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Efficient design of FIR filter based low-pass differentiators for biomedical signal processing

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Abstract: This paper describes an alternative design of linear phase low-pass differentiators with a finite impulse response (type III FIR filter). To reduce the number of necessary filter coefficients, the differentiator’s transfer function is approximated by a Fourier series of a triangle function. Thereby the filter’s transition steepness towards the stopband is intentionally reduced. It can be shown that the proposed design of low-pass differentiators yields to similar results as other published design recommendations, while the filter order can be significantly reduced.

Keywords: biomedical signal processing; event detection; FIR filter design; low-pass differentiator.

1 Introduction

When processing different types of biomedical signals one is often interested in extracting information about rapid transient characteristics in the signal like local extrema or inflection points. For example common methods for the determination of foot points in arterial pulse waves are based on calculating the first and second derivatives [1]. If the signal is disturbed by broadband noise or short time interferences classical numerical differentiators for calculating first or higher order derivatives are difficult to use because all differentiators have an inherent high-pass characteristic and amplify higher frequency components in the signal. The frequency response of an ideal n-th order continuous-time differentiator is given by

\[ H_{D,n}(j\omega) = (j\omega)^n. \] (1)

As it can be seen in (1), an ideal first order differentiator has a pure imaginary transfer function with a linear slope. To compensate the high-pass characteristics, a low-pass filter can be used to suppress the amplification of high frequency disturbances in the signal. This combination yields to a low-pass differentiator. The frequency response of an ideal first order low-pass differentiator is given by (2), where \( \Omega_c \) denotes the cut-off frequency of the low-pass filter. Figure 1 illustrates the frequency responses of an ideal differentiator and an ideal low-pass differentiator.

\[ H_{LPD}(j\omega) = \begin{cases} j\omega & \text{if } |\omega| \leq \Omega_c \\ 0 & \text{otherwise} \end{cases} \] (2)

Under the constraints of linear and time-invariant system behavior, low-pass differentiators can be implemented either as a cascade of a low-pass filter and a differentiator [2, 3] or as a combination in one filter [4, 5]. The proposed design in this paper leads to a single linear phase FIR filter. Its characteristics are compared to the design of maximally flat low-pass differentiators by Selesnick [5].

2 Methods

To avoid filters with higher orders because of the sharp transition between pass- and stopband, the proposed design approximates the transfer function by a Fourier series expansion of a triangle function. Thereby the reduced transition steepness towards the stopband is chosen on purpose as a tradeoff for the smaller filter order required. Let \( T \) denote the sampling interval of a discrete-time system, the angular sampling frequency is given by

\[ \Omega = \frac{2\pi}{T}. \] (3)

The ideal approximation of the proposed filter as a discrete-time system can be formulated piecewise as follows:

\[ H_A(j\omega) = \begin{cases} 0 & -\Omega_c < \omega < -2\Omega_c \\ -j(\omega + 2\Omega_c) & -2\Omega_c \leq \omega < -\Omega_c \\ j\omega & -\Omega_c \leq \omega < \Omega_c \\ -j(\omega - 2\Omega_c) & \Omega_c \leq \omega < 2\Omega_c \\ 0 & 2\Omega_c < \omega < \frac{\Omega}{2} \\ \Omega\text{-periodic} & \text{otherwise} \end{cases} \] (4)
Figure 1: Amplitude responses of an ideal differentiator and an ideal low-pass differentiator (both first order).

Figure 2 illustrates the ideal approximation of an $\Omega$-periodic triangle function. Because of the periodicity in $\Omega$, the triangle function defined in (4) can be approximated by a Fourier series with

$$H_\Delta(j\omega) = H_\Delta(j\omega + j\Omega) = \sum_{n=-\infty}^{\infty} h_n e^{-jn\omega T}.$$ (5)

The calculation of the Fourier coefficients $h_n$ is given by

$$h_n = \frac{1}{\Omega/2} \int_{-\Omega/2}^{\Omega/2} H_\Delta(j\omega)e^{jn\omega T} d\omega.$$ (6)

Inserting the piecewise definition from (4) into (6) yields to the final definition of the coefficients for the Fourier series as an approximation of the triangle function.

$$h_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{2}{\pi T n^2} \sin (n\Omega_c T) [1 - \cos (n\Omega_c T)] & \text{if } n \neq 0 \end{cases}$$ (7)

From (7) follows that the Fourier coefficients converge by $1/n^2$ to zero. Taking into account that the cut-off frequency $\Omega_c$ must be in the range of $0 < \Omega_c \leq \Omega/4$, a factor $\gamma$ can be defined.

$$0 < \Omega_c \leq \frac{\Omega}{4} \Rightarrow \Omega_c = \gamma \frac{\Omega}{4} \text{ with } 0 < \gamma \leq 1$$ (8)

Using $\gamma$ in (7) and considering the definition of $\Omega$ from (3) yields to a definition of the Fourier coefficients, where the arguments of the trigonometric functions become independent of the sampling interval:

$$h_n = \begin{cases} 0 & \text{if } n = 0 \\ -\frac{2}{\pi T n^2} \sin \left(\frac{\gamma n\pi}{2}\right) [1 - \cos \left(\frac{\gamma n\pi}{2}\right)] & \text{if } n \neq 0 \end{cases}$$ (9)

From (7) and (9) it can be seen that the coefficients have an odd symmetry ($h_n = -h_{-n}$). To realize a filter with an finite impulse response (FIR filter), the Fourier series has to be truncated after a finite number of elements. Considering the symmetry, the series can be truncated after $2L + 1$ elements for $-L \leq n \leq L$. This results in an approximation by

$$\hat{H}_\Delta(j\omega) = \sum_{n=-L}^{L} h_n e^{-jn\omega T}.$$ (10)

Figure 3 illustrates the Fourier series based approximation of the triangle function from Figure 2 with $\gamma = 2/3$ and 9 coefficients. Due to the truncation of the Fourier series expansion, ripples in the pass- and stopband of the proposed filter’s transfer function are present. The amplitude of these ripples decreases with increasing number of coefficients. To finish the filter design process the complex exponentials in (10) have to be substituted by $z = e^{j\omega T}$. Furthermore, the resulting impulse response has to be shifted by $L$ sampling intervals to realize a causal system. This leads to the z-transform of the system

$$\hat{H}_\Delta(z) = \sum_{n=0}^{N=2L} \tilde{h}_n z^{-n} \text{ with } z = e^{j\omega T}.$$ (11)

The filter coefficients $\tilde{h}_n$ can directly be computed via the Fourier coefficients by

$$\tilde{h}_n = h_{n-L} \text{ with } n = 0..N \text{ and } N = 2L.$$
3 Results

The proposed design of triangle approximated low-pass differentiators (Δ-LPD) creates type III FIR filters (even filter order, odd symmetry). This implies that the frequency response has a bandpass characteristic with linear phase and a constant group delay of \( N/2 \) samples. Figure 4A shows the amplitude responses of six implementations of the proposed design with identical filter order \( N=14 \) but varying cut-off frequencies. The truncation of the Fourier series yields to obvious ripples in the shape of the amplitude responses. These ripples lead to regions where the slope in the each filter’s passband is larger than one. Compared to the ideal differentiator this means an amplification of the corresponding frequency components. To avoid these amplifications, the coefficients have to be scaled to the maximum slope of the amplitude response as shown in Figure 4B.

Compared to the well published design of maximally-flat low-pass differentiators by Selesnick [5], the filter order can significantly be reduced even if a similar bandwidth of the filter has to be realized. Figure 5 illustrates a comparison between the proposed design and two implementations of Selesnick’s design. It can be seen that in this case the filter order can be reduced by approximately 36% while bandwidths and cut-off frequencies are similar.

To verify that the proposed design provides an efficient way to calculate the derivative of a signal, Figure 6 illustrates the exemplary usage for processing pulse pressure signals. Figure 6A shows a synthesized waveform of an arterial blood pressure signal. A single blood pressure wave is generated by superimposing three Gaussian functions as shown in (12). The parameters are obtained by minimizing the least squared errors to a real waveform from a clinical dataset of a healthy person (sampling frequency \( f_s = 128 \) Hz). The single pressure wave is repeated several times while the transition between two adjacent waves is smoothed by a spline interpolation over six samples.

\[
x(t) = a_0 + \sum_{m=1}^{3} a_m e^{-b_m(t-c_m)^2} \tag{12}
\]

\[
\frac{dx(t)}{dt} = \dot{x}(t) = \sum_{m=1}^{3} -2a_mb_m(t-c_m)e^{-b_m(t-c_m)^2} \tag{13}
\]

The exact mathematical derivative of the signal is given by (13) and also shown in Figure 6B. Figure 6C illustrates a comparison between this derivative and the output of an implementation of the proposed filter design. The filter order was chosen to \( N=14 \) and the cut-off frequency was \( \Omega_c = 0.289 \). The results of filtering the same signal by an implementation of Selesnick’s design is illustrated in Figure 6D where the filter order was set to \( N=22 \). The shapes of both filtering results are quite similar, as it can be seen by the zero crossings in Figure 6C and D. The sum of squared errors of the proposed design is about 7.1% larger than for the Selesnick design which can be attributed to the slightly smaller cut-off frequency and bandwidth as well as the greater attenuation of the Δ-LPD filter. As a consequence the deviation of the ideal differentiator’s amplitude response begins at smaller frequencies compared to Selesnick’s design. Furthermore the minor ripples in the passband lead to somewhat greater deviation in the waveform of the filter’s output.

The results of using the same two filters to detect extreme values and inflection points in noise disturbed signals are shown in Table 2. Almost all present extreme

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Figure 4: (A) Amplitude responses of the proposed filter design (varying cut-off frequencies, identical filter order); (B) same amplitude responses scaled to the maximum slope in the passband.

Figure 6: (A) Amplitude responses of the proposed filter design (varying cut-off frequencies, identical filter order); (B) same amplitude responses scaled to the maximum slope in the passband.
values in the signal are robustly found by both filters. The advantage of the proposed design is the reduction of falsely detected additional extreme and inflection points (~9.5% and 14.2% at 40 dB AWGN) compared to the Selesnick design (~13.5% and 18.4% at 40 dB AWGN) with increasing noise.

Table 1: Parameters for the synthesized blood pressure signal.

<table>
<thead>
<tr>
<th>m</th>
<th>$a_m$ (mmHg)</th>
<th>$b_m$ (s$^{-2}$)</th>
<th>$c_m$ (s)</th>
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<tbody>
<tr>
<td>0</td>
<td>76.25</td>
<td>–</td>
<td>–</td>
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<td>1</td>
<td>33.64</td>
<td>351.96</td>
<td>0.0851</td>
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<td>2</td>
<td>21.04</td>
<td>75.205</td>
<td>0.1783</td>
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<tr>
<td>3</td>
<td>14.78</td>
<td>58.569</td>
<td>0.4196</td>
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Table 2: Detection of extreme values and inflection points in the synthesized waveform with different levels of additive noise (AWGN).

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>40 dB AWGN</th>
<th>60 dB AWGN</th>
<th>80 dB AWGN</th>
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<tr>
<td></td>
<td>(\Delta)-LPD</td>
<td>Sel.-LPD</td>
<td>(\Delta)-LPD</td>
</tr>
<tr>
<td>Maxima</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Detection rate</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>False detection rate</td>
<td>0.095</td>
<td>0.135</td>
<td>0</td>
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<tr>
<td>Mean deviation [samples]</td>
<td>-0.623</td>
<td>-0.648</td>
<td>-0.499</td>
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<tr>
<td>Minima</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Detection rate</td>
<td>0.999</td>
<td>0.999</td>
<td>1</td>
</tr>
<tr>
<td>False detection rate</td>
<td>0.096</td>
<td>0.136</td>
<td>0</td>
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<tr>
<td>Mean deviation [samples]</td>
<td>-1.309</td>
<td>-1.274</td>
<td>-1.111</td>
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<tr>
<td>I.-points</td>
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<td></td>
<td></td>
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<tr>
<td>Detection rate</td>
<td>0.873</td>
<td>0.887</td>
<td>0.916</td>
</tr>
<tr>
<td>False detection rate</td>
<td>0.142</td>
<td>0.184</td>
<td>0</td>
</tr>
<tr>
<td>Mean deviation [samples]</td>
<td>0.216</td>
<td>0.204</td>
<td>0.152</td>
</tr>
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</table>

4 Conclusion

The proposed design for low-pass differentiators shows similar performance as published designs but considerably reduces the number of filter coefficients. Because of the triangle approximation, the coefficients converge by \(1/n^2\) to zero and additional windowing of the coefficients as known in standard FIR filter designs is not necessary. Analysis of synthesized waveforms indicates that the slightly greater deviation from the ideal differentiator in the passband has no considerable impact on the reliability and accuracy of detected events. The reduced filter order allows for a memory-efficient implementation with small filter delay which makes the method particularly attractive for real-time operations in mobile applications.

Author’s Statement

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References