RMS errors in Jacobi constant are graphed in Figure 11. We observe that the geometric method is faster for given level of accuracy as measured by the error in Jacobi constant. As the time step is lowered, the accuracies of both methods plateau at level \( \sim 10^{-14} \) in RMS error of Jacobi constant. The reason is accumulating round-off errors. Neither method is immune to these, as no form of compensated summation, such as Kahan summation (Kahan 1965), is used to minimize accumulating round-off errors in either method. The trend of the RK78 method is steeper, as is to be expected for a higher order method, but the symplectic method is still superior at all given levels of accuracy all the way to the plateau level, and the superiority is more pronounced at lower accuracies.

The errors in coordinate \( y \) and velocity \( v_y = \dot{y} \) behave similarly, as observed in Figure 12. From Figure 13 we see that the \( x \) coordinate errors stay well below the tolerance parameter \( \epsilon_{\text{cut}} \), with RMS errors in the lowest quintile. We conclude
that both methods converge adequately to the Poincaré section.

In Figure 14 we have plotted the RMS and maximum errors in the Jacobi constant for the SYMP method after a crossing of the Poincaré section has been detected, but before the secant method iteration has been used to find the cut point. To investigate the relative amount of error caused by the iteration we also plotted the difference in errors after the iteration procedure was completed. From the figure we see that the secant method iteration process causes only a negligible change in the Jacobi constant for the SYMP method across all considered time step values. For run 1 we see no change in the maximum error at all, and thus the corresponding data point is off the scale. Thus, the majority of the error is produced by the numerical propagator, and not by the iteration for the SYMP method.
Fig. 13. The root mean square and maximum errors of the $x$-coordinate in the rotating frame with respect to the Poincaré section $x = 0$.

Further proof of this can be seen in Figure 15 which shows a recalculation of the runs in Table 1, but with $\epsilon_{\text{cut}} = 10^{-8}$, overlaid on the results for $\epsilon_{\text{cut}} = 10^{-10}$. We find that the relaxation in cut tolerance leads to minimal change in the error in Jacobi constant and the run times.

As we are primarily interested in the shape of the invariant curves, the pointwise error in $y$ and $\dot{y}$ is not necessarily representative of the graphical accuracy of the method. In Figure 16 we have plotted extreme closeups of runs with a comparable accuracy in Jacobi constant. In the figure we can see that the calculations with both methods may result in points that clearly lie on the invariant curve of the truth model, or a curve close to it, but are shifted along the curve from the truth model points. This shift along the invariant curves compared to the Runge-Kutta truth model causes pointwise errors in $y$ and $\dot{y}$ to be overestimated.
We devised a metric which better approximates the distance from the invariant curve as given by the truth model. First, we note that since the values of $\dot{y}$ are in general much lower in magnitude than values of $y$, a standard Euclidean metric is not usable, as it would reduce to a metric on $y$ only. Instead we note that since for circular two-body orbits the orbital velocity $v$ scales as $v^2 \propto r^{-1}$, we get $r^{3/2} dv \propto dr$ and thus that small differences in velocity and position are comparable when the velocity difference is scaled by $r^{3/2} \propto P$, where $P$ is the Keplerian two-body period. We then let $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a non-Euclidean measure of distance

$$d((y_1, \dot{y}_1), (y_2, \dot{y}_2)) = \sqrt{(y_2 - y_1)^2 + y_1^2(\dot{y}_2 - \dot{y}_1)}. \quad (56)$$

Using this function we calculate the deviation from the invariant curve of the truth model for each point $\vec{x} = (y, \dot{y})$ of a trajectory in the following manner.

1. Find the point $\vec{x}_1$ of the corresponding trajectory of the truth model that minimizes $d(\vec{x}, \vec{x}_1)$.
2. Find the point $\vec{x}_2$ of the same truth model trajectory that minimizes $d(\vec{x}_2, \vec{x}_1)$.
3. Fit a line through $\vec{x}_1$ and $\vec{x}_2$ and project $\vec{x}$ on this line, yielding point $\vec{x}'$.
4. Calculate the difference $\vec{x} - \vec{x}'$ yielding approximate deviations in $y$ and $\dot{y}$ from the invariant curve.

Thus we are essentially doing a comparison against a piecewise linear approximation of the invariant curve obtained from the truth model points. We find an \textit{a posteriori} justification for the use of the function (56) in that by using it we correctly identify the truth model point that is in the same branch of the invariant curve in step 2 of the above process.

Using this process we calculated maximum and root mean square deviations for each run. The results are seen in Figure 17. From these results and those in Figure 11 we find that in general the SYMP method is markedly superior for
computations with tolerances no stricter than $10^{-14}$ in Jacobi constant and $10^{-9}$ in $y$ and $\dot{y}$.

To compare the kind of errors low tolerances can cause in the Poincaré maps, we plotted a sequence of runs for both methods starting from a low tolerance and converging to the truth model. The runs were chosen to have a comparable accuracy in $y$-coordinate when calculated using equation (56). Figure 18 shows the results for both methods.
Fig. 17. The root mean square and maximum deviations of the $y$-coordinate and its derivative $\dot{y}$ from the linearly approximated invariant curve.

We see that in general the SYMP method at large time steps distorts the shape of the invariant curves, but the curves stay intact. This is due to the lack of secular energy error growth and phase space invariance mentioned in section 6. As there is a bounded variation in the energy however, it is possible for the invariant curves to cross. This places a limit on how densely the initial values can be chosen and how large a time step can be used. In our results we do not observe this effect even at largest time steps. In addition, from Yoshida (1993) we know that we might expect a transition from a periodic or quasi-periodic to a chaotic orbit at large time steps. This type of error was not encountered in our results. This is not surprising, as a chaotic orbit is more likely to escape or collide, and we have culled all runs not making the 4000 cuts.

In contrast, the RK78 method causes an accumulating drift away from the invariant curves, causing the curves to disintegrate and smearing the structure of the Poincaré map. This drift is due to the accumulating local energy error, a byproduct of non-geometric integrators that is well known (Hairer et al. 2006; Sanz-Serna & Calvo 1994). The drift is apparent in Figure 16 also. In general, we observe that adequate convergence can be obtained in the accuracy regime of $\sim 10^{-11}$ or greater maximum error in Jacobi constant.

Finally we note that the central invariant curve of plots in Figure 18 is the smallest single feature measured by maximum extent produced by any trajectory, measuring $\Delta y \approx 1.2 \times 10^{-3}$ by $\Delta \dot{y} \approx 9.8 \times 10^{-5}$. Furthermore, from the same Figures we see that the qualitative features of this trajectory, as well as the others, are in the Figures indistinguishable from the truth model by run 7 of both methods. This is at the accuracy level of max $\Delta y \sim 10^{-7}$ and max $\Delta \dot{y} \sim 10^{-7}$, as seen in Figure 17. At this level of accuracy we find from Table 2 that the SYMP method was faster than the RK78 method by a factor of approximately two.

7. CONCLUSIONS

We investigated the characteristics of the Lagrange point $L_3$ in the short axis equilibrium configuration of RF3BP in comparison with the R3BP. We found
that the stability of $L_3$ proceeds from saddle-center to stable to complex unstable as the primary massive body is progressively elongated. These changes result in qualitatively different families of orbits, parametrised by the shape parameter $\alpha_x$, with the structural changes in phase space apparent in Poincaré maps of the system.
We also looked at simple methods for fast computation of Poincaré maps in RF3BP, and compared a traditional 7th and 8th order step size controlled Runge-Kutta scheme combined with Newton iteration with a simple 6th order time transformed geometric scheme combined with secant method iteration. We found the geometric scheme to be superior in the accuracy regime considered, being faster by a factor of two to four for a comparable level of accuracy and demonstrating better conservation of the invariant curves of the Poincaré maps. We also note that the geometric scheme was conceptually simpler to implement. Further, we found that the fast Poincaré maps produced could satisfactorily reproduce the qualitative features of a high precision truth model. We conclude that computing accurate realisations of Poincaré maps with lower error tolerances is feasible and that geometric methods are a computationally efficient tool to this end.

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