Fekete-Szegö inequalities associated with $k$th root transformation based on quasi-subordination

Abstract. Recently, Haji Mohd and Darus [1] revived the study of coefficient problems for univalent functions associated with quasi-subordination. Inspired largely by this article, we provide coefficient estimates with $k$-th root transform for certain subclasses of $S$ defined by quasi-subordination.

1. Introduction

Denote by $A$ the class of all analytic functions of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}),$$

where $\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$. Also denote by $S$ the class of all analytic univalent functions of the form $[1]$ in $\mathbb{U}$. Let $k$ be a positive integer. A domain $\mathbb{D}$ is said to be $k$-fold symmetric if a rotation of $\mathbb{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathbb{D}$ to itself. A function $f$ is said to be $k$-fold symmetric in $\mathbb{U}$, if $f(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} f(z)$ for every $z \in \mathbb{U}$. If $f$ is regular and $k$-fold symmetric in $\mathbb{U}$, then

$$f(z) = b_1 z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \ldots.$$  \hspace{1cm} (2)

Conversely, if $f$ is given by (2), then $f$ is $k$-fold symmetric inside the circle of convergence of the series. For $f \in S$ given by (1), the $k$th root transformation is defined by

$$F(z) = [f(z^k)]^{\frac{1}{k}} = z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \ldots.$$ \hspace{1cm} (3)

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For two analytic functions $f$ and $g$, the function $f$ is quasi-subordinate to $g$ in the open unit disc $U$, if there exist analytic functions $h$ and $w$, with $|h(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$, such that \( \frac{f(z)}{h(z)} \prec g(z) \) is analytic in $U$ and written as

\[
\frac{f(z)}{h(z)} \prec g(z) \quad (z \in U)
\]

and it is denoted by

\[
f(z) \prec_q g(z) \quad (z \in U)
\]

and equivalently

\[
f(z) = h(z)g(w(z)) \quad (z \in U).
\]

It is interesting to note that if $h(z) \equiv 1$, then $f(z) = g(w(z))$, so that $f(z) \prec g(z)$ in $U$, where $\prec$ is a subordination between $f$ and $g$ in $U$. Also notice that if $w(z) = z$, then $f(z) = h(z)g(z)$ and it is said that $f$ is majorized by $g$ and written as $f(z) \ll g(z)$ in $U$ (see [2]).

Let $\varphi$ be an analytic and univalent function with positive real part in $U$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and let $\varphi$ map the unit disk $U$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The Taylor’s series expansion of such a function is

\[
\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots,
\]

where all coefficients are real and $B_1 > 0$.

Recently, El-Ashwah and Kanas [3] introduced and studied the following two subclasses:

\[
S^*_q (\gamma, \varphi) := \left\{ f \in \mathcal{A} : \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec_q \varphi(z) - 1, \quad z \in U, \quad \gamma \in \mathbb{C} \setminus \{0\} \right\}
\]

and

\[
K_q (\gamma, \varphi) := \left\{ f \in \mathcal{A} : \frac{1}{\gamma} \frac{zf''(z)}{f'(z)} \prec_q \varphi(z) - 1, \quad z \in U, \quad \gamma \in \mathbb{C} \setminus \{0\} \right\}.
\]

We note that, when $h(z) \equiv 1$, the classes $S^*_q (\gamma, \varphi)$ and $K_q (\gamma, \varphi)$ reduce respectively, to the familiar classes $S^*(\gamma, \varphi)$ and $K(\gamma, \varphi)$ of Ma-Minda starlike and convex functions of complex order $\gamma$ ($\gamma \in \mathbb{C} \setminus \{0\}$) in $U$ (see [4]). For $\gamma = 1$, the classes $S^*_q (\gamma, \varphi)$ and $K_q (\gamma, \varphi)$ reduce to the classes $S^*_q (\varphi)$ and $K_q (\varphi)$ studied by Haji Mohd and Darus [4]. When $h(z) \equiv 1$, the classes $S^*_q (\varphi)$ and $K_q (\varphi)$ reduce respectively, to well known subclasses $S^*(\varphi)$ and $K(\varphi)$ introduced and studied by Ma and Minda [3].

By specializing

\[
\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1)
\]

or

\[
\varphi(z) = \left( \frac{1 + z}{1 - z} \right)^{\beta} \quad (0 < \beta \leq 1)
\]

the classes $S^*(\varphi)$ and $K(\varphi)$ consist of functions known as the starlike (respectively convex) functions of order $\alpha$ or strongly starlike (respectively convex) functions of order $\beta$, respectively.
A function \( f \in A \) given by (1) is said to be in the class \( M^\delta,\lambda q(\gamma,\varphi) \), \( 0 \neq \gamma \in \mathbb{C}, \delta \geq 0 \), if the following quasi-subordination condition is satisfied

\[
\frac{1}{\gamma} \left( (1 - \delta) \frac{zf'(z)}{F(z)} + \delta \left( 1 + \frac{zf''(z)}{F'(z)} \right) - 1 \right) \prec_q \varphi(z) - 1 \quad (z \in \mathbb{U}),
\]

where

\[
F(z) = (1 - \lambda)f(z) + \lambda zf'(z) \quad (0 \leq \lambda \leq 1).
\]

We note that,

1. \( M^\delta,0 q(\gamma,\varphi) := M^\delta q(\gamma,\varphi) \),
2. \( M^\delta(1,\varphi) := M^\delta q(\varphi) \), \[1\] Definition 1.7, p.3,
3. \( M^0,0 q(\gamma,\varphi) := S^* q(\gamma,\varphi) \), \[3\] Definition 1.1, p.680,
4. \( S^*(1,\varphi) := S^* q(\varphi) \), \[1\] Definition 1.1, p.2,
5. \( M^{1,0} q(\gamma,\varphi) := K q(\gamma,\varphi) \), \[3\] Definition 1.3, p.681,
6. \( K(1,\varphi) := K q(\varphi) \), \[1\] Definition 1.3, p.2,
7. For \( 0 \neq \gamma \in \mathbb{C}, 0 \leq \lambda \leq 1, \)

\[
M^{0,\lambda} q(\gamma,\varphi) \equiv P q(\gamma,\lambda,\varphi)
\]

\[
= \left\{ f \in A : \frac{1}{\gamma} \left( \frac{zf'(z) + \lambda zf''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)} - 1 \right) \prec_q \varphi(z) - 1, z \in \mathbb{U} \right\},
\]

8. For \( 0 \neq \gamma \in \mathbb{C}, 0 \leq \lambda \leq 1, \)

\[
M^{1,\lambda} q(\gamma,\varphi) \equiv K q(\gamma,\lambda,\varphi)
\]

\[
= \left\{ f \in A : \frac{1}{\gamma} \left( \frac{zf'(z) + (1 + 2\lambda)zf''(z) + \lambda^2zf'''(z)}{zf'(z) + \lambda^2zf''(z)} - 1 \right) \prec_q \varphi(z) - 1, z \in \mathbb{U} \right\},
\]

Inspired by the papers of \[1\] \[3\] \[6\] \[7\] \[8\], we obtain the upper bounds \( |b_{k+1}| \) and \( |b_{2k+1}| \) for \( f \in M^{0,\lambda} q(\gamma,\varphi) \). Also, we investigate the Fekete-Szegö results for the class \( M^{0,\lambda} q(\gamma,\varphi) \) and its special cases. In order to discuss our results we provide the following lemmas.

**Lemma 1.1** \([9]\)

*Let \( w \) be an analytic function with \( w(0) = 0, |w(z)| < 1 \) and let*

\[
w(z) = u_1 z + u_2 z^2 + \ldots \quad (z \in \mathbb{U}).
\]

*Then for \( t \in \mathbb{C}, *

\[
|u_2 - tu_1^2| \leq \max[1; |t|].
\]
Lemma 1.2 \([9]\)

Let \(h\) be an analytic function with \(|h(z)| < 1\) and let
\[
h(z) = h_0 + h_1 z + h_2 z^2 + \ldots \quad (z \in \mathbb{U}).
\]  
(6)

Then
\[
|h_0| \leq 1 \quad \text{and} \quad |h_n| \leq 1 - |h_0|^2 \leq 1 \quad (n > 0).
\]

Lemma 1.3 \([10]\)

Let \(w\) be the analytic function with \(w(0) = 0\), \(|w(z)| < 1\) and given by (5). Then
\[
|w_1| \leq 1 \quad \text{and for any integer } n \geq 2,
\]
\[
|u_n| \leq 1 - |u_1|^2.
\]

2. Main result

Unless otherwise stated, throughout the sequel, we set \(f\) is of the form (1) and \(\varphi, h\) and \(w\) are given by (4), (6) and (5), respectively.

In the following theorem, we find Fekete-Szegő result for \(f \in \mathcal{M}_q^{\delta,\lambda}(\gamma, \varphi)\).

Theorem 2.1

Let \(f \in \mathcal{M}_q^{\delta,\lambda}(\gamma, \varphi)\) and let \(F\) be given by (3). Then
\[
|b_{k+1}| \leq \frac{|\gamma| B_1}{k(1 + \delta)(1 + \lambda)},
\]
\[
|b_{2k+1}| \leq \frac{|\gamma| \{B_1 + \max\{B_1, \frac{\gamma(1+3\delta)k+(1-k)\gamma(1+2\delta)(1+2\lambda)}{k(1+\delta)^2(1+\lambda)^2} |B_1^2 + |B_2|\}\}}{2k(1 + 2\delta)(1 + 2\lambda)}
\]

and for \(\mu \in \mathbb{C}\),
\[
|b_{2k+1} - \mu b_{2k+1}| \leq \frac{|\gamma| \{B_1 + \max\{B_1, \frac{\gamma(1+3\delta)k+(1-2\mu-k)\gamma(1+2\delta)(1+2\lambda)}{k(1+\delta)^2(1+\lambda)^2} |B_1^2 + |B_2|\}\}}{2k(1 + 2\delta)(1 + 2\lambda)}.
\]

Proof. Since \(f \in \mathcal{M}_q^{\delta,\lambda}(\gamma, \varphi)\), there exist \(\varphi\) and \(w\) with
\[
|\varphi(z)| \leq 1, \quad w(0) = 0 \quad \text{and} \quad |w(z)| < 1
\]
such that
\[
\frac{1}{\gamma} \left( (1 - \delta) \frac{z F_{\lambda} (z)}{F_{\lambda} (z)} + \delta \left( 1 + \frac{z F_{\lambda} '' (z)}{F_{\lambda}' (z)} \right) - 1 \right) = h(z)(\varphi(w(z)) - 1)
\]
(7)
and
\[
h(z)(\varphi(w(z)) - 1) = h_0 B_1 u_1 z + [h_1 B_1 u_1 + h_0 (B_1 u_2 + B_2 u_1^2)] z^2 + \ldots
\]
(8)
From (7) and (8) we get
\[
\frac{1}{\gamma} (1 + \delta)(1 + \lambda) a_2 = h_0 B_1 u_1
\]
(9)
\(k^{th}\)-root transformation based on quasi-subordination \[11\]

and

\[
\frac{1}{\gamma} \left[ 2(1 + 2\delta)(1 + 2\lambda) a_3 - (1 + 3\delta)(1 + \lambda)^2 a_2^2 \right] = h_1 B_1 u_1 + h_0 B_1 u_2 + h_0 B_2 u_1^2. \tag{10} \]

Equation (9) yields

\[
a_2 = \frac{\gamma h_0 B_1 u_1}{(1 + \delta)(1 + \lambda)}. \tag{11} \]

By subtracting (10) from (9) and using (11) we obtain

\[
a_3 = \frac{\gamma}{2(1 + 2\delta)(1 + 2\lambda)} \left[ h_1 B_1 u_1 + h_0 B_1 u_2 + \left( h_0 B_2 + \frac{\gamma h_0^2 B_1^2 (1 + 3\delta)}{(1 + \delta)^2} u_1^2 \right) \right]. \tag{12} \]

For a given \(f \in S\) of the form (1), we define \(F\) by

\[
F(z) = [f(z^k)]^\frac{1}{k} = z + a_2 z + \left( a_2 \right) \frac{z^2}{k} + \left( a_2 \right) \frac{z^3}{k} + \ldots
\]

where

\[
b_{k+1} = a_2 \frac{z}{k}, \quad b_{2k+1} = a_3 \frac{z^2}{k} - \left( a_2 \right) \frac{z^3}{k} \quad \text{and so on}. \tag{13} \]

It follows from (11), (12) and (13) that

\[
b_{k+1} = a_2 \frac{z}{k} = \frac{\gamma h_0 B_1 u_1}{k(1 + \delta)(1 + \lambda)}
\]

and

\[
b_{2k+1} = a_3 \frac{z^2}{k} - \left( a_2 \right) \frac{z^3}{k} = \frac{\gamma [h_1 B_1 u_1 + h_0 B_1 u_2 + (h_0 B_2 + \frac{\gamma h_0^2 B_1^2 (1 + 3\delta)}{(1 + \delta)^2} u_1^2)]}{2k(1 + 2\delta)(1 + 2\lambda)} \]

\[
- \left( \frac{k - 1}{2k^2} \right) \frac{\gamma^2 h_0^2 B_1^2 u_1^2}{(1 + \delta)^2(1 + \lambda)^2}.
\]

For \(\mu \in \mathbb{C}\) we get

\[
b_{2k+1} - \mu b_{k+1}^2 = \frac{\gamma B_1}{2k(1 + 2\delta)(1 + 2\lambda)} \left\{ h_1 u_1 + h_0 \left( u_2 + \left[ \frac{B_2}{B_1} + \frac{\gamma h_0 B_1 (1 + 3\delta)}{(1 + \delta)^2} \right] \right) \right\}
\]

\[
- \left( \frac{k - 1}{2k^2} \right) \frac{\gamma^2 h_0^2 B_1^2 u_1^2}{(1 + \delta)^2(1 + \lambda)^2} + \frac{\gamma h_0 B_1 (1 - 2\mu)(1 + 2\delta)(1 + 2\lambda)}{k(1 + \delta)^2(1 + \lambda)^2} u_1^2 \right\},
\]

Since \(h\) is analytic and bounded in \(U\) we have

\[
|h_n| \leq 1 - |h_0|^2 \leq 1 \quad (n > 0).
\]
By using this fact and the well-known inequality 
\[ |u_1| \leq 1, \]
from Lemma 1.3, we conclude that 
\[ |b_{k+1}| \leq \frac{|\gamma|B_1}{k(1+\delta)(1+\lambda)} \]
and 
\[ |b_{k+1} - \mu b_{k+1}^2| \leq \frac{|\gamma| B_1}{2k(1+2\delta)(1+2\lambda)} \left( 1 + |u_2 - \left[ \frac{-B_2}{B_1} \right] \right) - \frac{\gamma(1+3\delta)k - \gamma(1+2\delta)(1+2\lambda)k + \gamma(1-2\mu)(1+2\delta)(1+2\lambda)}{k(1+\delta)^2(1+\lambda)^2} \left| h_0 B_1 \right| u_1^2 \right) \}

In view of Lemma 1.1, we have 
\[ |b_{k+1} - \mu b_{k+1}^2| \leq \frac{|\gamma| \{ B_1 + \max \{ B_1, \frac{\gamma(1+3\delta)k + (1-2\mu-k)\gamma(1+2\delta)(1+2\lambda)}{k(1+\delta)^2(1+\lambda)^2} \} |B_1^2 + |B_2| \} \}}{2k(1+2\delta)(1+2\lambda)} \]

When \( \mu = 0 \), we obtain 
\[ |b_{k+1}| \leq \frac{|\gamma| \{ B_1 + \max \{ B_1, \frac{\gamma(1+3\delta)k + (1-2\mu)(1+2\delta)(1+2\lambda)}{k(1+\delta)^2(1+\lambda)^2} \} |B_1^2 + |B_2| \} \}}{2k(1+2\delta)(1+2\lambda)} \]
Hence we obtained the required inequalities of Theorem 2.1.

3. Concluding remarks and corollaries

In light of the special subclasses of the class \( M_q^{6,\lambda}(\gamma, \varphi) \), we have the following corollaries and remarks.

Remark 3.1
For \( \delta = \lambda = 0 \) and \( \gamma = 1 \), Theorem 2.1 reduces to [6, Theorem 2.1, p.619]. For \( \delta = \lambda = 0 \) and \( \gamma = k = 1 \), Theorem 2.1 reduces to [1, Theorem 2.1, p.4].

Corollary 3.2
If \( f \in K_q(\gamma, \varphi) \), then
\[ |b_{k+1}| \leq \frac{|\gamma|B_1}{2k}, \]
\[ |b_{2k+1}| \leq \frac{|\gamma|}{6k} \left( B_1 + \max \left\{ B_1, \frac{|(k+3)B_1^2 + |B_2|}{4k} \right\} \right) \]
and for \( \mu \in \mathbb{C} \),
\[ |b_{2k+1} - \mu b_{k+1}^2| \leq \frac{|\gamma|}{6k} \left( B_1 + \max \left\{ B_1, \frac{|k+3(1-2\mu)|B_1^2 + |B_2|}{4k} \right\} \right) \]
Remark 3.3
For $\gamma = k = 1$, Corollary 3.2 reduces to [1] Theorem 2.4, p.7.

Remark 3.4
Taking $\lambda = 0$ and $\gamma = 1$, Theorem 2.1 coincides with [6] Theorem 2.2, p.620. Also, for $\lambda = 0$ and $\gamma = k = 1$, Theorem 2.1 reduces to [1] Theorem 2.10, p.10.

Corollary 3.5
If $f \in P_q(\gamma, \lambda, \varphi)$, then

$$|b_{k+1}| \leq \frac{\gamma |B_1|}{k(1 + \lambda)},$$

$$|b_{2k+1}| \leq \frac{\gamma}{2k(1 + 2\lambda)} \left[ B_1 + \max \left\{ B_1, \frac{|\gamma| \left| \frac{1 - k - 2\lambda}{(1 - k + 2\lambda)k(1 + \lambda)} \right| B_1^2 + |B_2| \right\} \right]$$

and for $\mu \in \mathbb{C}$,

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{\gamma}{2k(1 + 2\lambda)} \left[ B_1 + \max \left\{ B_1, \frac{|\gamma| \left| \frac{3 - 2\mu - 2k\lambda}{(3 - 2\mu - 2k\lambda)(1 + 2\lambda)} \right| B_1^2 + |B_2| \right\} \right].$$

Corollary 3.6
If $f \in K_q(\gamma, \lambda, \varphi)$, then

$$|b_{k+1}| \leq \frac{\gamma |B_1|}{2k(1 + \lambda)},$$

$$|b_{2k+1}| \leq \frac{\gamma}{6k(1 + 2\lambda)} \left[ B_1 + \max \left\{ B_1, \frac{|\gamma| \left| \frac{3 - 2\mu - 2k\lambda}{4(3 - 2\mu - 2k\lambda)(1 + 2\lambda)} \right| B_1^2 + |B_2| \right\} \right]$$

and for $\mu \in \mathbb{C}$,

$$|b_{2k+1} - \mu b_{k+1}^2| \leq \frac{\gamma}{6k(1 + 2\lambda)} \left[ B_1 + \max \left\{ B_1, \frac{|\gamma| \left| \frac{3 - 2\mu - 2k\lambda}{4(3 - 2\mu - 2k\lambda)(1 + 2\lambda)} \right| B_1^2 + |B_2| \right\} \right].$$

Remark 3.7
For $\gamma = 1$ and $k = 1$, Corollary 3.6 corrects the results in [8] Theorem 2.1, p.195.

Remark 3.8
For $k = 1$, the results discussed in present paper coincide with the results obtained in [11].

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