On the Inners Formulation of stability condition within the shifted unit circle

Die Formulierung der Stabilitätsbedingung für den verschobenen Einheitskreis mittels der inneren Determinanten

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The concept of Inners recently introduced by Jury [1]—[2] is applied to stability conditions within the shifted unit circle as formulated by Tschauner [3]—[4]. The conditions becomes simply the coefficients of the characteristic polynomial to be positive and a certain $(n - 1) \times (n - 1)$ matrix be positive innerwise, i.e. (p.i). The symmetry of this matrix can be utilized to obtain an algorithm which can be easily computed using the double triangularization procedure discussed in [5].


1. Introduction

The recently introduced concept of Inners of an $[n \times n]$ matrix has found many applications in problems of stability and root distribution. It offers a unified approach to many problems of system theory. In this paper, this concept is further applied to problems of roots, lying within the shifted unit circle in the complex plane. This represents the stability condition of sampled-data systems described in the $\zeta$-plane as formulated by Tschauner [3].

To formulate the form of these conditions, we require to introduce the following definitions:

Definition 1:

Let a square $[n \times n]$ matrix be denoted as $[A_n]$. If we delete the first row and column and the $n^{th}$ row and column, we obtain $(n - 2) \times (n - 2)$ matrix $[A_{n-2}]$. We designate this matrix as an "Inner", if we repeat this process of deletion in the same manner, we thus obtain all the inner of $[A_n]$. If $n$, is even the inner of $[2 \times 2]$ matrix is referred to as the first inner. Similarly the $[4 \times 4]$ matrix is the second inner and so forth.

If $n$, is odd. The first inner is $[1 \times 1]$ matrix. The second inner is $[3 \times 3]$ matrix and so forth.

Example:

Let $n = 6$, the inners of a $[6 \times 6]$ matrix are formed as follows:

\[
[D_6] = \begin{bmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
 a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
 a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
 a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
 a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
 a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
\]

The two inners are $[D_2]$ and $[D_4]$.

Definition 2:

If the determinants of all the "inners", as well as of the matrix are all positive, we designate this matrix as "Positive Innerwise", or. (p.i).

Based on the above simple definitions we can reformulate the stability criterion in a direct and simplified form as follows.

2. Formulation of the stability criterion

Given the following real polynomial,

\[ F(z) = a_0z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1z + a_0, \]

with $a_0 \equiv 1$,

the necessary and sufficient condition that the roots of the above polynomial to lie inside the shifted unit circle (with center at $-1,0$) in the $z$-plane is formulated by Tschauner [3], [6]. It is expressed in inners form as follows:

a) For $n$-even:

\[
F(0) > 0, \quad F(-2) > 0,
\]

and the following $(n - 1) \times (n - 1)$ matrix be positive innerwise as well as the coefficients $A_{mv}$ (or half of them) given below be positive.

\[
[D_{n-1}] =
\begin{bmatrix}
 A_{0,0} & A_{0,1} & A_{0,2} & \ldots & 0 \\
 0 & A_{0,0} & A_{0,1} & \ldots & \ldots \\
 0 & 0 & A_{0,0} & A_{1,1} & \ldots \\
 0 & 0 & 0 & A_{1,1} & \ldots \\
 0 & 0 & 0 & 0 & A_{1,1}
\end{bmatrix}
\]


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where
\[ q = r = \frac{n - 2}{2} \]
and
\[ A_{mv} = \sum_{\mu=0}^{m} (-1)^{\mu} \binom{\mu + v}{\mu} a_{m-\mu}, \]
where
\[ m = 0, 1, 2, \ldots, n, \]
\[ r = 0, 1, 2, \ldots, q \text{ or } r. \]

b) For \( n \)-odd
\[ F(0) > 0, \quad F(-2) < 0, \]
and the following \((n - 1) \times (n - 1)\) matrix be positive inner-wise as well as the coefficients \( A_{mv} \) (or half of them) given below be positive.

\[
\begin{bmatrix}
A_{1,1} & A_{3,2} & A_{5,1} & A_{7,0} & 1 \\
0 & A_{1,1} & A_{3,2} & A_{5,1} & A_{7,0}
\end{bmatrix}
\]

where
\[ r = -\frac{n - 3}{2}, \quad q = r + 1 \]
and
\[ A_{mv} = \sum_{\mu=0}^{m} (-1)^{\mu} \binom{\mu + v}{\mu} a_{m-\mu}, \]
Note: \( A_{0,1} = a_{0,}, \)
\[ m = 0, 1, \ldots, n - 1, \]
\[ r = 0, 1, 2, \ldots, q \text{ or } r. \]

Remarks:
1. The above formulation is similar to Liénard-Chipart criteria for the roots to lie in the open left half plane. It is simpler than Hurwitz conditions, because we require conditions on \( n/2 \) inners and \( n/2 \) on the coefficients.
2. The generation of the matrix is simple and follows a certain pattern which forms a left triangle of zeros.
3. Based on this pattern, the double triangularization algorithm recently developed [5] can be applied very easily to this case.
4. Similar to other stability criteria one can also formulate the given conditions in terms of a positive definite symmetric matrix. This matrix can be used to prove the Lyapunov stability criterion for this case.

3. Illustrative Example (6)
Let \( n = 9, \)
\[ F(z) = \alpha_{9}z^9 + \alpha_{8}z^8 + \alpha_{7}z^7 + \alpha_{6}z^6 + \alpha_{5}z^5 + \alpha_{4}z^4 + \alpha_{3}z^3 + \ldots + \alpha_{2}z^2 + \alpha_1z + \alpha_0, \quad \alpha_9 = 1. \]
In this case \( n \) is odd, hence
\[ r = \frac{9 - 3}{2} = 3, \quad q = 3 + 1 = 4. \]

The \([d_8]\) matrix is as follows
\[
\begin{bmatrix}
A_{1,1} & A_{3,2} & A_{5,1} & A_{7,0} & 1 \\
0 & A_{1,1} & A_{3,2} & A_{5,1} & A_{7,0}
\end{bmatrix}
\]

The stability conditions are:

\[ F(0) > 0, \quad F(-2) < 0, \]
all the \( A_{mv} \)'s are positive and \([d_8]\) is positive inner-wise \([p. i] \), i.e
\[ \Delta_2 > 0, \quad \Delta_4 > 0, \quad \Delta_6 > 0, \quad \Delta_8 > 0. \]

4. Conclusion
In this paper a direct and simplified formulation of the stability criterion for linear sampled-data systems in the shifted unit circle is introduced. The conditions given are similar to Liénard-Chipart conditions for the stability of linear continuous systems. The inner representation is performed in a systematic way which shows a certain pattern that can be utilized for efficient computational procedure.

References: