Two-channel sampling in wavelet subspaces

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Abstract

We develop two-channel sampling theory in the wavelet subspace $V_1$ from the multi resolution analysis $\{V_j\}_{j \in \mathbb{Z}}$. Extending earlier results by G. G. Walter [11], W. Chen and S. Itoh [2] and Y. M. Hong et al [5] on the sampling theory in the wavelet or shift invariant spaces, we find a necessary and sufficient condition for two-channel sampling expansion formula to hold in $V_1$.

1 Introduction

The classical Whittaker-Shannon-Kotel’nikov (WSK) sampling theorem [4] states that any signal $f(t)$ with finite energy and the bandwidth $\pi$ can be completely reconstructed from its discrete values by the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}.$$ 

WSK sampling theorem has been extended in many directions (see [1], [2], [5], [6], [7], [8], [10], [11], [12] and references therein). G. G. Walter [11] developed a sampling theorem in wavelet subspaces, noting that the sampling function $\text{sinc} := \sin \pi t/\pi t$ in the WSK sampling theorem is a scaling function of a multi-resolution analysis. A. J. E. M. Janssen [6] used the Zak transform to generalize Walter’s work to regular shifted sampling. Later, W. Chen and S. Itoh [2] (see also [12]) extended Walter’s result further by relaxing conditions...
on the scaling function \( \phi(t) \). Recently in [5],[8] general sampling expansion are handled on shift invariant spaces([9]). In this work, we find a necessary and sufficient condition for two-channel sampling expansion to hold in the wavelet subspace \( V_1 \) of a multi resolution analysis \( \{ V_j \}_{j \in \mathbb{Z}} \).

2 Preliminaries

For a measurable function \( f(t) \) on \( \mathbb{R} \), we let

\[
\| f(t) \|_0 := \sup_{|E| = 0} \inf_{\mathbb{R} \setminus E} \| f(t) \| \quad \text{and} \quad \| f(t) \|_\infty := \inf_{|E| = 0} \sup_{\mathbb{R} \setminus E} | f(t) |
\]

be the essential infimum and the essential supremum of \( | f(t) | \) on \( \mathbb{R} \) respectively, where \( |E| \) is the Lebesgue measure of \( E \). For any \( f(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \), we let

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \int_{-\infty}^{\infty} f(t) e^{-it\xi} dt
\]

be the Fourier transform of \( f(t) \) so that \( \frac{1}{\sqrt{2\pi}} \mathcal{F}(\cdot) \) becomes a unitary operator on \( L^2(\mathbb{R}) \). A sequence \( \{ \phi_n : n \in \mathbb{Z} \} \) in a Hilbert space \( H \) is called a Riesz sequence if \( \{ \phi_n : n \in \mathbb{Z} \} \) is a Riesz basis of the closed subspace \( V := \overline{\text{span}} \{ \phi_n : n \in \mathbb{Z} \} \) of \( H \).

**Definition 1.** A function \( \phi(t) \in L^2(\mathbb{R}) \) is called a scaling function of a multi-resolution analysis (MRA in short) \( \{ V_j \}_{j \in \mathbb{Z}} \) if the closed subspaces \( V_j \) of \( L^2(\mathbb{R}) \),

\[
V_j := \overline{\text{span}} \{ \phi(2^j t - k) : k \in \mathbb{Z} \}, \quad j \in \mathbb{Z}
\]

satisfy the following properties;

1. \( \cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots \);
2. \( \bigcup V_j = L^2(\mathbb{R}) \);
3. \( \bigcap V_j = \{ 0 \} \);
4. \( f(t) \in V_j \) if and only if \( f(2t) \in V_{j+1} \);
5. \( \{ \phi(t - n) : n \in \mathbb{Z} \} \) is a Riesz basis of \( V_0 \).

The wavelet subspace \( W_j \) is the orthogonal complement of \( V_j \) in \( V_{j+1} \) so that \( V_{j+1} = V_j \oplus W_j \). Then there is a wavelet \( \psi(t) \in L^2(\mathbb{R}) \) that induces a Riesz basis \( \{ \psi(2^j t - k) : k \in \mathbb{Z} \} \) of \( W_j \). Moreover, \( \{ \phi(2^j t - k), \psi(2^j t - k) : k \in \mathbb{Z} \} \) forms a Riesz basis of \( V_{j+1} \).
For any $\phi(t) \in L^2(\mathbb{R})$, $\{\phi(t - n) : n \in \mathbb{Z}\}$ is a Bessel sequence if there is a constant $B > 0$ such that
\[
\sum_{n \in \mathbb{Z}} |\langle f(t), \phi(t - n) \rangle|^2 \leq B||f||^2_{L^2(\mathbb{R})}, \quad f \in L^2(\mathbb{R})
\]
or equivalently (see Theorem 7.2.3 in [3]) $G_\phi(\xi) := \sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)|^2 \leq B$ a.e. on $[0, 2\pi]$.

**Lemma 1.** (Lemma 2.2 in [5] and Lemma 7.2.1 in [3]) Let $\{c_k\}_{k \in \mathbb{Z}}$, $e_{\{2\}}$ on $[0, 2\pi]$. Then, for any $\{c_k\}_{k \in \mathbb{Z}} \in l^2$, $\sum_{k \in \mathbb{Z}} c_k \phi(t - k)$ converges in $L^2(\mathbb{R})$ and
\[
\mathcal{F}\left(\sum_{k \in \mathbb{Z}} c_k \phi(t - k)\right) = \left(\sum_{k \in \mathbb{Z}} c_k e^{-ik\xi}\right)\hat{\phi}(\xi).
\]

For any $c = \{c_k\}_{k \in \mathbb{Z}} \in l^2$, let $\hat{c}(\xi) := \sum_{k \in \mathbb{Z}} c_k e^{-ik\xi}$. Then, $\hat{c}(\xi) \in L^2[0, 2\pi]$ if $\{c_k\}_{k \in \mathbb{Z}} \in l^2$ or $C[0, 2\pi]$ if $\{c_k\}_{k \in \mathbb{Z}} \in l^2$ or $l^1$ respectively.

**Lemma 2.** If $a = \{a_k\}_{k \in \mathbb{Z}}$, $b = \{b_k\}_{k \in \mathbb{Z}} \in l^2$ and $\hat{a}(\xi) \in L^\infty[0, 2\pi]$, then $a * b := \{\sum_{j \in \mathbb{Z}} a_j b_{k-j}\}_{k \in \mathbb{Z}} \in l^2$ and $\hat{a * b}(\xi) = \hat{a}(\xi) \hat{b}(\xi)$.

**Proof.** Since $\hat{a}(\xi) \in L^\infty[0, 2\pi]$ and $\hat{b}(\xi) \in L^2[0, 2\pi]$, $\hat{a}(\xi) \hat{b}(\xi) \in L^2[0, 2\pi]$, $\hat{a}(\xi) \hat{b}(\xi) \in L^2[0, 2\pi]$.

Hence we can expand $\hat{a}(\xi) \hat{b}(\xi)$ into its Fourier series $\sum_n c_n e^{-in\xi}$ in $L^2[0, 2\pi]$, where
\[
c_n = \frac{1}{2\pi} \left\langle \hat{a}(\xi) \hat{b}(\xi), e^{-in\xi}\right\rangle_{L^2[0, 2\pi]} = \frac{1}{2\pi} \left\langle \sum_{k \in \mathbb{Z}} a_k e^{-ik\xi}, \left(\sum_{k \in \mathbb{Z}} b_k e^{ik\xi}\right)e^{-in\xi}\right\rangle_{L^2[0, 2\pi]}
\]
\[
= \frac{1}{2\pi} \left(\sum_{k \in \mathbb{Z}} a_k e^{-ik\xi}, \sum_{k \in \mathbb{Z}} b_{n-k} e^{-i(k-n)\xi}\right)_{L^2[0, 2\pi]} = \sum_{k \in \mathbb{Z}} a_k b_{n-k}
\]
by Parseval’s identity. Hence the conclusion follows. \qed

For any $\phi(t) \in L^2(\mathbb{R})$, let
\[
H_\phi(\xi) := \sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2n\pi)| \quad \text{and} \quad C_\phi(t) := \sum_{n \in \mathbb{Z}} |\phi(t + n)|^2.
\]
3 Main Result and an example

In the following, let \( \phi(t) \in L^2(\mathbb{R}) \) be a scaling function of an MRA \( \{V_j\}_{j \in \mathbb{Z}} \) and \( \psi(t) \) the associated wavelet, of which we always assume that \( H_\phi(\xi) \) and \( H_\psi(\xi) \) are in \( L^\infty[0,2\pi] \). Then (cf. Proposition 2.4 in [8]) \( \phi(t) \) and \( \psi(t) \) are in \( L^2(\mathbb{R}) \cap C(\mathbb{R}) \) and \( \sup_{R \in \mathbb{R}} C_\phi(t) < \infty \), \( \sup_{R \in \mathbb{R}} C_\psi(t) < \infty \). Hence for any \( c = \{c_n\}_{n \in \mathbb{Z}} \in l^2 \),

\[
(e * \phi)(t) := \sum_{n \in \mathbb{Z}} c_n \phi(t - n)
\]

converges both in \( L^2(\mathbb{R}) \) and uniformly in \( \mathbb{R} \) so that each \( V_j \subset L^2(\mathbb{R}) \cap C(\mathbb{R}) \), \( j \in \mathbb{Z} \).

Let \( L_j[\cdot] \) be the LTI (linear time invariant) systems with frequency responses \( M_j(\xi) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) for \( j = 1, 2 \). Then

\[
L_j[f](t) := F^{-1}(\hat{f}M_j)(t) \in L^2(\mathbb{R}) \cap C(\mathbb{R}), f \in L^2(\mathbb{R}) \text{ and } j = 1, 2
\]

and \( \lim_{|\xi| \to \infty} L_j[f](t) = 0 \) by the Riemann-Lebesgue Lemma since \( \hat{f}(\xi)M_j(\xi) \in L^1(\mathbb{R}) \).

Moreover by the Poisson summation formula (cf. Lemma 5.1 in [8]), for any fixed \( t \in \mathbb{R} \) and \( j = 1, 2 \),

\[
\sum_{n \in \mathbb{Z}} L_j[\phi](t + n)e^{-in\xi} = \sum_{n \in \mathbb{Z}} \hat{\phi}(\xi + 2n\pi)M_j(\xi + 2n\pi)e^{it(\xi + 2n\pi)}
\]

and

\[
\sum_{n \in \mathbb{Z}} L_j[\psi](t + n)e^{-in\xi} = \sum_{n \in \mathbb{Z}} \hat{\psi}(\xi + 2n\pi)M_j(\xi + 2n\pi)e^{it(\xi + 2n\pi)}
\]

are in \( L^\infty[0,2\pi] \) as functions in \( \xi \) since \( H_\phi(\xi), H_\psi(\xi) \in L^\infty[0,2\pi] \).

In particular, \( A_{i,j}(\xi) = A_{i,j}(\xi + 2\pi) \in L^\infty[0,2\pi] \) for \( i, j = 1, 2 \), where

\[
A_{1,j}(\xi) := \sum_{n \in \mathbb{Z}} L_j[\phi](n)e^{-in\xi}, \quad A_{2,j}(\xi) := \sum_{n \in \mathbb{Z}} L_j[\psi](n)e^{-in\xi}.
\]

Let

\[
A(\xi) = \|A_{i,j}(\xi)\|^2_{l^2, j=1}.
\]

Then for any \( f(t) = \sum_{k \in \mathbb{Z}} c_{1,k}\phi(t - k) + \sum_{k \in \mathbb{Z}} c_{2,k}\psi(t - k) \in V_1 \), where \( \{c_{1,k}\}_{k \in \mathbb{Z}} \) and \( \{c_{2,k}\}_{k \in \mathbb{Z}} \in l^2 \), we have for \( i = 1, 2 \) and \( n \in \mathbb{Z} \)

\[
L_i(f)(t) := \sum_{k \in \mathbb{Z}} c_{1,k}L_i(\phi)(t - k) + \sum_{k \in \mathbb{Z}} c_{2,k}L_i(\psi)(t - k),
\]

which converges both in \( L^2(\mathbb{R}) \) and absolutely on \( \mathbb{R} \). In particular

\[
L_i(f)(n) = \sum_{k \in \mathbb{Z}} c_{1,k}L_i(\phi)(n-k) + \sum_{k \in \mathbb{Z}} c_{2,k}L_i(\psi)(n-k), \quad n \in \mathbb{Z} \text{ and } i = 1, 2.
\]
Lemma 3. Assume that \( \det A(\xi) \neq 0 \) a.e. in \([0, 2\pi]\). Let \( \lambda_{1,B}(\xi) \leq \lambda_{2,B}(\xi) \) be eigenvalues of the Hermitian matrix \( B(\xi) = (A(\xi)^*A(\xi))^{-1} \). If \( \| \det A(\xi) \|_0 > 0 \), then

\[
0 < \| \lambda_{1,B}(\xi) \|_0 \leq \| \lambda_{2,B}(\xi) \|_\infty < \infty.
\]

Proof. Since \( B(\xi) \) is a nonsingular positive semi-definite Hermitian matrix a.e. in \([0, 2\pi]\),

\[
0 < \lambda_{1,B}(\xi) \leq \lambda_{2,B}(\xi) \quad \text{a.e. in } [0, 2\pi].
\]

Since \( A_{i,j}(\xi) \in L^\infty[0, 2\pi] \) and \( \| \det A(\xi) \|_0 > 0 \), all entries of \( B(\xi) \) are also in \( L^\infty[0, 2\pi] \) so that the characteristic equation of \( B(\xi) \) is of the form

\[
\lambda(\xi)^2 + f(\xi)\lambda(\xi) + g(\xi) = 0
\]

where \( f(\xi) \) and \( g(\xi) \) are real-valued functions in \( L^\infty[0, 2\pi] \). Hence

\[
0 < \| \lambda_{2,B}(\xi) \|_\infty < \infty.
\]

Since \( \lambda_{1,B}(\xi)\lambda_{2,B}(\xi) = \det B(\xi) = |\det A(\xi)|^{-2} \),

\[
\| \det A(\xi) \|_\infty^2 \leq \lambda_{1,B}(\xi)\lambda_{2,B}(\xi) \leq \| \det A(\xi) \|_\infty^{-2} \quad \text{a.e. in } [0, 2\pi]
\]

so that \( 0 < \| \det A(\xi) \|_\infty^2 \| \lambda_{2,B}(\xi) \|_\infty^{-1} \leq \lambda_{1,B}(\xi) \) a.e. in \([0, 2\pi]\). □

Definition 2. For any \( f_1(t) \) and \( f_2(t) \) in \( L^2(\mathbb{R}) \), let \( F(\xi) := [F_{i,j}(\xi)]_{i,j=1}^2 \) be the Gramian of \( \{f_1, f_2\} \), where \( F_{i,j}(\xi) := \sum_{n \in \mathbb{Z}} \hat{f}_i(n + 2k\pi)\hat{f}_j(n + 2k\pi) \).

Then as a Hermitian matrix, \( F(\xi) \) has real eigenvalues.

Proposition 1. ([9]) Let \( \lambda_{1,F}(\xi) \leq \lambda_{2,F}(\xi) \) be eigenvalues of the Gramian \( F(\xi) \) of \( \{f_1, f_2\} \). Then \( \{f_1(t-n), f_2(t-n) : n \in \mathbb{Z}\} \) is a Riesz sequence if and only if

\[
0 < \| \lambda_{1,F}(\xi) \|_0 \leq \| \lambda_{2,F}(\xi) \|_\infty < \infty.
\]

Lemma 4. Assume \( \| \det A(\xi) \|_0 > 0 \). Let

\[
\begin{bmatrix}
\hat{S}_1(\xi) \\
\hat{S}_2(\xi)
\end{bmatrix} := A(\xi)^{-1} \begin{bmatrix}
\hat{\phi}(\xi) \\
\hat{\psi}(\xi)
\end{bmatrix}
\]

and \( S_i(t) := \mathcal{F}^{-1}(\hat{S}_i(t)) \) for \( i = 1, 2 \). Then \( S_i(t) \in V_1 \) for \( i = 1, 2 \) and \( \{S_i(t-n) : n = 1, 2 \text{ and } t \in \mathbb{R}\} \) is a Riesz sequence.

Proof. Let \( A(\xi)^{-1} = C(\xi) = [C_{i,j}(\xi)]_{i,j=1}^2 \). Since \( C_{i,j}(\xi) \in L^\infty(\mathbb{R}) \), \( \hat{S}_i(\xi) = C_{i,1}(\xi)\hat{\phi}(\xi) + C_{i,2}(\xi)\hat{\psi}(\xi) \in L^2(\mathbb{R}) \) for \( i = 1, 2 \). Since \( C_{i,j}(\xi) = C_{i,j}(\xi + 2\pi) \in L^\infty[0, 2\pi] \), we may expand \( C_{i,j}(\xi) \) into its Fourier series \( C_{i,j}(\xi) = \sum_{k \in \mathbb{Z}} c_{i,j,k}e^{-ik\xi} \) where \( \{c_{i,j,k}\}_{k \in \mathbb{Z}} \in l^2 \). Then by Lemma 1,

\[
\hat{S}_i(\xi) = \sum_{k \in \mathbb{Z}} \left( c_{i,1,k}e^{-ik\xi}\hat{\phi}(\xi) + c_{i,2,k}e^{-ik\xi}\hat{\psi}(\xi) \right)
\]
so that
\[ S_i(t) := \mathcal{T}^{-1}(\hat{S}_i)(t) = \sum_{k \in \mathbb{Z}} \left( c_{i,1,k}\phi(t-k) + c_{i,2,k}\psi(t-k) \right) \in V_1. \]

Let \( G(\xi) \) and \( S(\xi) \) be the Gramians of \( \{\phi, \psi\} \) and \( \{S_1, S_2\} \) respectively and \( \lambda_{1,G}(\xi) \leq \lambda_{2,G}(\xi) \) and \( \lambda_{1,S}(\xi) \leq \lambda_{2,S}(\xi) \) the eigenvalues of \( G(\xi) \) and \( S(\xi) \) respectively. Then \( S(\xi) = C(\xi)G(\xi)C(\xi)^* \). Let \( U_S(\xi) \) and \( U_G(\xi) \) be unitary matrices, which diagonalize \( S(\xi) \) and \( G(\xi) \) respectively, i.e.,
\[
S(\xi) = U_S(\xi) \begin{bmatrix} \lambda_{1,S}(\xi) & 0 \\ 0 & \lambda_{2,S}(\xi) \end{bmatrix} U_S(\xi)^*
\]
and
\[
G(\xi) = U_G(\xi) \begin{bmatrix} \lambda_{1,G}(\xi) & 0 \\ 0 & \lambda_{2,G}(\xi) \end{bmatrix} U_G(\xi)^*.
\]
Then
\[
\begin{bmatrix} \lambda_{1,S}(\xi) & 0 \\ 0 & \lambda_{2,S}(\xi) \end{bmatrix} = R(\xi) \begin{bmatrix} \lambda_{1,G}(\xi) & 0 \\ 0 & \lambda_{2,G}(\xi) \end{bmatrix} R(\xi)^*
\]
where
\[
R(\xi) = U_S(\xi)^*C(\xi)U_G(\xi) := \begin{bmatrix} R_{1,1}(\xi) & R_{1,2}(\xi) \\ R_{2,1}(\xi) & R_{2,2}(\xi) \end{bmatrix}
\]
so that
\[
\lambda_{1,S}(\xi) = \lambda_{1,G}(\xi)|R_{1,1}(\xi)|^2 + \lambda_{2,G}(\xi)|R_{1,2}(\xi)|^2; \quad (2)
\]
\[
\lambda_{2,S}(\xi) = \lambda_{1,G}(\xi)|R_{2,1}(\xi)|^2 + \lambda_{2,G}(\xi)|R_{2,2}(\xi)|^2. \quad (3)
\]
On the other hand,
\[
R(\xi)R(\xi)^* = U_S(\xi)^*C(\xi)C(\xi)^*U_S(\xi) = U_S(\xi)^*B(\xi)U_S(\xi) \quad (4)
\]
\[
= U_S(\xi)^*U_B(\xi) \begin{bmatrix} \lambda_{1,B}(\xi) & 0 \\ 0 & \lambda_{2,B}(\xi) \end{bmatrix} U_B(\xi)^*U_S(\xi),
\]
where \( U_B(\xi) \) is the unitary matrix such that
\[
B(\xi) = U_B(\xi) \begin{bmatrix} \lambda_{1,B}(\xi) & 0 \\ 0 & \lambda_{2,B}(\xi) \end{bmatrix} U_B(\xi)^*
\]
with \( \lambda_{1,B}(\xi) \leq \lambda_{2,B}(\xi) \). Set \( U_S(\xi)^*U_B(\xi) = [D_{i,j}(\xi)]_{i,j=1}^2 \), which is also a unitary matrix. Then we have from diagonal entries of both sides of (4),
\[
|R_{1,1}(\xi)|^2 + |R_{1,2}(\xi)|^2 = \lambda_{1,B}(\xi)|D_{1,1}(\xi)|^2 + \lambda_{2,B}(\xi)|D_{1,2}(\xi)|^2; \quad (5)
\]
\[
|R_{2,1}(\xi)|^2 + |R_{2,2}(\xi)|^2 = \lambda_{1,B}(\xi)|D_{2,1}(\xi)|^2 + \lambda_{2,B}(\xi)|D_{2,2}(\xi)|^2. \quad (6)
\]
Then we have a.e. in [0, 2\pi] (from (2), (3), (5), and (6))
\[
\lambda_{1,S}(\xi) \geq \lambda_{1,G}(\xi) \left( |R_{1,1}(\xi)|^2 + |R_{1,2}(\xi)|^2 \right) \geq \lambda_{1,G}(\xi) \lambda_{1,B}(\xi);
\]
\[
\lambda_{2,S}(\xi) \leq \lambda_{2,G}(\xi) \left( |R_{2,1}(\xi)|^2 + |R_{2,2}(\xi)|^2 \right) \leq \lambda_{2,G}(\xi) \lambda_{2,B}(\xi)
\]
since \(|D_{1,1}(\xi)|^2 + |D_{1,2}(\xi)|^2 = |D_{2,1}(\xi)|^2 + |D_{2,2}(\xi)|^2 = 1\) a.e. in [0, 2\pi]. Hence
\[
0 < \|\lambda_{1,G}(\xi)\|_0 \|\lambda_{1,B}(\xi)\|_0 \leq \|\lambda_{1,S}(\xi)\|_0 \leq \|\lambda_{2,S}(\xi)\|_0 \leq \|\lambda_{2,G}(\xi)\|_0 \|\lambda_{2,B}(\xi)\|_0 \leq \|\lambda_{2,S}(\xi)\|_0 < \infty
\]
by Lemma 3 and Proposition 4 so that \(\{S_i(t - n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}\) is a Riesz sequence by Proposition 4.

Now we are ready to give the main result of this work.

**Theorem 1.** There exist \(S_i(t) \in V_1\) \((i = 1, 2)\) such that \(\{S_i(t - n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}\) is a Riesz basis of \(V_1\) for which two-channel sampling formula
\[
f(t) = \sum_{n \in \mathbb{Z}} \mathcal{L}_1(f)(n) S_1(t - n) + \sum_{n \in \mathbb{Z}} \mathcal{L}_2(f)(n) S_2(t - n) \tag{7}
\]
holds for \(f \in V_1\) if and only if \(\|\det A(\xi)\|_0 > 0\). In this case
\[
S_i(t) = \mathcal{F}^{-1} \left( C_{i,1}(\xi) \hat{\phi}(\xi) + C_{i,2}(\xi) \hat{\psi}(\xi) \right)(t) \text{ for } i = 1, 2. \tag{8}
\]

**Proof.** Assume \(\|\det A(\xi)\|_0 > 0\) and define \(S_i(t)\) by (8). Then \(S_i(t) \in V_1\) \((i = 1, 2)\) and \(\{S_i(t - n) : i = 1, 2 \text{ and } n \in \mathbb{Z}\}\) is a Riesz sequence by Lemma 5.

For any \(f(t) \in V_1\)
\[
f(t) = \sum_{k \in \mathbb{Z}} c_{1,k} \phi(t - k) + \sum_{k \in \mathbb{Z}} c_{2,k} \psi(t - k)
\]
where \(\{c_{i,k}\}_{k \in \mathbb{Z}} \in l^2\) for \(i = 1, 2\), we have by Lemma 1,
\[
\hat{f}(\xi) = \left( \sum_{k \in \mathbb{Z}} c_{1,k} e^{-ik\xi} \right) \hat{\phi}(\xi) + \left( \sum_{k \in \mathbb{Z}} c_{2,k} e^{-ik\xi} \right) \hat{\psi}(\xi). \tag{9}
\]
Since
\[
\begin{bmatrix}
\hat{\phi}(\xi) \\
\hat{\psi}(\xi)
\end{bmatrix} = A(\xi)
\begin{bmatrix}
\hat{S}_1(\xi) \\
\hat{S}_2(\xi)
\end{bmatrix}
\]
we have by (1), (9) and Lemma 1
\[
\hat{f}(\xi) = \sum_{j=1}^{2} \left[ \left( \sum_{k \in \mathbb{Z}} c_{1,k} e^{-ik\xi} \right) A_{1,j}(\xi) + \left( \sum_{k \in \mathbb{Z}} c_{2,k} e^{-ik\xi} \right) A_{2,j}(\xi) \right] \hat{S}_j(\xi) \tag{10}
\]
\[
= \sum_{n \in \mathbb{Z}} \mathcal{L}_1[f](n) e^{-in\xi} \hat{S}_1(\xi) + \sum_{n \in \mathbb{Z}} \mathcal{L}_2[f](n) e^{-in\xi} \hat{S}_2(\xi).
\]
Taking the inverse Fourier transform on (10) gives (7), which implies \( V_1 = \text{span} \{ S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z} \} \) so that \( \{ S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z} \} \) is a Riesz basis of \( V_1 \). Conversely assume that there exist \( S_i(t) \in V_1 \ (i = 1, 2) \) such that \( \{ S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z} \} \) is a Riesz basis of \( V_1 \) and (7) holds. In particular,

\[
\phi(t) = \sum_{n \in \mathbb{Z}} \mathcal{L}_1[\phi](n) S_1(t - n) + \sum_{n \in \mathbb{Z}} \mathcal{L}_2[\phi](n) S_2(t - n);
\]

\[
\psi(t) = \sum_{n \in \mathbb{Z}} \mathcal{L}_1[\psi](n) S_1(t - n) + \sum_{n \in \mathbb{Z}} \mathcal{L}_2[\psi](n) S_2(t - n).
\]

By taking Fourier transform and using Lemma 1, we have

\[
\begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix} = A(\xi) \begin{bmatrix} \hat{S}_1(\xi) \\ \hat{S}_2(\xi) \end{bmatrix}, \tag{11}
\]

We then have as in the proof of Lemma 5, \( G(\xi) = A(\xi) S(\xi) A(\xi)^* \) where \( G(\xi) \) and \( S(\xi) \) are Gramians of \( \{ \phi, \psi \} \) and \( \{ S_1, S_2 \} \) respectively. Hence \( \det G(\xi) = \det S(\xi) |\det A(\xi)|^2 \) so that

\[
|\det A(\xi)|^2 = \frac{\det G(\xi)}{\det S(\xi)} = \frac{\lambda_{1,G}(\xi) \lambda_{2,G}(\xi)}{\lambda_{1,S}(\xi) \lambda_{2,S}(\xi)} \geq \frac{\lambda_{1,G}(\xi)^2}{\lambda_{2,S}(\xi)^2} \quad \text{a.e. in } [0, 2\pi],
\]

where \( \lambda_{1,G}(\xi) \leq \lambda_{2,G}(\xi) \) and \( \lambda_{1,S}(\xi) \leq \lambda_{2,S}(\xi) \) are eigenvalues of \( G(\xi) \) and \( S(\xi) \) respectively. Therefore,

\[
|\det A(\xi)| \geq \frac{\lambda_{1,G}(\xi)}{\lambda_{2,S}(\xi)} \geq \frac{\|\lambda_{1,G}(\xi)\|_0}{\|\lambda_{2,S}(\xi)\|_{\infty}} \quad \text{a.e. in } [0, 2\pi]
\]

so that \( \|\det A(\xi)\|_0 > 0 \) since both \( \{ \phi(t-n), \psi(t-n) : n \in \mathbb{Z} \} \) and \( \{ S_i(t-n) : i = 1, 2 \text{ and } n \in \mathbb{Z} \} \) are Riesz sequences. Finally (8) comes from (11) immediately.

Note that if \( \{ \mathcal{L}_i(\phi)(n) \}_{n \in \mathbb{Z}} \) and \( \{ \mathcal{L}_j(\psi)(n) \}_{n \in \mathbb{Z}} \in l^1 \), then \( A_{i,j}(\xi) \in C[0, 2\pi] \) for \( i, j = 1, 2 \) so that \( A_{i,j}(\xi) \in L^\infty[0, 2\pi] \) and \( \|\det A(\xi)\|_0 > 0 \) is equivalent to \( \det A(\xi) \neq 0 \) on \([0, 2\pi]\).

**Example.** (2-channel sampling in Paley-Wiener space)

Let \( \phi(t) = \text{sinc} \) so that \( V_0 = \text{span} \{ \phi(t-n) : n \in \mathbb{Z} \} = PW_\pi \) and \( V_1 = \text{span} \{ \phi(2t-n) : n \in \mathbb{Z} \} = PW_{2\pi} \). Then \( V_1 = V_0 \oplus W_0 \) where \( W_0 = \text{span} \{ \psi(t-n) : n \in \mathbb{Z} \} \) and \( \psi(t) = (\cos \frac{3}{2} \pi t)(\text{sinc}\frac{1}{2} t) \).

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Note that $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \chi_{[-\pi,\pi]}(\xi)$ and $\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}} \chi_{[-2\pi,-\pi] \cup [\pi,2\pi]}(\xi)$, where $\chi_E(\cdot)$ is the characteristic function of a set $E$ in $\mathbb{R}$. We have by (12)

$$\sum_{n \in \mathbb{Z}} \mathcal{L}_i[\phi](n)e^{-in\xi} = \begin{cases} M_i(\xi), & \xi \in [0, \pi) \\ M_i(\xi - 2\pi), & \xi \in [\pi, 2\pi] \end{cases}$$

and

$$\sum_{n \in \mathbb{Z}} \mathcal{L}_i[\psi](n)e^{-in\xi} = \begin{cases} M_i(\xi - 2\pi), & \xi \in [0, \pi) \\ M_i(\xi), & \xi \in [\pi, 2\pi] \end{cases}.$$

Hence

$$A(\xi) = \begin{cases} \left[ \begin{array}{cc} M_1(\xi) & M_2(\xi) \\ M_1(\xi - 2\pi) & M_2(\xi - 2\pi) \end{array} \right] & \text{on } [0, \pi) \\ \left[ \begin{array}{cc} M_1(\xi - 2\pi) & M_2(\xi - 2\pi) \\ M_1(\xi) & M_2(\xi) \end{array} \right] & \text{on } [\pi, 2\pi] \end{cases}$$

so that the determinant condition $\|\det A(\xi)\|_0 > 0$ is equivalent to $\|\det M(\xi)\|_0 > 0$ where

$$M(\xi) = \left[ \begin{array}{cc} M_1(\xi) & M_1(\xi - 2\pi) \\ M_2(\xi) & M_2(\xi - 2\pi) \end{array} \right].$$

Take $M_1(\xi) = 1$ and $M_2(\xi) = -\text{sgn} \xi$ so that $\mathcal{L}_1[f](t) = f(t)$ and $\mathcal{L}_2[f](t) = \tilde{f}(t)$ where $\tilde{f}(t)$ is the Hilbert transform of $f(t)$. Then

$$M(\xi) = \left[ \begin{array}{cc} 1 & 1 \\ -i & i \end{array} \right]$$

so that $\|\det M(\xi)\|_0 = 2$. As a consequence, the sampling formula holds on $V_1 = PW_{2\pi}$. In fact, we have from (11),

$$\left[ \begin{array}{c} \hat{S}_1(\xi) \\ \hat{S}_2(\xi) \end{array} \right] = \begin{cases} \frac{1}{2\sqrt{2\pi}} \left[ \begin{array}{c} 1 \\ -i \end{array} \right] & \text{on } [-2\pi, 0) \\ \frac{1}{2\sqrt{2\pi}} \left[ \begin{array}{c} 1 \\ i \end{array} \right] & \text{on } [0, 2\pi] \\ \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] & \text{otherwise} \end{cases}$$

so that

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) \text{sinc} 2(t - n) - \sum_{n \in \mathbb{Z}} \tilde{f}(n) \sin \pi(t - n) \text{sinc}(t - n), \ f \in PW_{2\pi}.$$
As another example, take $M_1(\xi) = 1$ and $M_2(\xi) = i\xi$ so that $L_1[f](t) = f(t)$ and $L_2[f](t) = f'(t)$. Then

$$M(\xi) = \begin{bmatrix} 1 & 1 \\ i\xi & i(\xi - 2\pi) \end{bmatrix}$$

so that $\|\det M(\xi)\|_0 = 2\pi$. By the similar procedure as above, we obtain a sampling formula

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) \text{sinc}^2 2(t - n) + \frac{1}{\pi} \sum_{n \in \mathbb{Z}} f'(n) \sin \pi(t - n) \text{sinc}(t - n), \quad f \in PW_{2\pi}.$$

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