On a subclass of analytic functions involving harmonic means

Andreea-Elena Tudor and Dorina Răducanu

Abstract

In the present paper, we consider a generalised subclass of analytic functions involving arithmetic, geometric and harmonic means. For this function class we obtain an inclusion result, Fekete-Szegő inequality and coefficient bounds for bi-univalent functions.

1 Introduction

Let \( U_r = \{ z \in \mathbb{C} : |z| < r \} \) \((r > 0)\) and let \( U = U_1 \) denote the unit disk.

Let \( A \) be the class of all analytic functions \( f \) in \( U \) of the form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U.
\]

Further, by \( S \) we shall denote the class of all functions in \( A \) which are univalent in \( U \). It is known (see [6]) that if \( f \in S \), then \( f(U) \) contains the disk \( \{ |w| < \frac{1}{4} \} \). Here \( \frac{1}{4} \) is the best possible constant known as the Koebe constant for \( S \). Thus every univalent function \( f \) has an inverse \( f^{-1} \) defined on some disk containing the disk \( \{ |w| < \frac{1}{4} \} \) and satisfying:

\[
f^{-1}(f(z)) = z, \quad z \in U \quad \text{and} \quad f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4},
\]

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267
Lemma 3.

\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \]  

(2)

We denote by \( S^* \) the class of analytic functions which are starlike in \( U \).

Let \( f \in A \) and \( \alpha, \beta \in \mathbb{R} \). We define the following function:

\[ F(z) = [f(z)^{1-\alpha}(zf'(z))^\alpha]^{1-\beta} \cdot [(1 - \alpha)f(z) + \alpha zf'(z)]^\beta, \quad \alpha, \beta \in \mathbb{R}. \]  

(3)

Remark 1. It is easy to observe that for specific values of \( \beta \), the function \( F(z) \) reduces to some generalised means. If \( \beta = 0 \) we obtain generalised geometric means, if \( \beta = 1 \) we obtain generalised arithmetic means and if \( \beta = -1 \) we obtain generalised harmonic means of functions \( f(z) \) and \( zf'(z) \).

Definition 1. A function \( f \in A \) is said to be in the class \( H_{\alpha,\beta} \), \( \alpha, \beta \in \mathbb{R} \), if the function \( F(z) \) defined by (3) is starlike, that is

\[ \Re\left\{ (1 - \beta)\left[ (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] + \beta \frac{zf'(z) + \alpha zf''(z)}{(1 - \alpha)f(z) + \alpha zf'(z)} \right\} > 0, \quad z \in U. \]  

(4)

In order to prove our main results we will need the following lemmas.

Lemma 1. [9, p.24] Let \( q \in Q \), with \( q(0) = a \), and let \( p(z) = a + a_n z^n + \ldots \) be analytic in \( U \) with \( p(z) \not\equiv a \) and \( n \geq 1 \). If \( p \) is not subordinate to \( q \) then there exist \( z_0 = r_0 e^{i\theta_0} \in U \) and \( \zeta_0 \in \partial U \setminus E(q) \) and \( m \geq n \geq 1 \) for which \( p(U_{r_0}) \subset q(U) \) and:

1. \( p(z_0) = q(\zeta_0) \),
2. \( z_0 p'(z_0) = m \zeta_0 q'(\zeta_0) \),
3. \( \Re\left\{ \frac{z_0 q''(\zeta_0)}{p'(z_0)} + 1 \right\} \geq m \Re\left\{ \frac{q''(\zeta_0)}{q'(\zeta_0)} + 1 \right\} \).

Denote by \( \mathcal{P} \) the class of analytic functions \( p \) normalized by \( p(0) = 1 \) and having positive real part in \( U \).

Lemma 2. [6] Let \( p \in \mathcal{P} \) be of the form \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \), \( z \in U \). Then the following estimates hold

\[ |p_n| \leq 2, \quad n = 1, 2, \ldots \]

Lemma 3. [4] If \( p \in \mathcal{P} \) is of the form \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \), \( z \in U \). Then

\[ |p_2 - vp_1^2| \leq \begin{cases} -4v + 2, & v \leq 0, \\ 2, & 0 \leq v \leq 1, \\ 4v - 2, & v \geq 1. \end{cases} \]
When \( v < 0 \) or \( v > 1 \), the equality holds if and only if \( p_1(z) = \frac{1+z}{1-z} \) or one of its rotations. If \( 0 < v < 1 \) then the equality holds if and only if \( p_1(z) = \frac{1+z^2}{1-z^2} \) or one of its rotations. If \( v = 0 \), the equality holds if and only if

\[
p_1(z) = \left(\frac{1}{2} + \frac{1}{2} \lambda\right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2} \lambda\right) \frac{1-z}{1+z}, \quad \lambda \in [0,1]
\]

or one of its rotations. If \( v = 1 \), the equality holds if and only if \( p_1 \) is the reciprocal of one of the functions such that the equality holds in the case of \( v = 0 \).

### 2 Inclusion result

In this section we show that the new class \( H_{\alpha,\beta} \) is a subclass of the class of starlike functions.

**Theorem 1.** Let \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha \beta (1 - \alpha) \geq 0 \). Then

\[ H_{\alpha,\beta} \subset S^* \subset S. \]

**Proof.** Let \( f \) be in the class \( H_{\alpha,\beta} \) and let \( p(z) = \frac{zf'(z)}{f(z)} \). Then from (4) we obtain that \( f \in H_{\alpha,\beta} \) if and only if

\[
\Re \left\{ \alpha(1 - \beta) \frac{zp'(z)}{p(z)} + \alpha \beta \frac{zp'(z)}{1 - \alpha + \alpha p(z)} \right\} > 0. \tag{5}
\]

Let

\[
q(z) = \frac{1+z}{1-z} = 1 + q_1 z + \cdots. \tag{6}
\]

Then \( \Delta = q(\mathbb{D}) = \{w : \Re w > 0\} \), \( q(0) = 1, E(q) = \{1\} \) and \( q \in Q \). To prove that \( f \in S^* \) it is enough to show that

\[
\Re \left\{ \alpha(1 - \beta) \frac{zp'(z)}{p(z)} + \alpha \beta \frac{zp'(z)}{1 - \alpha + \alpha p(z)} \right\} > 0 \Rightarrow p(z) < q(z).
\]

Suppose that \( p(z) \not< q(z) \). Then, from Lemma 1, there exist a point \( z_0 \in U \) and a point \( \zeta_0 \in \partial U \setminus \{1\} \) such that \( p(z_0) = q(\zeta_0) \) and \( \Re p(z) > 0 \) for all \( z \in U_{x_0} \). This implies that \( \Re p(z_0) = 0 \), therefore we can choose \( p(z_0) \) of the form \( p(z_0) := ix \), where \( x \) is a real number. Due to symmetry, it is sufficient to consider only the case where \( x > 0 \). We have

\[
\zeta_0 = q^{-1}(p(z_0)) = \frac{p(z_0) - 1}{p(z_0) + 1},
\]
then \( z_0p'(z_0) = m \zeta_0q'(\zeta_0) = -m(x^2 + 1) = y \), where \( y < 0 \).

Thus, we obtain:

\[
\mathbb{R} \left( \alpha(1 - \beta) \frac{z_0p'(z_0)}{p(z_0)} \right) + \mathbb{R} \left( \alpha \beta \frac{z_0p'(z_0)}{1 - \alpha + \alpha p(z_0)} \right) = \mathbb{R} \left( \alpha(1 - \beta) \frac{y}{ix} \right) + \mathbb{R} \left( \frac{\alpha \beta y}{1 - \alpha + \alpha ix} \right) = 0 + \frac{y |\alpha \beta(1 - \alpha)|}{|1 - \alpha + \alpha ix|^2} \leq 0.
\]

This contradicts the hypothesis of the theorem, therefore \( p < q \) and the proof of Theorem 1 is complete. \( \Box \)

3 Fekete-Szegö problem

In 1933 M. Fekete and G. Szegö obtained sharp upper bounds for \( |a_3 - \mu a_2^2| \) for \( f \in S \) and \( \mu \) real number. For this reason, the determination of sharp upper bounds for the non-linear functional \( |a_3 - \mu a_2^2| \) for any compact family \( F \) of functions \( f \in A \) is popularly known as the Fekete-Szegö problem for \( F \).

For different subclasses of \( S \), the Fekete-Szegö problem has been investigated by many authors (see [2], [4], [11]).

In this section we will solve the Fekete-Szegö problem for the class \( H_{\alpha,\beta} \), where \( \alpha \) and \( \beta \) are positive real numbers.

**Theorem 2.** Let \( \alpha, \beta, \mu \) be positive real numbers. If the function \( f \) given by (1) belongs to the class \( H_{\alpha,\beta} \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
-4\mu \frac{1}{(1 + \alpha)^2} + \frac{2(\alpha - 1)(1 + \alpha \beta) + (\alpha + 1)(5 + \alpha)}{(1 + 2\alpha)(1 + \alpha)^2}, & \mu \leq \sigma_1, \\
\frac{1}{1 + 2\alpha}, & \sigma_1 \leq \mu \leq \sigma_2, \\
\frac{4\mu}{(1 + \alpha)^2} - \frac{2(\alpha - 1)(1 + \alpha \beta) + (\alpha + 1)(3 - \alpha)}{(1 + 2\alpha)(1 + \alpha)^2}, & \mu \geq \sigma_2.
\end{cases}
\]

where

\[
\sigma_1 = \frac{1 + 3\alpha - \alpha \beta + \alpha^2 \beta}{2(1 + 2\alpha)}, \quad \sigma_2 = \frac{2 + 5\alpha + \alpha^2 - \alpha \beta + \alpha^2 \beta}{2(1 + 2\alpha)}.
\]

**Proof.** Let \( f \) be in the class \( H_{\alpha,\beta} \) and let \( p \in \mathcal{P} \). From (4) we obtain

\[
\left\{ (1 - \beta) \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] + \beta \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha) f(z) + \alpha z f'(z)} \right\} = p(z).
\]

Since \( f \) has the Taylor series expansion (1) and \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots, z \in U \), we have

\[
1 + (1 + \alpha)a_2 z + \left[ 2(1 + 2\alpha)a_3 - (1 + 3\alpha - \alpha \beta + \alpha^2 \beta) a_2^2 \right] z^2 + \ldots = \begin{cases} 
1 + p_1 z + p_2 z^2 + \ldots, \quad (7)
\end{cases}
\]

(\( p = \left( \begin{array}{c} p_1 \ p_2 \end{array} \right) \))
Therefore, equating the coefficients of $z^2$ and $z^3$ in (7), we obtain

$$a_2 = \frac{p_1}{1 + \alpha}, \quad a_3 = \frac{1}{2(1 + 2\alpha)} \left[ p_2 + \frac{(1 + 3\alpha - \alpha\beta + \alpha^2\beta)p_1^2}{(1 + \alpha)^2} \right].$$

So, we have

$$a_3 - \mu a_2^2 = \frac{1}{2(1 + 2\alpha)} (p_2 - vp_1^2),$$

where

$$v = \frac{2(1 + 2\alpha)}{(1 + \alpha)^2} \mu - \frac{1 + 3\alpha - \alpha\beta + \alpha^2\beta}{(1 + \alpha)^2}. \tag{8}$$

Now, our result follows as an application of Lemma 3. $\Box$

4 Subclass of bi-univalent function

A function $f \in \mathcal{A}$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\sigma$ be the class of all functions $f \in \mathcal{S}$ such that the inverse function $f^{-1}$ has an univalent analytic continuation to $\{|w| < 1\}$. The class $\sigma$, called the class of bi-univalent functions, was introduced by Levin [7] who showed that $|a_2| < 1.51$. Branan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$. On the other hand, Netanyahu [10] showed that $\max_{f \in \sigma} |a_2| = \frac{4}{3}$. Several authors have studied similar problems in this direction (see [1] [5], [8], [12], [13]).

We notice that the class $\sigma$ is not empty. For example, the following functions are members of $\sigma$:

$$z, \quad \frac{z}{1 - z}, \quad -\log(1 - z), \quad \frac{1}{2} \log \frac{1 + z}{1 - z}.$$

However, the Koebe function is not a member of $\sigma$. Other examples of univalent functions that are not in the class $\sigma$ are

$$z - \frac{z^2}{2}, \quad \frac{z}{1 - z^2}.$$

In the sequel we assume that $\varphi$ is an analytic function with positive real part in the unit disk $U$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and such that $\varphi(U)$ is symmetric with respect to the real axis. Assume also that:

$$\varphi(z) = 1 + B_1z + B_2z^2 + \ldots, \quad B_1 > 0. \tag{9}$$
Definition 2. A function \( f \in A \) is said to be in the class \( H_{\alpha,\beta}(\varphi) \), \( \alpha \in [0,1] \), \( \beta \geq 0 \), if \( f \in \sigma \) and satisfies the following conditions:

\[
(1 - \beta) \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] + \beta \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha zf'(z)} \leq \varphi(z),
\]

and

\[
(1 - \beta) \left[ (1 - \alpha) \frac{wg'(w)}{g(w)} + \alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] + \beta \frac{wg'(w) + \alpha w^2 g''(w)}{(1 - \alpha)g(w) + \alpha wg'(w)} \leq \varphi(w),
\]

where \( g \) is the extension of \( f^{-1} \) to \( U \).

Theorem 3. If \( f \in H_{\alpha,\beta}(\varphi) \) is in \( A \) then

\[
|a_2| \leq \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{|(1 + \alpha) + \alpha\beta(1 - \alpha)|B_1^2 - (B_2 - B_1)(1 + \alpha)^2|}}, \quad (10)
\]

and

\[
|a_3| \leq B_1 \left[ \frac{1}{1 + \alpha} + \frac{\alpha\beta(1 - \alpha)}{2(1 + 2\alpha)(1 + \alpha)} \right] + \frac{|B_2 - B_1|}{1 + \alpha}. \quad (11)
\]

Proof. Let \( f \in H_{\alpha,\beta}(\varphi) \) and \( g = f^{-1} \). Then there exist two analytic functions \( u, v : U \to U \) with \( u(0) = v(0) = 0 \) such that:

\[
(1 - \beta) \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] + \beta \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha)f(z) + \alpha zf'(z)} = \varphi(u(z)) \quad \text{and}
\]

\[
(1 - \beta) \left[ (1 - \alpha) \frac{wg'(w)}{g(w)} + \alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right] + \beta \frac{wg'(w) + \alpha w^2 g''(w)}{(1 - \alpha)g(w) + \alpha wg'(w)} = \varphi(v(w)). \quad (12)
\]

Define the functions \( p \) and \( q \) by

\[
p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + \ldots, \quad q(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + \ldots
\]

or equivalently,

\[
u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[ p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \ldots \right], \quad (13)
\]

and

\[
v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[ q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \ldots \right]. \quad (14)
\]
We observe that \( p, q \in \mathcal{P} \) and, in view of Lemma 2, we have that \(|p_n| \leq 2\) and \(|q_n| \leq 2\), for \( n \geq 1\).

Further, using (13) and (14) together with (9), it is evident that

\[
\varphi(u(z)) = 1 + \frac{1}{2} B_1 p_1 z + \left( \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \ldots
\]

and

\[
\varphi(v(z)) = 1 + \frac{1}{2} B_1 q_1 z + \left( \frac{1}{2} B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2 \right) z^2 + \ldots.
\]

Therefore, in view of (12), (15) and (16) we have

\[
(1 - \beta) \left[ (1 - \alpha) \frac{z f'(z)}{f(z)} + \alpha \left( 1 + \frac{z f''(z)}{f'(z)} \right) \right] + \beta \frac{z f'(z) + \alpha z^2 f''(z)}{(1 - \alpha) f(z) + \alpha z f'(z)}
\]

\[= 1 + \frac{1}{2} B_1 p_1 z + \left( \frac{1}{2} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \ldots,
\]

and

\[
(1 - \beta) \left[ (1 - \alpha) \frac{w g'(w)}{g(w)} + \alpha \left( 1 + \frac{w g''(w)}{g'(w)} \right) \right] + \beta \frac{w g'(w) + \alpha w^2 g''(w)}{(1 - \alpha) g(w) + \alpha w g'(w)}
\]

\[= 1 + \frac{1}{2} B_1 q_1 w + \left( \frac{1}{2} B_1 \left( q_2 - \frac{1}{2} q_1^2 \right) + \frac{1}{4} B_2 q_1^2 \right) w^2 + \ldots.
\]

Since \( f \in \sigma \) has the Taylor series expansion (1) and \( g = f^{-1} \) the series expansion (2), we have

\[
(1 - \beta) \left[ (1 - \alpha) \frac{z f'(z)}{f(z)} + \alpha \left( 1 + \frac{z f''(z)}{f'(z)} \right) \right] + \beta \frac{z f'(z) + \alpha z^2 f''(z)}{(1 - \alpha) f(z) + \alpha z f'(z)}
\]

\[= 1 + (1 + \alpha) a_2 z + \left[ 2(1 + 2\alpha)a_3 - (1 + 3\alpha - \alpha \beta + \alpha^2 \beta) a_2^2 \right] z^2 + \ldots,
\]

and

\[
(1 - \beta) \left[ (1 - \alpha) \frac{w g'(w)}{g(w)} + \alpha \left( 1 + \frac{w g''(w)}{g'(w)} \right) \right] + \beta \frac{w g'(w) + \alpha w^2 g''(w)}{(1 - \alpha) g(w) + \alpha w g'(w)}
\]

\[= 1 - (1 + \alpha) a_2 w - \left[ 2(1 + 2\alpha)a_3 - (1 + 3\alpha - \alpha \beta + \alpha^2 \beta)(a_3 - 2a_2^2) \right] w^2 + \ldots.
\]
Equating the coefficients in (17), (19) and (18), (20), we obtain

\[
\begin{cases}
(1 + \alpha)a_2 = \frac{1}{2}B_1p_1,
\end{cases}
\]

\[
2(1 + 2\alpha)a_3 - (1 + 3\alpha - \alpha\beta + \alpha^2\beta)a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2,
\]

\[-(1 + \alpha)a_2 = \frac{1}{2}B_1q_1,
\]

\[-2(1 + 2\alpha)(a_3 - 2a_2^2) - (1 + 3\alpha - \alpha\beta + \alpha^2\beta)a_2^2 = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2q_1^2.
\]

From the first and the third equation of the system (21) it follows that

\[p_1 = -q_1,
\]

and

\[a_2^2 = \left(\frac{B_1p_1\tau}{4(1 + \gamma)}\right)^2.
\]

Now, (22), (23) and the next two equations of the system (21) lead to

\[a_2^2 = \frac{B_1^2(p_2 + q_2)}{4[(1 + \alpha) + \alpha\beta(1 - \alpha)]B_1^2 - 4(B_2 - B_1)(1 + \alpha)^2}.
\]

Thus, in view of Lemma 2, we obtain the desired estimation of \(|a_2|\).

From the third and the fourth equation of (21), we obtain

\[a_3 = \frac{1}{2}B_1p_2\frac{3 + 5\alpha + \alpha\beta(1 - \alpha)}{4(1 + 2\alpha)(1 + \alpha)} + \frac{1}{4}B_1^2(B_2 - B_1)\frac{1}{1 + \alpha},
\]

which yields to the estimate given by (11) and so the proof of Theorem 3 is completed.

\[\square\]

References


Andreea-Elena TUDOR,
Department of Mathematics,
Transilvania University of Brașov,
Str.Iuliu Maniu 50, 500091, Brasov, Romania.
Email: tudor_andreea_elena@yahoo.com

Dorina RĂDUCANU,
Department of Mathematics,
Transilvania University of Brașov,
Str.Iuliu Maniu 50, 500091, Brasov, Romania.
Email: draducanu@unitbv.ro