

An extension of Klamka's method of minimum energy control to fractional positive discrete-time linear systems with bounded inputs

T. KACZOREK*

Faculty of Electrical Engineering, Bialystok University of Technology, 45D Wiejska St., 15-351 Bialystok, Poland

Abstract. The Klamka's method of minimum energy control problem is extended to fractional positive discrete-time linear systems with bounded inputs. Sufficient conditions for the existence of solution to the problem are established. A procedure for solving of the problem is proposed and illustrated by numerical example.

Key words: fractional, discrete-time, positive, minimum energy control, bounded inputs, procedure.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs [1, 2]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of the fractional calculus are given in the monographs [3–5]. The positive fractional linear systems have been investigated in [6–9]. Stability of fractional linear 1D discrete-time and continuous-time systems has been investigated in the papers [9–11] and of 2D fractional positive linear systems in [12]. The notion of practical stability of positive fractional discrete-time linear systems has been introduced in [13, 14] and the stability problem of systems with delays and systems consisting of n subsystems have been investigated in [15–19]. Some recent interesting results in fractional systems theory and its applications can be found in [20–25]. The minimum energy control problem for standard linear systems has been formulated and solved by J. Klamka [26–28] and for 2D linear systems with variable coefficient in [29]. The controllability and minimum energy control problem of fractional discrete-time linear systems has been investigated by Klamka in [30]. The minimum energy control of fractional positive continuous-time linear systems has been addressed in [31]. The Klamka's method of minimum energy control to positive continuous-time linear systems has been extended in [9, 31] and to positive discrete-time linear systems in [32].

In this paper the method is extended to fractional positive discrete-time linear systems with bounded-inputs.

The paper is organized as follows. In Sec. 2 some definitions and theorems concerning fractional positive discrete-time linear systems are recalled. The main result of the paper

is presented in Sec. 3, where the Klamka's method of minimum energy control is extended to fractional positive discrete-time linear systems with bounded inputs. Concluding remarks are given in Sec. 4.

The following notation is used: \mathfrak{R} – the set of real numbers, $\mathfrak{R}^{n \times m}$ – the set of $n \times m$ real matrices, $\mathfrak{R}_+^{n \times m}$ – the set of $n \times m$ matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$, I_n – the $n \times n$ identity matrix.

2. Preliminaries

The following definition of the fractional difference will be used

$$\Delta^\alpha x_k = \sum_{j=1}^k (-1)^j \binom{\alpha}{j} x_{k-j}, \quad (1)$$

$$0 < \alpha < 1, \quad k \in \mathbb{Z}_+ = \{0, 1, \dots\},$$

where $\alpha \in \mathfrak{R}$ is the order of the fractional difference, and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{\alpha(\alpha-1) \cdots (\alpha-j+1)}{j!} & \text{for } j = 1, 2, \dots \end{cases} \quad (2)$$

Consider the fractional discrete-time linear system, described by the state-space equation

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+, \quad (3)$$

where $x_k \in \mathfrak{R}^n$ is the state vector, $u_k \in \mathfrak{R}^m$ is the input vector and $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$.

Using (1) we can write the equation (3) in the following form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k. \quad (4)$$

*e-mail: kaczorek@isep.pw.edu.pl

Theorem 1. The solution of Eq. (4) is given by

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} B u_i, \quad k \in Z_+ \quad (5)$$

and the matrix Φ_i can be computed from the formula

$$\Phi_{k+1} = (A + I_n \alpha) \Phi_k + \sum_{j=2}^{k+1} (-1)^{j+1} \binom{\alpha}{j} \Phi_{k-j+1}, \quad (6)$$

$$\Phi_0 = I_n.$$

Definition 1. The system (3) (or (4)) is called the fractional (internally) positive system if $x_k \in \mathbb{R}_+^n$, $k \in Z_+$ for any initial conditions $x_0 \in \mathbb{R}_+^n$ and all input sequences $u_k \in \mathbb{R}_+^m$, $k \in Z_+$.

Theorem 2. The fractional system (3) for $0 < \alpha < 1$ is positive if and only if

$$A\alpha = A + I_n \alpha \in \mathbb{R}_+^{n \times n}, \quad B \in \mathbb{R}_+^{n \times m}. \quad (7)$$

Proof is given in [9].

Definition 2. The fractional positive system (3) is called reachable in q steps if for every given final state $x_f \in \mathbb{R}_+^n$ there exists an input sequence $u_k \in \mathbb{R}_+^m$ for $k = 0, 1, \dots, q-1$ such that $x_q = x_f$.

A column of the form $a e_i$, $i = 1, \dots, n$, $a > 0$ is called monomial, where e_i , $i = 1, \dots, n$ is the i -th column of the identity matrix. A square matrix A is called monomial if in each row and in each column only one entry is positive and the remaining entries are zero.

Theorem 3. The fractional positive system (3) is reachable in q steps if and only if the matrix

$$R_q = [B \quad \Phi_1 B \quad \dots \quad \Phi_{q-1} B] \quad (8)$$

contains n linearly independent monomial columns.

Proof is given in [9].

3. Minimum energy control problem

Consider the fractional positive discrete-time system (3). If the system is reachable in q steps then there exist many ($m > 1$) input sequences that steer the state of the system from $x_0 = 0$ to the given final state $x_f \in \mathbb{R}_+^n$. Among these input sequences we are looking for sequence $u_i \in \mathbb{R}_+^m$ for $i = 0, 1, \dots, q-1$ that satisfy the condition

$$u_i \leq U \in \mathbb{R}_+^m \quad \text{for } i = 0, 1, \dots, q-1 \quad (9)$$

which minimizes the performance index

$$I(u) = \sum_{j=0}^{q-1} u_j^T Q u_j, \quad (10)$$

where $Q \in \mathbb{R}_+^{n \times n}$ is a symmetric positive defined matrix.

The minimum energy control problem for the fractional positive discrete-time linear systems (3) can be stated as follows: Given the matrices A , B , degree α of the system (3), the final state $x_f \in \mathbb{R}_+^n$ and the matrix $Q \in \mathbb{R}_+^{n \times n}$ of

the performance index (2). Find an input sequence $u_i \in \mathbb{R}_+^m$ for $i = 0, 1, \dots, q-1$ satisfying the condition (9) that steers the state of the system (3) from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (10).

To solve the problem we define the matrix

$$W_q = W(q, Q) = R_q Q_q^{-1} R_q^T \in \mathbb{R}^{n \times n}, \quad (11)$$

where R_q is defined by (8) and

$$Q_q^{-1} = \text{blockdiag} [Q^{-1}, \dots, Q^{-1}] \in \mathbb{R}_+^{qm \times qm}. \quad (12)$$

If all columns of the matrix R_q are monomial and the matrix Q is diagonal then the matrix (11) is also diagonal. All columns of the matrix R_q are monomial only if the pair (A, B) is reachable and only if all columns of the matrix $[A, B]$ are monomial [9, 31].

If the fractional positive system (3) is reachable in q steps and

$$W_q x_f^{-1} \in \mathbb{R}_+^n \quad (13)$$

then the input sequence

$$\hat{u}_q = \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix} = Q_q^{-1} R_q^T W_q^{-1} x_f \in \mathbb{R}_+^{qm} \quad (14)$$

steers the fractional positive system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ since

$$x_q = R_q \hat{u}_q = R_q Q_q^{-1} R_q^T W_q^{-1} x_f = x_f. \quad (15)$$

Theorem 4. Let the fractional positive system (3) be reachable in q steps and the conditions (12) and (13) be satisfied. Let $\bar{u}_k \in \mathbb{R}_+^m$ for $k = 0, 1, \dots, q-1$ be an input sequence satisfying (9) that steers the state of the system (3) from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Then the input sequence (14) satisfying (9) also steers the state of the system (3) from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (10), i.e. $I(\hat{u}) \leq I(\bar{u})$. The minimal value of the performance index (10) is given by

$$I(\hat{u}) = x_f^T W_q^{-1} x_f. \quad (16)$$

Proof. If the fractional system (3) is positive, reachable in q and the conditions (12) and (13) are met then for $x_f \in \mathbb{R}_+^n$ we have $\hat{u}_q \in \mathbb{R}_+^{qm}$ and (15) holds. Both input sequences \hat{u}_q and \bar{u}_q steer the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and we have $x_f = R_q \hat{u}_q = R_q \bar{u}_q$, i.e.

$$R_q [\hat{u}_q - \bar{u}_q] = 0. \quad (17)$$

Using (17) and (14) we shall show that

$$[\hat{u}_q - \bar{u}_q]^T Q_q \hat{u}_q = 0, \quad (18)$$

where $Q_q = \text{blockdiag} [Q, \dots, Q] \in \mathbb{R}_+^{qm \times qm}$.

Transposing (17) and postmultiplying it by $W_q^{-1} x_f$ we obtain

$$[\hat{u}_q - \bar{u}_q]^T R_q^T W_q^{-1} x_f = 0. \quad (19)$$

Using (14) and (18) we obtain (19) since

$$u \hat{u}_q - \bar{u}_q^T Q_q \hat{u}_q = [\hat{u}_q - \bar{u}_q]^T Q_q Q_q^{-1} R_q^T W_q^{-1} x_f \quad (20)$$

$$= [\hat{u}_q - \bar{u}_q]^T R_q^T W_q^{-1} x_f = 0$$

and $Q_q Q_q^{-1} = I_{qm}$.

Taking into account (18) it is easy to show that

$$\bar{u}_q^T Q_q \bar{u}_q = \hat{u}_q^T Q_q \hat{u}_q + [\hat{u}_q - \bar{u}_q]^T Q_q [\bar{u}_q - \hat{u}_q]. \quad (21)$$

From (21) it follows that the inequality $I(\hat{u}) \leq I(\bar{u})$ holds since

$$[\bar{u}_q - \hat{u}_q]^T Q_q [\bar{u}_q - \hat{u}_q] \geq 0. \quad (22)$$

To find the minimal value of the performance index (16) we substitute (14) into (10) and using (11) we obtain

$$\begin{aligned} I(\hat{u}) &= \hat{u}_q^T Q_q \hat{u}_q = [Q_q^{-1} R_q^T W_q^{-1} x_f]^T Q_q [Q_q^{-1} R_q^T W_q^{-1} x_f] \\ &= x_f^T W_q^{-1} R_q Q_q^{-1} R_q^T W_q^{-1} x_f = x_f^T W_q^{-1} x_f \end{aligned} \quad (23)$$

since $W_q^{-1} R_q Q_q^{-1} R_q^T = I_n$.

Remark 1. If the components of U in (9) decrease then the number q of steps needed to transfer the state of the system (3) from $x_0 = 0$ to $x_f \in \mathfrak{R}_+^n$ increases.

Therefore from Theorem 4 and Remark 1 we have the following theorem.

Theorem 5. If the fractional positive system (3) is reachable in q steps, all columns of the reachability matrix (9) are monomial and the conditions (12) and (13) are met then the minimum energy control problem has a solution satisfying the condition (9) for arbitrary given U .

The optimal input sequence (14) and the minimal value of the performance index (16) can be computed by the use of the following procedure.

Procedure 1.

Step 1. Knowing A, B, Q and α and using (6), (8), (11) compute the matrices R_q and W_q for a chosen q such that the matrix R_q contains at least n linearly independent monomial columns.

Step 2. Using (14) find an input sequence $u_k \in \mathfrak{R}_+^m$ for $k = 0, 1, \dots, q - 1$ satisfying (9). If the condition (9) is not satisfied then increase q by one and repeat the computation for $q + 1$. Note that if the matrix W_q is diagonal then after some number of steps we obtain the desired input sequence satisfying (9).

Step 3. Using (16) compute the minimal value of the performance index $I(\hat{u})$.

Example 1. Consider the fractional discrete-time linear system (3) with $\alpha = 0.5$ and the matrices

$$A = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (24)$$

and the performance index (10) with $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Find an input sequence $u_k \in \mathfrak{R}_+^m$ for $k = 0, 1, \dots, q - 1$ satisfying (9) with

$$u_k \leq U = \begin{bmatrix} \frac{1}{1.09} \\ \frac{1}{1.16} \end{bmatrix} \quad \text{for } k = 0, 1, \dots \quad (25)$$

that steers the state of the system from $x_0 = 0$ to final state $x_f \in [1 \ 1]^T$ and minimizes the performance index.

Note that the fractional system (3) with (24) is positive since

$$A_\alpha = A + I_2 \alpha = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix} \in \mathfrak{R}_+^{2 \times 2} \quad (26)$$

$$\text{and } B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathfrak{R}_+^{2 \times 2}.$$

The system is also reachable in q steps for $q = 1, 2, \dots$ since the reachability matrix

$$\begin{aligned} R_q &= [B \quad \Phi_1 B \quad \dots \quad \Phi_{q-1} B] \\ &= \begin{bmatrix} 0 & 1 & 0 & 0.4 & 0 & 0.215 & \dots \\ 1 & 0 & 0.3 & 0 & 0.285 & 0 & \dots \end{bmatrix} \end{aligned} \quad (27)$$

has only monomial columns.

Using Procedure 1. we obtain the following:

Step 1. Using (11) we obtain

$$\begin{aligned} W_2 &= R_2 Q_2^{-1} R_2^T \\ &= \begin{bmatrix} 0 & 1 & 0 & 0.4 \\ 1 & 0 & 0.3 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0.3 \\ 0.4 & 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1.16 & 0 \\ 0 & 1.09 \end{bmatrix}. \end{aligned} \quad (28)$$

Step 2. Using (14), (27) and (28) we obtain

$$\begin{aligned} \hat{u}_2 &= \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = Q_2^{-1} R_2^T W_2^{-1} x_f \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0.3 \\ 0.4 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{1.16} & 0 \\ 0 & \frac{2}{1.09} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned} \quad (29)$$

$$= \begin{bmatrix} \frac{1}{1.09} \\ \frac{1}{1.16} \\ \frac{0.3}{1.09} \\ \frac{0.4}{1.16} \end{bmatrix}.$$

The input sequence (29) does not satisfy the condition (25) and we compute

$$W_3 = R_3 Q_3^{-1} R_3^T = \begin{bmatrix} 0 & 1 & 0 & 0.4 & 0 & 0.215 \\ 1 & 0 & 0.3 & 0 & 0.285 & 0 \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0.3 \\ 0.4 & 0 \\ 0 & 0.285 \\ 0.215 & 0 \end{bmatrix} \quad (30)$$

$$= \frac{1}{2} \begin{bmatrix} 0.6031 & 0 \\ 0 & 0.5856 \end{bmatrix}$$

and

$$\hat{u}_3 = \begin{bmatrix} u_2 \\ u_1 \\ u_0 \end{bmatrix} = Q_3^{-1} R_3^T W_3^{-1} x_f$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0.3 \\ 0.4 & 0 \\ 0 & 0.285 \\ 0.215 & 0 \end{bmatrix} \quad (31)$$

$$\frac{1}{2} \begin{bmatrix} 0.6031 & 0 \\ 0 & 0.5856 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8538 \\ 0.8290 \\ 0.2561 \\ 0.3316 \\ 0.2433 \\ 0.1782 \end{bmatrix}$$

The input sequence (31) satisfies the condition (25) and by Theorem 4 is the optimal one that steers the state of the system in 3-steps from $x_0 = 0$ to final state $x_f \in [1 \ 1]^T$ and

minimizes the performance index (16) for $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Step 3. The minimal value (16) of the performance index (10) is equal to

$$I(\hat{u}) = x_f^T W_3^{-1} x_f = [1 \ 1]$$

$$\frac{1}{2} \begin{bmatrix} 0.6031 & 0 \\ 0 & 0.5856 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3.3657. \quad (32)$$

4. Concluding remarks

The Klamka’s method of the minimum energy control problem has been extended to fractional positive discrete-time linear systems with bounded inputs. Sufficient conditions for the existence of solution to the problem have been established

(Theorem 4 and 5). A procedure (Procedure 1) for computation of the optimal input sequence (14) and the minimal value of the performance index (10) has been proposed. The effectiveness of the procedure has been demonstrated on a numerical example (Example 1).

The considerations can be extended to fractional positive continuous-time linear systems with bounded inputs and to fractional positive linear systems with delays and bounded inputs. An open problem is an extension of the method to continuous-discrete 2D linear systems.

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