Abstract. The investigation of properties of generalized Toeplitz operators with respect to the pairs of doubly commuting contractions (the abstract analogue of classical two variable Toeplitz operators) is proceeded. We especially concentrate on the condition of existence such a non-zero operator. There are also presented conditions of analyticity of such an operator.

1. Introduction

Let \( L(H_1, H_2) \) denote the algebra of all bounded linear operators from \( H_1 \) into \( H_2 \), where \( H_1, H_2 \) are complex, separable Hilbert spaces. If \( H_1 = H_2 \) we will use the notation \( L(H_1) \).

The classical Toeplitz operators on the Hardy space on the unit disc are well known and they are fully characterized by the relation \( X = T_2^* XT_2 \), where \( T_2 \) is the shift operator – the multiplication operator by the independent variable on the Hardy space \( H^2 \) on the circle \( T \). This notion can be generalized when instead of the backward shift \( T_2^* \) in the equation above we will put arbitrary, possibly different, contractions. Namely, for given contractions \( S \in L(H_1) \) and \( T \in L(H_2) \), an operator \( X \in L(H_2, H_1) \) is called generalized Toeplitz operator if \( X = SXT^* \). These type of operators were studied in [1, 6, 11].

The classical Toeplitz operators are also considered on the Hardy space on the torus \( H^2(T^2) \). The space \( H^2(T^2) \) can be seen as a subspace of \( L^2(T^2) = L^2(T^2, m \otimes m) \) (\( m \) denotes the normalized Lebesgue measure on \( T \)) and \( P_{H^2(T^2)} \) is the appropriate projection. For any \( \varphi \in L^\infty(T^2) = L^\infty(T^2, m \otimes m) \) we define the Toeplitz operator \( T_\varphi \in L(H^2(T^2)) \) by \( T_\varphi f = P_{H^2(T^2)}(\varphi f) \) (\( f \in H^2(T^2) \)).

AMS (2010) Subject Classification: Primary 47B35; secondary 47B38.

Keywords and phrases: Pair of contractions, isometric dilation, doubly commuting, Toeplitz operator, symbol operator.

The research of the authors was financed by the Ministry of Science and Higher Education of the Republic of Poland.
The function \( \varphi \) is called the symbol of the Toeplitz operator. The multiplication operators by the independent variables in this space we denote by \( T_{z_1}, T_{z_2} \). As it was shown in [10, Proposition 3.3], the set of Toeplitz operators on \( H^2(T^2) \) can be characterized as a set of operators \( X \in L(H^2(T^2)) \) such that \( X = T_{z_1}^* T_{z_2} \) and \( X = T_{z_2} T_{z_2} \). In [10], for given pairs of contractions \( S_1, S_2 \in L(H_1) \) and \( T_1, T_2 \in L(H_2) \), there were considered operators \( X \in L(H_2, H_1) \) such that \( X = S_1 X T_{z_1}^* \) and \( X = S_2 X T_{z_2}^* \) and they were called generalized Toeplitz operators with respect to the pairs \( S_1, S_2 \) and \( T_1, T_2 \). A general assumption was that \( S_1, S_2 \) doubly commute (i.e. not only \( S_1, S_2 \) commute but also \( S_1^*, S_2 \) do) and \( T_1, T_2 \) also doubly commute. Observe that in the previous case operators \( T_{z_1}, T_{z_2} \) doubly commute. One of the results, see [10] and Theorem 3.3 claims that for every generalized Toeplitz operator \( X \) there is an operator \( Y \in L(K_1^+, K_2^+) \) with \( X = P_H Y |_{H_2} \), \( Y = W_1 Y V_1^* \) and \( Y = W_2 Y V_2^* \), where pair \( W_1, W_2 \) and pair \( V_1, V_2 \) are minimal isometric dilations of the pairs of operators \( S_1, S_2 \) and \( T_1, T_2 \), respectively, defined on spaces \( K_1^+, K_2^+ \), respectively. Such an operator \( Y \) is called the symbol of \( X \).

In this paper we continue the investigation of properties of generalized Toeplitz operator with respect to pairs of doubly commuting contractions. We especially concentrate on the condition of existing such a non zero operator (Section 3). There are also presented conditions of analyticity of such an operator (Section 4). The dilation theory of the pairs of contractions is the main tool. Hence, in Section 2, we recall results on dilations of pairs of doubly commuting contractions. In both sections 3 and 4 examples are given.

2. Preliminaries on dilations of pairs of operators

In what follows some properties of a minimal isometric dilation for a pair of contractions will be needed. For a pair \( T_1, T_2 \in L(H) \) of commuting contractions, by Ando’s theorem [8, Theorem I.6.1], there is a pair of commuting isometries \( V_1, V_2 \in L(K^+) \), \( H \subset K^+ \), being a minimal isometric dilation of the given pair \( T_1, T_2 \), i.e. for all non-negative integers \( n, m \) the following holds

\[
T_{i_1} T_{i_2}^m = P_H V_1^n V_2^m |_{H} \quad \text{and} \quad K^+ = \bigvee_{n, m \geq 0} V_1^n V_2^m H. \quad (1)
\]

A minimal isometric dilations of a pair of commuting contractions is not unique but for each minimal isometric dilation we have (see [8] 10)

\[
T_i P_H = P_H V_i, \quad V_i^* H \subset H \quad \text{and} \quad V_i^* |_{H} = T_i^*, \quad i = 1, 2. \quad (2)
\]

The aim of the paper is to consider a doubly commuting pairs of contractions \( T_1, T_2 \in L(H) \), i.e. we assume that not only \( T_1, T_2 \) commute but also \( T_1^*, T_2^* \) commute. The important observation, which was made in [12] (Lemma 1 and remarks afterwards), is that in this case the isometries \( V_1, V_2 \) can be also chosen doubly commuting. Then as it was noticed in [10] 12 13 we have the specific properties. Let \( R \) be a maximal subspace of \( K^+ \), such that \( R_1 = V_1 |_{R} \), \( R_2 = V_2 |_{R} \), is a pair of unitary operators. As it was shown in [10] the projection \( P_R \) can be defined as follows

\[
P_R k = \lim_{n, m \to \infty} V_1^n V_2^m V_1^{*n} V_2^{*m} k \quad \text{for} \quad k \in K^+. \quad (3)
\]
Properties of two variables Toeplitz type operators

and that

\[ V_i P_R = P_R V_i, \quad V_i^* P_R = P_R V_i^* \quad \text{for } i = 1, 2. \]  

The following will be used later.

**Lemma 2.1**

Let \( T_1, T_2 \) be a pair of doubly commuting contractions. Using the notations introduced above the doubly index sequence of closed subspaces \( R_{1}^{n_1} R_{2}^{m_2} P_R H \) is increasing according to the natural order in \( \mathbb{N} \times \mathbb{N} \). Moreover,

\[ \mathcal{R} = \bigvee_{n,m \geq 0} R_{1}^{n_1} R_{2}^{m_2} P_R H. \]  

**Proof.** By (2) and (3)

\[ P_R h = \lim_{n,m \to \infty} V_{1}^{n_1} V_{2}^{m_2} T_{1}^{*n_1} T_{2}^{*m_2} h \quad \text{for } h \in H. \]

Hence

\[ V_1 P_R T_1^* h = \lim_{n,m \to \infty} V_{1}^{n_1+1} V_{2}^{m_2} T_{1}^{*n_1+1} T_{2}^{*m_2} h = P_R h \]

and

\[ P_R T_1^* h = R_{1}^{n_1} P_R h \quad \text{for } h \in H. \]

Similarly

\[ P_R T_2^* h = R_{2}^{m_2} P_R h \quad \text{for } h \in H, \]

which implies \( R_{1}^{n_1} P_R H \subset P_R H \) thus \( P_R H \subset R_{i} P_R H \) for \( i = 1, 2 \). In consequence

\[ R_{1}^{n_1} R_{2}^{m_2} P_R H \subset R_{1}^{n_1} R_{2}^{m_2} P_R H \quad \text{for } n_1 \leq m_1, n_2 \leq m_2. \]

Applying [1] and [4] we have

\[ \mathcal{R} = P_R K^+ = P_R \bigvee_{n,m \geq 0} V_{1}^{n_1} V_{2}^{m_2} H = \bigvee_{n,m \geq 0} P_R V_{1}^{n_1} V_{2}^{m_2} H = \bigvee_{n,m \geq 0} R_{1}^{n_1} R_{2}^{m_2} P_R H. \]

3. **Review on existence of a symbol**

Let us consider two pairs of doubly commuting contractions

\[ S_1, S_2 \in L(H_1), \quad T_1, T_2 \in L(H_2). \]

Let the pairs \( W_1, W_2 \in L(K_1^+), \quad V_1, V_2 \in L(K_2^+) \) be minimal isometric dilations of the pairs \( S_1, S_2 \) and \( T_1, T_2 \), respectively. Chose using [10] [12], as above, the pairs \( W_1, W_2 \) and \( V_1, V_2 \) doubly commuting. Let \( \mathcal{R}_1, \mathcal{R}_2 \) be maximal subspaces of \( K_1^+, K_2^+ \), respectively, such that both pairs \( W_1|_{\mathcal{R}_1}, W_2|_{\mathcal{R}_1} \) and \( V_1|_{\mathcal{R}_2}, V_2|_{\mathcal{R}_2} \) are unitary. An operator \( Y \in L(K_2^+, K_1^+) \), following [10], is called a *symbol with respect to the pairs* \( S_1, S_2 \) and \( T_1, T_2 \) if \( Y = W_1 Y V_1^* \) and \( Y = W_2 Y V_2^* \).

Recall after [10] some basic fact about the symbols.
Remark 3.1 (Remark 3.1, [10])

If an operator \( Y \in L(K^+_2, K^+_1) \) is a symbol with respect to the pairs \( S_1, S_2 \in L(H_1) \) and \( T_1, T_2 \in L(H_2) \), then the operator \( X = P_{H_1} Y |_{H_2} \) is a generalized Toeplitz operator with respect to the pairs \( S_1, S_2 \) and \( T_1, T_2 \).

Now we recall a characterization of the symbol.

Proposition 3.2 (Proposition 3.2, [10])

Let \( Y \in L(K^+_2, K^+_1) \). Then the following are equivalent

(i) \( Y \) is a symbol with respect to the pairs \( S_1, S_2 \) and \( T_1, T_2 \),
(ii) \( YV_i = W_i Y, \ i = 1, 2 \) and \( Y = Y P_{R_2} \),
(iii) \( YV_i^* = W_i^* Y, \ i = 1, 2 \) and \( Y = P_{R_1} Y \),
(iv) \( Y = \lim_{n,m \to \infty} W_1^n W_2^m P_{H_1} Y P_{H_2} V_1^* V_2^m \) in SOT.

Now let us recall the theorem about an existence of a symbol.

Theorem 3.3 (Theorem 3.5, [10])

Suppose \( X \in L(H_2, H_1) \). Let \( S_1, S_2 \in L(H_1) \) and \( T_1, T_2 \in L(H_2) \) be pairs of doubly commuting contractions. Assume that \( X = S_1 T_1^* \), \( X = S_2 T_2^* \). Then there exists exactly one operator \( Y \in L(K^+_2, K^+_1) \) such that

(i) \( Y \) is a symbol with respect to the pairs \( S_1, S_2 \) and \( T_1, T_2 \),
(ii) \( X = P_{H_1} Y |_{H_2} \),
(iii) \( \|X\| = \|Y\| \).

4. Existence of non-zero generalized Toeplitz operators

The next two theorems characterize when a non-zero generalized Toeplitz operator with respect to the pairs of doubly commuting contractions can exists.

Theorem 4.1

Let \( T_1, T_2 \in L(H_2) \) be a pair of doubly commuting contractions, then the following are equivalent.

(i) The only operator \( X \in L(H_2) \) satisfying \( X = T_1 X T_1^* \) and \( X = T_2 X T_2^* \) is the zero operator,
(ii) \( \lim_{n,m \to \infty} T_1^n T_2^* h = 0 \) for \( h \in H_2 \),
(iii) \( P_{R_2} H_2 = 0 \),
(iv) \( P_{H_2} R_2 = 0 \),
(v) \( P_{H_2} P_{R_2} P_{H_2} = 0 \),
(vi) \( R_2 = 0 \).

Proof. Note firstly that projection \( P_{R_2} \) satisfies condition (ii) in Proposition 3.2, thus \( P_{R_2} \) is a symbol by condition (i) of this Proposition. Hence, by Remark 3.1, \( X = P_{H_2} P_{R_2} |_{H_2} \) is a generalized Toeplitz operator with respect to the pair \( T_1, T_2 \). If (i) is satisfied, then \( X = 0 \) and \( P_{H_2} P_{R_2} P_{H_2} = X = 0 \) and (v) is fulfilled. Note
Properties of two variables Toeplitz type operators

that \( 0 = P_{R_2}P_{R_2}P_{H_2} = P_{H_2}P_{R_2}(P_{H_2}P_{R_2})^* \) implies \( P_{H_2}P_{R_2} = 0 \), i.e. \( (v) \Rightarrow (iv) \).

If \( (iv) \) is satisfied, then also \( P_{R_2}P_{H_2} = 0 \) and we obtain \( (iii) \).

Assuming \( (iii) \) by \( (5) \) we get \( (vi) \). When we assume \( (vi) \) and apply \( (5) \) we obtain \( P_{R_2}H_2 = 0 \).

Using \( (3) \) and isometric properties of \( V_1, V_2 \) we get \( (ii) \). The implication \( (iii) \Rightarrow (i) \) is straightforward.

**Example 4.2**

Let us now consider \( T_{z_1}^*, T_{z_2}^* \) the adjoints to the multiplication operators by the independent variables in the space \( H^2(T^2) \) as a pair of contractions. Note that the operators \( T_{z_1}, T_{z_2} \) doubly commute. It was shown in [10] Example 3.7 that a minimal isometric dilation for the pair \( T_{z_1}, T_{z_2} \) is the pair \( M_{z_1}^*, M_{z_2}^* \) of multiplication operators by the conjugates of the independent variables in \( L^2(T^2) \). Hence \( R_2 = L^2(T^2) \), so it is far from being zero. On the other hand, if \( X \in L(H^2(T^2)) \) fulfills the equations \( X = T_{z_i}^* XT_{z_i} \), for \( i = 1, 2 \), then, as it was shown in [10] Example 3.7, the symbol \( Y \in L(L^2(T^2)) \) for \( X \) is represented by a function \( \varphi \in L^\infty(T^2) \) such that \( Y = M_{\varphi} \), \((M_{\varphi}f)(z_1, z_2) = \varphi(z_1, z_2)f(z_1, z_2) \) for \( f \in H^2(T^2) \). Hence \( X = P_{H^2(T^2)}M_{\varphi} \) \((\text{Theorem 3.3})\). Thus the set of \( X \) fulfilling the equations \( X = T_{z_1}^* XT_{z_1} \), \( X = T_{z_2}^* XT_{z_2} \) can be identify with \( \varphi \in L^\infty(T^2) \). Hence the set of generalized Toeplitz operators with respect to both pairs equal to \( T_{z_1}, T_{z_2} \) is "rich". This is the case of the classical Toeplitz operator of two variables.

**Example 4.3**

Let \( T_{z_1}, T_{z_2} \) be the multiplication operators by the independent variables in the space \( H^2(T^2) \). Note that the operators \( T_{z_1}, T_{z_2} \) doubly commute. Since they are isometries, a minimal isometric dilation is the same pair \( T_{z_1}, T_{z_2} \) and \( K^+ = H^2(T^2) \). It is easy to see that \( \lim_{n,m \rightarrow \infty} T_{z_1}^{*n}T_{z_2}^{*m}h = 0 \) for \( h \in H^2(T^2) \) so that \( R_2 = 0 \). Hence the only operator \( X \in L(H^2(T^2)) \) satisfying \( X = T_{z_1}^* XT_{z_1} \) and \( X = T_{z_2}^* XT_{z_2} \) is the zero operator.

Now let us consider the general case.

**Theorem 4.4**

Let \( S_1, S_2 \in L(H_1) \) and \( T_1, T_2 \in L(H_2) \) be pairs of doubly commuting contractions.

Then the following are equivalent.

(i) The only operator \( X \in L(H_2, H_1) \) satisfying \( X = S_1 XT_1^* \) and \( X = S_2 XT_2^* \) is the zero operator.

(ii) One of the subspaces \( R_1, R_2 \) is trivial or the pairs of operators \( W_1|_{R_1}, W_2|_{R_2} \) are relatively singular.

**Proof.** Let \( X \in L(H_2, H_1) \) satisfying \( X = S_1 XT_1^* \), \( X = S_2 XT_2^* \). Then there exists its symbol \( Y \in L(K_2^+, K_1^+) \) such that

\[
Y = W_1 YV_1^* , \quad Y = W_2 YV_2^* .
\]

Let \( Z = Y|_{R_2} \). By definition of \( R_1, R_2 \) we have

\[
ZV_1|_{R_2} = W_1|_{R_1} Z , \quad ZV_2|_{R_2} = W_2|_{R_1} Z .
\]
Let $Z = AU$ be a polar decomposition of $Z$. By [1 Lemma 4.1] ker $Z^\perp$ reduces $V_1|_{\mathcal{R}_1}$ and $V_2|_{\mathcal{R}_2}$ and the subspace $\text{Ran} \, Z$ reduces $W_1|_{\mathcal{R}_1}, W_2|_{\mathcal{R}_1}$. Moreover, operators $V_1|_{\ker Z^\perp}, W_1|_{\text{Ran} \, Z}$ are unitarily equivalent taking $U|_{\ker Z^\perp}: \ker Z^\perp \rightarrow \text{Ran} \, Z$ and $V_2|_{\ker Z^\perp}, W_2|_{\text{Ran} \, Z}$ are unitarily equivalent taking the same unitary operator $U|_{\ker Z^\perp}$. Thus pairs $V_1|_{\ker Z^\perp}, V_2|_{\ker Z^\perp}$ and $W_1|_{\text{Ran} \, Z}, W_2|_{\text{Ran} \, Z}$ are unitarily equivalent taking $U|_{\ker Z^\perp}$.

By (i) $X$ have to be the zero operator. Hence $Y$ and $Z$ have to be zero operators. If $Z = Y|_{\mathcal{R}_2}$ is zero operator, then $\mathcal{R}_1 = 0$ or $\mathcal{R}_2 = 0$.

For the proof of the converse implication we assume that $\mathcal{R}_1 \neq 0, \mathcal{R}_2 \neq 0$. Let $A(\mathbb{D}^2)$ be the algebra of all holomorphic functions on $\mathbb{D}^2$ and continuous in $\mathbb{D}^2$ ($\mathbb{D}$ is the unit disc). It is a standard technique (see [4,5]) that the pair $W_1|_{\mathcal{R}_1}, W_2|_{\mathcal{R}_1}$ generate the representation $\Phi_W : A(\mathbb{D}^2) \rightarrow L(\mathcal{R}_1)$, i.e. $\Phi_W$ is linear, $\Phi_W(w) = \Phi_W(u), \Phi_W(v)$ and $\|\Phi_W(w)\| \leq \|w\|_\infty$ for $w, v \in A(\mathbb{D}^2)$. For any polynomial $p$ of two variables the representation $\Phi_W$ is defined as $\Phi_W(p) := p(W_1|_{\mathcal{R}_1}, W_2|_{\mathcal{R}_1})$. Next $\Phi_W$ is uniquely extended to $A(\mathbb{D}^2)$. Then, for any $x \in \mathcal{R}_1$, there exists a positive regular Borel measure $\mu_x$ on $\mathbb{T}^2$ such that

$$\langle \Phi_W(u)x, x \rangle = \int u \, d\mu_x \quad \text{for } x \in \mathcal{R}_1, \ u \in A(\mathbb{D}^2)$$

and $\|\mu_x\| \leq \|x\|^2$. Let $\mathcal{M}_\mu$ be a band of measures generated (for definition see [4]) by $\{\mu_x\}_{x \in \mathcal{R}_1}$. Similarly the pair $V_1|_{\mathcal{R}_2}, V_2|_{\mathcal{R}_2}$ generate the representation $\Phi_V$ and there are measures $\nu_y, y \in \mathcal{R}_2$, such that

$$\langle \Phi_V(v)y, y \rangle = \int v \, d\nu_y \quad \text{for } y \in \mathcal{R}_2, \ u \in A(\mathbb{D}^2).$$

Let $\mathcal{M}_\nu$ be a band of measures generated by $\nu_y, y \in \mathcal{R}_2$. If the pairs $W_1|_{\mathcal{R}_1}, W_2|_{\mathcal{R}_1}$ and $V_1|_{\mathcal{R}_2}, V_2|_{\mathcal{R}_2}$ are not singular, there is a measure $\eta \in \mathcal{M}_\mu \cap \mathcal{M}_\nu$. By [4 Proposition 1.4] there is $x \in \mathcal{R}_1$ such that $\eta \ll \mu_x$. By the theory of spectral multiplicity (see [4]), mainly by [2 §65, Theorem 3], there are vectors $x_0 \in \mathcal{R}_1, y_0 \in \mathcal{R}_2$ and a unitary operator $U : Z(y_0) \rightarrow Z(x_0)$, where $Z(x_0)$ is the smallest closed subspace containing $x_0$ and reducing for $W_1|_{\mathcal{R}_1}$ and $W_2|_{\mathcal{R}_1}$ and $Z(y_0)$ is the smallest closed subspace containing $y_0$ and reducing for $V_1|_{\mathcal{R}_2}$ and $V_2|_{\mathcal{R}_2}$. Moreover, $UV_i|_{Z(y_0)} = W_i|_{Z(x_0)}U$ for $i = 1, 2$. Let us define nonzero operator $Y \in L(K_2^+, K_1^+)$ as $Y = U$ on $Z(y_0)$ and $Y = 0$ on $K_2^+ \ominus Z(y_0)$. Clearly $Y = W_iYV_i^*$ for $i = 1, 2$, i.e. $Y$ is a symbol with respect to the pairs $S_1, S_2$ and $T_1, T_2$. By Theorem 3.3 and Remark 3.4 the operator $X = P_{H_1}Y|_{H_2}$ fulfils equalities $X = S_1XT_1^*$ and $X = S_2XT_2^*$. Moreover, $X \neq 0$, since $\|X\| = \|Y\|$

**Example 4.5**

Let the first pair of contractions be the pair $M_{z_1}, M_{z_2}$ of multiplication operators by the independent variables in the space $L^2(\mathbb{T}^2, \mu \otimes \mu)$, where $\mu$ is a non-atomic normalized measure concentrated on the Cantor set on the unit circle $T$ of the Lebesgue measure zero. The operators $M_{z_1}, M_{z_2}$ on $L^2(\mathbb{T}^2, \mu \otimes \mu)$ doubly commute as unitary. The pair $M_{z_1}, M_{z_2}$ is its own isometric (and unitary) dilation. Hence the space $\mathcal{R}_1 = L^2(\mathbb{T}^2, \mu \otimes \mu)$ is non-zero. Let the second pair of contractions be as in Example 4.2 i.e. $T_{z_1}^*, T_{z_2}^*$ in the space $H^2(\mathbb{T}^2)$. As we have noticed above $\mathcal{R}_2 = \{0\}$.
Properties of two variables Toeplitz type operators

$L^2(\mathbb{T}^2, m \otimes m)$, so it is also non-zero. There is no non-zero generalized Toeplitz operator with respect to this two pairs since the pair $M_{z_1}, M_{z_2}$ on $L^2(\mathbb{T}^2, \mu \otimes \mu)$ and the pair $M_{z_1}^*, M_{z_2}^*$ on $L^2(\mathbb{T}^2, m \otimes m)$ are relatively singular ($\mu \otimes \mu$ and $m \otimes m$ are singular measures).

5. Analytic generalized Toeplitz operators

Let us above $S_1, S_2 \in L(H_1)$ and $T_1, T_2 \in L(H_2)$ be two pairs of doubly commuting contractions. Let $Y \in L(K_+^+, K_-^+)$ be a symbol with respect to this pairs. We call a symbol $Y$ analytic if $Y H_2 \subset H_1$. The following theorem characterizes the analyticity of the symbol.

**Theorem 5.1**

Let $S_1, S_2 \in L(H_1)$ and $T_1, T_2 \in L(H_2)$ be pairs of doubly commuting contractions. Assume that $X \in L(H_2, H_1)$ such that

$$S_1^* X = X T_1^* \quad \text{and} \quad S_2^* X = X T_2^*. \quad (6)$$

Then the operator $P_{H_1} P_{R_1} X$ is a generalized Toeplitz operator with respect to the pairs $S_1, S_2$ and $T_1, T_2$, and the following are equivalent

(i) $X$ is a generalized Toeplitz operator with respect to the pairs $S_1, S_2$ and $T_1, T_2$,

(ii) $X = P_{H_1} P_{R_1} X$,

(iii) $X = P_{H_1 \cap R_1} X$,

(iv) $X(H_2) \subset H_1 \cap R_1$.

Additionally, if the operator $X$ satisfies (6) and one of the above conditions is fulfilled then $X$ is a generalized Toeplitz operator whose symbol is analytic.

Adversely, if $Y$ is an analytic symbol with respect to the pairs $S_1, S_2$ and $T_1, T_2$, then the related Toeplitz $X = P_{H_1} Y |_{H_2}$ operator satisfies (6) and conditions (iii) and (iv).

**Proof.** Let $X \in L(H_2, H_1)$ be such that $S_1^* X = X T_1^*$ and $S_2^* X = X T_2^*$. Then, by (2), we have

$$S_1 P_{H_1} P_{R_1} X = S_1 P_{H_1} P_{R_1} S_1^* X = P_{H_1} W_1 P_{R_1} S_1^* X$$

$$= P_{H_1} W_1 P_{R_1} W_1^* X = P_{H_1} W_1^* P_{R_1} X$$

$$= P_{H_1} P_{R_1} X,$$

since $R_1$ is reducing for $W_1$. Similarly we prove that $S_2 P_{H_1} P_{R_1} X T_2^* = P_{H_1} P_{R_1} X$. Hence the operator $P_{H_1} P_{R_1} X$ is a generalized Toeplitz operator with respect to the pairs $S_1, S_2$ and $T_1, T_2$.

Let now (i) be fulfilled. For nonnegative integers $n, m$ and $h_2 \in H_2$ we have

$$X h_2 = S_1^n S_2^m X T_1^n T_2^m h_2 = S_1^n S_2^m S_1^* X T_2^* T_1^m h_2 = S_1^n S_2^m S_1^* S_2^* X h_2 = P_{H_1} W_1^n W_2^m X h_2,$$

where $W_1 = P_{H_1} W_1, W_2 = P_{H_2} W_2, h_2 \in H_2.$
by (i) and (ii). Taking the limit when $n,m \to \infty$, by (3), we obtain
\[ P_{H_1} W_1^n W_2^m W_1^* W_2^* h_2 \to P_{H_1} P_{R_1} X h_2 \]
and (iii) is satisfied. Assume that (iii) holds, then
\[ X = (P_{H_1} P_{R_1})^n P_{H_1} X \to P_{H_1 \cap R_1} X, \]
by [7] p.192. The remaining implications in the equivalence (i) (iv) are straightforward.

Assume that (iv) hold and $Y$ is a symbol for $X$. Using Proposition 3.2 [10, Theorem 3.3] we have the following
\[ Y h_2 = \lim_{n,m \to \infty} W_1^n W_2^m P_{H_1} Y P_{H_2} V_1^* V_2^* h_2 = \lim_{n,m \to \infty} W_1^n W_2^m P_{H_1} Y T_1^* T_2^* h_2 \]
\[ = \lim_{n,m \to \infty} W_1^n W_2^m X T_1^* T_2^* h_2 = \lim_{n,m \to \infty} W_1^n W_2^m S_1^* S_2^* X h_2 \]
\[ = \lim_{n,m \to \infty} W_1^n W_2^m S_1^* S_2^* V_1^* V_2^* h_2 = P_{R_1} X h_2 \]
\[ = X h_2 \]
by (3) and (iv)]. Consequently $Y$ is an analytic symbol.

For the proof of the converse implication we assume that $Y$ is an analytic symbol, $X = Y|_{H_2}$ and $X(H_2) = Y(H_2) \subset H_1 \cap R_1$ by Proposition 3.2 [10, Proposition 3.2,(3)]. Moreover, for $i = 1,2$ and $h_2 \in H_2$, we have
\[ XT_i h_2 = YT_i h_2 = YV_i^* h_2 = W_i^* X h_2 = S_i^* X h_2, \]
which finishes the proof of the theorem.

**Example 5.2**
Let, as in Example 4.2, $T_{z_1}^*, T_{z_2}^*$ be the adjoints to multiplication operators by the independent variables in the space $H^2(T^2)$ as both pairs of contractions. Looking from one point of view, if an operator $X$ fulfills (i), then $T_{z_1} X = X T_{z_2}$ and $T_{z_2} X = X T_{z_2}$, which means that $X \in \{ T_{z_1}, T_{z_2} \}$. Hence, by [3] Theorem 11, the operator $X$ have to be equal to a Toeplitz operator $X = T_\psi$ with $\psi$ being a bounded holomorphic function on $D^2$. On the other hand the symbol $Y \in L(L^2(T^2))$ for $X$ is represented by a function $\psi \in L^\infty(T^2)$ such that $Y = M_\psi$. The analyticity of the symbol means that $Y(H_2(T^2)) = M_\psi(H_2(T^2)) \subset H_2(T^2)$. It forces $\psi$ to be holomorphic.

**Example 5.3**
Let $M_{z_1}, M_{z_2}$ be the multiplication operators by the independent variables in the space $L^2(T^2)$. Note that the operators $M_{z_1}, M_{z_2}$ doubly commute. Since they are unitary operators a minimal isometric dilation is the same pair $M_{z_1}, M_{z_2}$ and $K^+ = L^2(T^2)$. Hence the operator $X \in L(L^2(T^2))$ satisfies $X = M_{z_1} X M_{z_2}^*$ and $X = M_{z_2} X M_{z_2}^*$ if and only if $X$ belongs to the commutant $\{ M_{z_1}, M_{z_2} \}$. The commutant equals to the set of all multiplication operators $M_\psi$ with $\psi \in L^\infty(T^2)$. Thus each generalized Toeplitz operator $X$ with respect to both pairs being $M_{z_1}, M_{z_2}$ equals to its symbol $Y$ and there is a function $\psi \in L^\infty(T^2)$ such that $X = Y = M_\psi$. Moreover, the symbol $Y$ is analytic in our sense, since $Y L^2(T^2) \subset L^2(T^2)$.
Properties of two variables Toeplitz type operators

References


Eżbieta Król-Klimkowska
Institute of Mathematics
Pedagogical University
ul. Podchorążych 2
30-084 Kraków
Poland
E-mail: e.krol@onet.eu
Marek Ptak
Department of Applied Mathematics
University of Agriculture
ul. Balieka 253c
30-198 Kraków
Poland
and Institute of Mathematics
Pedagogical University
ul. Podchorążych 2
30-084 Kraków
Poland
E-mail: rmptak@cyf-kr.edu.pl

Received: May 23, 2016; final version: October 14, 2016;
available online: November 14, 2016.