(α, β, λ, δ, m, Ω)p—Neighborhood for some families of analytic and multivalent functions

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Abstract

In the present investigation, we give some interesting results related with neighborhoods of p-valent functions. Relevant connections with some other recent works are also pointed out.

1 Introduction and Definitions

Let A demonstrate the family of functions f(z) of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \}. \)

We denote by \( A_p(n) \) the class of functions f(z) normalized by

\[ f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (n, p \in \mathbb{N} := \{1, 2, 3, \ldots\}) \quad (1) \]

which are analytic and p-valent in \( U. \)
Upon differentiating both sides of (1) \( m \) times with respect to \( z \), we have
\[
 f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} \frac{(k+p)!}{(k+p-m)!} a_{k+p} z^{k+p-m} 
\]
(2)
\( (n, p \in \mathbb{N}; m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; p > m) \).

We show by \( A_p(n, m) \) the class of functions of the form (2) which are analytic and \( p \)-valent in \( \mathbb{U} \).

The concept of neighborhood for \( f(z) \in A \) was first given by Goodman [7]. The concept of \( \delta \)-neighborhoods \( N_\delta(f) \) of analytic functions \( f(z) \in A \) was first studied by Ruscheweyh [8]. Walker [12], defined a neighborhood of analytic functions having positive real part. Later, Owa et al. [13] generalized the results given by Walker. In 1996, Altıntaş and Owa [14] gave \( (n, \delta) \)-neighborhoods for functions \( f(z) \in A \) with negative coefficients. In 2007, \( (n, \delta) \)-neighborhoods for \( p \)-valent functions with negative coefficients were considered by Srivastava et al. [4], and Orhan [5]. Very recently, Orhan et al. [1], introduced a new definition of \( (n, \delta) \)-neighborhood for analytic functions \( f(z) \in A \). Orhan et al.’s [1] results were generalized for the functions \( f(z) \in A \) and \( f(z) \in A_p(n) \) by many author (see, [6, 9, 10, 15]).

In this paper, we introduce the neighborhoods \( (\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g) \) and \( (\alpha, \beta, \lambda, m, \delta, \Omega)_p - M(g) \) of a function \( f^{(m)}(z) \) when \( f(z) \in A_p(n) \).

Using the Salagean derivative operator [3], we can write the following equalities for the function \( f^{(m)}(z) \) given by
\[
 D^0 f^{(m)}(z) = f^{(m)}(z) 
\]
\[
 D^1 f^{(m)}(z) = \frac{z}{(p-m)} \left( f^{(m)}(z) \right)' 
\]
\[
 = \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} \frac{(k+p-m)(k+p)!}{(p-m)(k+p-m)!} a_{k+p} z^{k+p-m} 
\]
\[
 D^2 f^{(m)}(z) = D(Df^{(m)}(z)) 
\]
\[
 = \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} \frac{(k+p-m)^2(k+p)!}{(p-m)^2(k+p-m)!} a_{k+p} z^{k+p-m} 
\]
We define $F : \mathcal{A}_p(n, m) \to \mathcal{A}_p(n, m)$ such that

$$F(f(m)(z)) = (1 - \lambda) \left( D^\Omega f(m)(z) \right) + \frac{\lambda z}{(p - m)} \left( D^\Omega f(m)(z) \right)'$$

$$= \frac{p!}{(p - m)!} z^{p - m} + \sum_{k=n}^{\infty} \frac{(k + p - m)\Omega(k + p)!}{(p - m)!\Omega(k + p - m)!} a_{k+p} z^{k+p-m}$$

We show this neighborhood by $(\alpha, \beta, \lambda, m, \delta, \Omega)_p - \mathcal{N}(g)$.

Also, we say that $f \in (\alpha, \beta, \lambda, m, \delta, \Omega)_p - M(g)$ if it satisfies

$$\left| \frac{e^{i\alpha} F(f(m)(z))}{z^{p - m - 1}} - \frac{e^{i\beta} F(g(m)(z))}{z^{p - m - 1}} \right| < \delta \quad (z \in \mathbb{U})$$

for some $-\pi \leq \alpha - \beta \leq \pi$ and $\delta > \frac{p!}{(p - m - 1)!} \sqrt{2(1 - \cos(\alpha - \beta))}$. We give some results for functions belonging to $(\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g)$ and $(\alpha, \beta, \lambda, m, \delta, \Omega)_p - M(g)$. 

\[(\alpha, \beta, \lambda, m, \delta, \Omega)_p - \text{NEIGHBORHOOD FOR SOME FAMILIES OF ANALYTIC AND MULTIVALENT FUNCTIONS} \]
2 Main Results

Now we can establish our main results.

**Theorem 2.1.** If \( f \in F(\lambda, m, \Omega) \) satisfies

\[
\sum_{k=n}^{\infty} \frac{(k + p - m)^{\Omega+1}(k + p)!}{(p - m)^{\Omega}(k + p)!} \left( 1 + \lambda k(p - m)^{-1} \right) \left| e^{i\alpha} a_{k+p} - e^{i\beta} b_{k+p} \right|
\]

\[
\leq \delta - \frac{p!}{(p - m - 1)!} \sqrt{2[1 - \cos(\alpha - \beta)]}
\]

for some \(-\pi \leq \alpha - \beta \leq \pi\); \( p > m \) and \( \delta > \frac{p!}{(p - m - 1)!} \sqrt{2[1 - \cos(\alpha - \beta)]} \), then \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g) \).

**Proof.** By virtue of (3), we can write

\[
\left| \frac{e^{i\alpha} F'(f(m)(z))}{z^{p-m-1}} - \frac{e^{i\alpha} F'(g(m)(z))}{z^{p-m-1}} \right|
\]

\[
= \left| \frac{p!(p - m)}{(p - m)!} e^{i\alpha} + \frac{e^{i\alpha}}{(p - m)!} \sum_{k=n}^{\infty} \frac{(k + p - m)^{\Omega+1}(k + p)!}{(p - m)^{\Omega}(k + p)!} \left( 1 + \lambda k(p - m)^{-1} \right) a_{k+p} z^k \right|
\]

\[
- \left| \frac{p!(p - m)}{(p - m)!} e^{i\beta} - \frac{e^{i\beta}}{(p - m)!} \sum_{k=n}^{\infty} \frac{(k + p - m)^{\Omega+1}(k + p)!}{(p - m)^{\Omega}(k + p)!} \left( 1 + \lambda k(p - m)^{-1} \right) b_{k+p} z^k \right|
\]

\[
< \frac{p!}{(p - m - 1)!} \sqrt{2[1 - \cos(\alpha - \beta)]}
\]

\[
+ \sum_{k=n}^{\infty} \frac{(k + p - m)^{\Omega+1}(k + p)!}{(p - m)^{\Omega}(k + p)!} \left| e^{i\alpha} a_{k+p} - e^{i\beta} b_{k+p} \right|
\]

If

\[
\sum_{k=n}^{\infty} \frac{(k + p - m)^{\Omega+1}(k + p)!}{(p - m)^{\Omega}(k + p)!} \left| e^{i\alpha} a_{k+p} - e^{i\beta} b_{k+p} \right|
\]

\[
\leq \delta - \frac{p!}{(p - m - 1)!} \sqrt{2[1 - \cos(\alpha - \beta)]},
\]
then we observe that
\[
\left| \frac{e^{i\alpha}T'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}T'(g^{(m)}(z))}{z^{p-m-1}} \right| < \delta \quad (z \in \mathcal{U}).
\]

Thus, \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g) \). This evidently completes the proof of Theorem 2.1.

\[\square\]

**Remark 2.2.** In its special case when \( m = \Omega = \lambda = \alpha = 0 \) and \( \beta = \alpha \), (5) in Theorem 2.1 yields a result given earlier by Altuntaş et al. ([9] p.3, Theorem 1).

We give an example for Theorem 2.1.

**Example 2.1.** For given
\[
g(z) = \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} B_{k+p}(\alpha, \beta, \lambda, m, \delta, \Omega) z^{k+p-m} \in \mathcal{F}(\lambda, m, \Omega)
\]

\((n, p \in \mathbb{N} = \{1, 2, 3, \ldots\}; \ p > m; \ \Omega, m \in \mathbb{N}_0)\)

we consider
\[
f(z) = \frac{p!}{(p-m)!} z^{p-m} + \sum_{k=n}^{\infty} A_{k+p}(\alpha, \beta, \lambda, m, \delta, \Omega) z^{k+p-m} \in \mathcal{F}(\lambda, m, \Omega)
\]

\((n, p \in \mathbb{N} = \{1, 2, 3, \ldots\}; \ p > m; \ \Omega, m \in \mathbb{N}_0)\)

with
\[
A_{k+p} = \frac{(p-m)\delta - \frac{p!}{(p-m-1)!} \sqrt{2[1 - \cos(\alpha - \beta)]}}{(1 + \lambda k(p-m)^{-1})[(k+p-m)!/(n+p-1)!][(k+p)!/(k+m)!]} e^{i\alpha} + e^{i(\beta - \alpha)} B_{k+p}.
\]

Then we have that
\[
\sum_{k=n}^{\infty} \frac{(k+p-m)\Omega + 1(k+p)![(1 + \lambda k(p-m)^{-1})]}{(p-m)\Omega[(k+p-m)!]} \left| e^{i\alpha} A_{k+p} - e^{i\beta} B_{k+p} \right|
\]
Finally, in view of the telescopic series, we obtain
\[
\sum_{k=n}^{\infty} \frac{1}{(k+p-1)(k+p)} = \lim_{\zeta \to \infty} \sum_{k=n}^{\zeta} \left[ \frac{1}{k+p-1} - \frac{1}{k+p} \right] = \frac{1}{n+p-1}.
\]

Using (7) in (6), we get
\[
\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega+1} |a_{k+p}| - |b_{k+p}|}{(p-m)^{\Omega}(k+p-m)!} \leq \delta - \frac{p!}{(p-m-1)!} \sqrt{2[1 - \cos(\alpha - \beta)]}
\]

Therefore, we say that \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega)^p - N(g) \).

Also, Theorem 2.1 gives us the following corollary.

**Corollary 2.3.** If \( f \in F(\lambda, m, \Omega) \) satisfies
\[
\sum_{k=n}^{\infty} \frac{(k+p-m)^{\Omega+1} |a_{k+p}| - |b_{k+p}|}{(p-m)^{\Omega}(k+p-m)!} \leq \delta - \frac{p!}{(p-m-1)!} \sqrt{2[1 - \cos(\alpha - \beta)]}
\]

for some \(-\pi \leq \alpha - \beta \leq \pi\) and \( \delta > \frac{p!}{(p-m-1)!} \sqrt{2[1 - \cos(\alpha - \beta)]} \), and \( \arg(a_{k+p}) - \arg(b_{k+p}) = \beta - \alpha \) \((m, p \in \mathbb{N} = \{1, 2, 3, \ldots\}; m \in \mathbb{N}_0, p > m)\), then \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega)^p - N(g) \).

**Proof.** By Theorem 2.1, we see the inequality (4) which implies that \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega)^p - N(g) \).

Since \( \arg(a_{k+p}) - \arg(b_{k+p}) = \beta - \alpha \), if \( \arg(a_{k+p}) = \alpha_{k+p} \), we see \( \arg(b_{k+p}) = \alpha_{k+p} - \beta + \alpha \). Therefore,
\[
e^{i\alpha} a_{k+p} - e^{i\beta} b_{k+p} = e^{i\alpha} |a_{k+p}| e^{i\alpha_{k+p}} - e^{i\beta} |b_{k+p}| e^{i(\alpha_{k+p} - \beta + \alpha)}
\]
implies that
\[ |e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}| = \|a_{k+p} - |b_{k+p}|. \] (8)

Using (8) in (4) the proof of the corollary is complete.

Next, we can prove the following theorem.

**Theorem 2.4.** If \( f \in \mathcal{F}(\lambda, m, \Omega) \) satisfies
\[
\sum_{k=n}^{\infty} \frac{(k + p - m)^{\Omega}(k + p)!(1 + \lambda k(p - m)^{-1})}{(p - m)^{\Omega}(k + p - m)!} |e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}| \leq \delta - \frac{p!}{(p - m)!} \sqrt{2[1 - \cos(\alpha - \beta)]} \quad (z \in \mathcal{U}).
\]
for some \(-\pi \leq \alpha - \beta \leq \pi; p > m \) and \( \delta > \frac{p!}{(p - m)!} \sqrt{2[1 - \cos(\alpha - \beta)]} \) then \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega)_p - M(g) \).

The proof of this theorem is similar with Theorem 2.1.

**Corollary 2.5.** If \( f \in \mathcal{F}(\lambda, m, \Omega) \) satisfies
\[
\sum_{k=n}^{\infty} \frac{(k + p - m)^{\Omega+1}(k + p)!(1 + \lambda k(p - m)^{-1})}{(p - m)^{\Omega+1}(k + p - m)!} \|a_{k+p} - |b_{k+p}|\|
\leq \delta - \frac{p!}{(p - m)!} \sqrt{2[1 - \cos(\alpha - \beta)]} \quad (z \in \mathcal{U}).
\]
for some \(-\pi \leq \alpha - \beta \leq \pi; p > m \) and \( \delta > \frac{p!}{(p - m)!} \sqrt{2[1 - \cos(\alpha - \beta)]} \) and \( \arg(a_{k+p}) - \arg(b_{k+p}) = \beta - \alpha \), then \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega)_p - M(g) \).

Our next result as follows.

**Theorem 2.6.** If \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega)_p - N(g), 0 \leq \alpha < \beta \leq \pi; p > m \) and \( \arg(e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}) = k\phi \), then
\[
\sum_{k=n}^{\infty} \frac{(k + p - m)^{\Omega+1}(k + p)!(1 + \lambda k(p - m)^{-1})}{(p - m)^{\Omega+1}(k + p - m)!} |e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}| \leq \delta - \frac{p!}{(p - m - 1)!} \left(\cos\alpha - \cos\beta\right).
\]
Proof. For \( f \in (\alpha, \beta, \lambda, \delta, \Omega)_p - N(g) \), we have
\[
\left| \frac{e^{i\beta}f'(m)(z)}{z^{p-m-1}} - \frac{e^{i\beta}f'(m)(z)}{z^{p-m-1}} \right| = \left| \frac{p! (e^{i\alpha} - e^{i\beta})}{(p - m - 1)!} + \sum_{k=n}^\infty (k + p - m)^{\Omega + 1}(k + p)(1 + \lambda k(p - m)^{-1})(e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p})z^k \right| < \delta.
\]

Let us consider \( z \) such that \( \arg z = -\phi \). Then \( z^k = |z|^k e^{-ik\phi} \). For such a point \( z \in U \), we see that
\[
\left| \frac{e^{i\beta}f'(m)(z)}{z^{p-m-1}} - \frac{e^{i\beta}f'(m)(z)}{z^{p-m-1}} \right| = \left| \frac{p! (e^{i\alpha} - e^{i\beta})}{(p - m - 1)!} + \sum_{k=n}^\infty (k + p - m)^{\Omega + 1}(k + p)(1 + \lambda k(p - m)^{-1})(e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p})|z|^k \right| \leq \delta.
\]

This implies that
\[
\left| \frac{p! (e^{i\alpha} - e^{i\beta})}{(p - m - 1)!} + \sum_{k=n}^\infty (k + p - m)^{\Omega + 1}(k + p)(1 + \lambda k(p - m)^{-1})(e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p})|z|^k \right| < \delta^2,
\]
or
\[
\frac{p! (e^{i\alpha} - e^{i\beta})}{(p - m - 1)!} + \sum_{k=n}^\infty (k + p - m)^{\Omega + 1}(k + p)(1 + \lambda k(p - m)^{-1})(e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p})|z|^k \leq \delta - \frac{p!}{(p - m - 1)!} (e^{i\alpha} - e^{i\beta}).
\]

\( \square \)
Remark 2.7. Applying the parametric substitutions listed in (5), Theorem 2.4 and 2.6 would yield a set of known results due to Altuntaş et al. ([9] p.5, p.6, Theorem 7).

Theorem 2.8. If \( f \in (\alpha, \beta, \lambda, m, \delta, \Omega) \cdot M(g), 0 \leq \alpha < \beta \leq \pi \) and \( \arg(e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p}) = k\phi \), then
\[
\sum_{k=n}^{\infty} \frac{(k + p - m)!}{(p - m)!} \left| e^{i\alpha}a_{k+p} - e^{i\beta}b_{k+p} \right| \leq \delta + \frac{p!}{(p - m - 1)!} (\cos \beta - \cos \alpha).
\]

The proof of this theorem is similar with Theorem 2.6.

Remark 2.9. Taking \( \lambda = \alpha = \Omega = m = 0, \beta = \alpha \) and \( p = 1 \), in Theorem 2.8, we arrive at the following theorem due to Orhan et al. [1].

Theorem 2.10. If \( f \in (\alpha, \delta) - N(g) \) and \( \arg(a_{n} - e^{i\alpha}b_{n}) = (n - 1)\phi \) \((n = 2, 3, 4, \ldots)\), then
\[
\sum_{n=2}^{\infty} n |a_{n} - e^{i\alpha}b_{n}| \leq \delta + \cos \alpha - 1.
\]

We give an application of following lemma due to Miller and Mocanu [2] (see also, [11]).

Lemma 2.1. Let the function
\[
w(z) = b_{n}z^{n} + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots \quad (z \in \mathbb{U})
\]
be regular in \( \mathbb{U} \) with \( w(z) \neq 0, (n \in \mathbb{U}) \). If \( z_{0} = re^{i\theta_{0}} (r_{0} < 1) \) and \( |w(z_{0})| = \max_{|z| \leq r_{0}} |w(z)| \), then \( z_{0}w'(z_{0}) = qw(z_{0}) \) where \( q \) is real and \( q \geq n \geq 1 \).

Applying the above lemma, we derive

Theorem 2.11. If \( f \in \mathcal{F}(\lambda, m, \Omega) \) satisfies
\[
\frac{|e^{i\alpha}f^{(m)}(z)|}{z^{p-m-1}} - \frac{|e^{i\beta}g^{(m)}(z)|}{z^{p-m-1}} < \delta(p + n - m) - \frac{p!}{(p - m - 1)!} \sqrt{2}\left[1 - \cos(\alpha - \beta)\right]
\]
for some \(-\pi \leq \alpha - \beta \leq \pi; p > m \) and \( \delta > \left( \frac{p!}{(p + n - m)(p - m - 1)!} \right) \sqrt{2}\left[1 - \cos(\alpha - \beta)\right] \), then
\[
\frac{|e^{i\alpha}f^{(m)}(z)|}{z^{p-m}} - \frac{|e^{i\beta}g^{(m)}(z)|}{z^{p-m}} < \delta + \frac{p!}{(p - m)!} \sqrt{2}\left[1 - \cos(\alpha - \beta)\right] \quad (z \in \mathbb{U}).
\]
Proof. Let us define \( w(z) \) by
\[
\frac{e^{i\alpha}F'(f^{(m)}(z))}{z^{p-m}} - \frac{e^{i\beta}F'(g^{(m)}(z))}{z^{p-m}} = \frac{p!}{(p-m)!} (e^{i\alpha} - e^{i\beta}) + \delta w(z),
\]
where
\[
\delta w(z) = \frac{\delta w'(z)}{e^{i\alpha} - e^{i\beta}} + \delta w(z).
\]

Then \( w(z) \) is analytic in \( \mathcal{U} \) and \( w(0) = 0 \). By logarithmic differentiation, we get from (9) that
\[
\frac{e^{i\alpha}F'(f^{(m)}(z))}{e^{i\alpha}F(f^{(m)}(z))} - \frac{e^{i\beta}F'(g^{(m)}(z))}{e^{i\beta}F(g^{(m)}(z))} = \frac{p-m}{z} = \frac{\delta w(z)}{e^{i\alpha} - e^{i\beta}} + \delta w(z).
\]

Since
\[
\frac{e^{i\alpha}F'(f^{(m)}(z))}{z^{p-m}} - \frac{e^{i\beta}F'(g^{(m)}(z))}{z^{p-m}} = \frac{p!}{(p-m)!} (e^{i\alpha} - e^{i\beta}) + \delta w(z)
\]
we see that
\[
\frac{e^{i\alpha}F'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}F'(g^{(m)}(z))}{z^{p-m-1}} = \frac{p!}{(p-m)!} (e^{i\alpha} - e^{i\beta}) + \delta w(z)
\]

This implies that
\[
\left| \frac{e^{i\alpha}F'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}F'(g^{(m)}(z))}{z^{p-m-1}} \right| = \left| \frac{p!}{(p-m-1)!} (e^{i\alpha} - e^{i\beta}) + \delta w(z) \right|
\]

We claim that
\[
\left| \frac{e^{i\alpha}F'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}F'(g^{(m)}(z))}{z^{p-m-1}} \right| < \delta (p - m + n) - \frac{p!}{(p-m-1)!} \sqrt{2[1 - \cos(\alpha - \beta)]}
\]
in \( \mathcal{U} \).

Otherwise, there exists a point \( z_0 \in \mathcal{U} \) such that \( z_0 w'(z_0) = qw(z_0) \) (by Miller and Mocanu’s Lemma) where \( w(z_0) = e^{i\theta} \) and \( q \geq n \geq 1 \).

Therefore, we obtain that
\[
\left| \frac{e^{i\alpha}F'(f^{(m)}(z))}{z^{p-m-1}} - \frac{e^{i\beta}F'(g^{(m)}(z))}{z^{p-m-1}} \right| = \left| \frac{p!}{(p-m-1)!} (e^{i\alpha} - e^{i\beta}) + \delta e^{i\theta} (p - m + q) \right|
\]
\[
\geq \delta (p - q - m) - \left| \frac{p!}{(p-m-1)!} (e^{i\alpha} - e^{i\beta}) \right|
\]
\[
\geq \delta (p + n - m) - \frac{p!}{(p-m-1)!} \sqrt{2[1 - \cos(\alpha - \beta)]}.
\]

This contradicts our condition in Theorem 2.11.
Hence, there is no \( z_0 \in U \) such that \( |w(z_0)| = 1 \). This means that \( |w(z)| < 1 \) for all \( z \in U \).

Thus, have that
\[
\left| \frac{e^{i\alpha} \wp'(f(z))}{z^p} - \frac{e^{i\beta} \wp'(g(z))}{z^p} \right| = \left| \frac{p!}{(p-m)!} \left( e^{i\alpha} - e^{i\beta} \right) + \delta w(z) \right| \\
\leq \frac{p!}{(p-m)!} \left| e^{i\alpha} - e^{i\beta} \right| + \delta |w(z)| \\
< \delta + \frac{p!}{(p-m)!} \sqrt{2[1 - \cos(\alpha - \beta)]}. 
\]

Upon setting \( m = 0, \alpha = \varphi, \wp = \wp \) and \( \beta = \alpha \) in Theorem 2.11, we have the following corollary given by Sağsöz et al.\([6]\).

**Corollary 2.12.** If \( f \in \wp(\Omega, \lambda) \) satisfies
\[
\left| \frac{e^{i\alpha} \wp'(f(z))}{z^p} - \frac{e^{i\beta} \wp'(g(z))}{z^p} \right| < \delta \left( p + n \right) - p \sqrt{2[1 - \cos(\varphi - \alpha)]}
\]
for some \(-\pi \leq \alpha - \beta \leq \pi \); and \( \delta > \left( \frac{p}{(p+n)} \right) \sqrt{2[1 - \cos(\alpha - \beta)]} \), then
\[
\left| \frac{e^{i\alpha} \wp(f(z))}{z^p} - \frac{e^{i\beta} \wp(g(z))}{z^p} \right| < \delta + \sqrt{2[1 - \cos(\varphi - \alpha)]} \quad (z \in U). 
\]

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