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Communications in Mathematics 24 (2016) 115–123

Homogeneous variational problems and Lagrangian sections

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Abstract. We define a canonical line bundle over the slit tangent bundle of a manifold, and define a Lagrangian section to be a homogeneous section of this line bundle. When a regularity condition is satisfied the Lagrangian section gives rise to local Finsler functions. For each such section we demonstrate how to construct a canonically parametrized family of geodesics, such that the geodesics of the local Finsler functions are reparametrizations.

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On 15 August 2016 Gennadi Sardanashvily gave the first of three lectures on 'Noether Theorems and Applications' at the 21st International Summer School on Global Analysis and its Applications, in Poprad, Slovakia. There were two other series of lectures at this summer school; I gave one and Salvatore Capozziello gave the other. Gennadi discussed with some of us the volume of Communications in Mathematics which he was planning to edit; we could not possibly imagine that two weeks later he would no longer be with us and that the volume would be issued in his memory.

1 Introduction

Some years ago Massa *et al.* proposed an approach to the study of time-dependent Lagrangian mechanics where the Lagrangian specifying the variational problem, rather than being a function or a 1-form defined on the first jet bundle, was instead a section of a line bundle [6]. A claimed advantage of this approach was that equivalent Lagrangians in the usual sense, namely Lagrangians differing by a total derivative and therefore giving rise to the same Euler-Lagrange equations, all corresponded to the same Lagrangian section.

2010 MSC: 53C60, 53C22
Key words: Finsler geometry, line bundle, geodesics
DOI: 10.1515/cm-2016-0008

In this paper we consider how a similar approach of using a section of a line bundle might provide an insight into 'pre-Finsler' geometry, where the function specifying the variational problem, now defined on a slit tangent manifold, is required to be positively homogeneous. True Finsler geometry requires, of course, that the Finsler function be positive and strongly convex; initially we do not impose these extra conditions and simply require the line bundle section to give rise to positively homogeneous functions. We are, however, able to specify a suitable regularity condition such that locally the section gives rise to Finsler functions (see [2], [3]), and we are also able to construct a family of parametrized geodesics such that the local Finsler geodesics are reparametrizations.

The background to our approach involves the construction of a manifold with an additional coordinate, proposed originally by T. Y. Thomas [8] and considered more recently in [5], [7]; we largely follow the construction and notations of [4], and summarise the details in Section 2. In Section 3 we introduce the idea of Lagrangian sections, and in Section 4 we explain the construction of geodesics by using sprays.

2 Preliminaries

Let M be a differentiable manifold (supposed as usual to be smooth, finite-dimensional, Hausdorff and paracompact); put $n = \dim M$. The bundle $\bigwedge^n T^*M \to M$, the bundle of 'oriented' volume elements, has a global section precisely when M is orientable. We shall construct from this a new bundle $\nu\colon \mathcal{V}M \to M$ of 'unoriented' volume elements which we shall call simply the *volume bundle* and which will always admit global sections.

Define an equivalence relation on the nonzero elements $\omega \in \bigwedge^n T^*M \to M$ by $\omega_1 \sim \omega_2$ if, and only if, $\omega_1 = \pm\omega_2$, and write $[\omega]$ for the equivalence class containing ω; then put

$$\mathcal{V}M = \{[\omega] : \omega \in \textstyle\bigwedge^n T^*M,\ \omega \neq 0\}$$

and

$$\nu([\omega]) = x\,, \qquad \omega \in \textstyle\bigwedge^n T^*_x M\,.$$

Any Riemannian metric g on M gives rise to its Riemannian 'volume form' σ_g which is determined only up to sign and is therefore a global section of the volume bundle $\mathcal{V}M \to M$ rather than of the oriented volume bundle. Furthermore, M admits Riemannian metrics (by, for example, using the Whitney Embedding Theorem and restricting the canonical Euclidean metric on $\mathbb{R}^{2n+1}$ to M) so that $\mathcal{V}M \to M$ admits global sections.

We note also that $\bigwedge^n T^*M$ supports a tautological n-form Θ given by $\Theta_\omega = \nu^*\omega$, and that $\mathrm{d}\Theta$ is a natural volume form on this $(n+1)$-dimensional manifold; we therefore obtain an odd scalar density $[\mathrm{d}\Theta]$ on $\mathcal{V}M$.

There is a natural (right) action μ^1 of $\mathbb{R}_+$ on the fibres of the volume bundle, given by multiplication, so that $\mu^1([\omega], s) = [s\omega]$; we may also consider weighted actions μ^p given by $\mu^p([\omega], s) = [s^p\omega]$ for $p \in \mathbb{R}_+$. For each $s \in \mathbb{R}_+$ we may consider the tangent map $T\mu^p_s\colon T\mathcal{V}M \to T\mathcal{V}M$; the relation $v_1 \sim v_2$ if $T\mu^p_s(v_1) = v_2$ for some $s \in \mathbb{R}_+$ defines an equivalence relation on the fibres of $T\mathcal{V}M \to \mathcal{V}M$ (independently of the choice of p) giving a quotient manifold $\mathcal{W}M$ fibred over M. We shall write $\chi\colon T\mathcal{V}M \to \mathcal{W}M$ and $\tau\colon \mathcal{W}M \to M$ for the projections.

Regarding $\mathbb{R}$ as the Lie algebra of $\mathbb{R}_+$, the fundamental vector field Υ^p on $\mathcal{V}M$ corresponding to $1 \in \mathbb{R}$ does, of course, depend on the choice of weight p; it is, however, projectable under the bundle map (χ, ν) to a global section e^p of $\mathcal{W}M \to M$. Where the choice of weight makes no difference to the discussion we shall omit the superscript and write μ, Υ and e.

Some of the geometric objects on $\mathcal{V}M$ may be lifted to $T\mathcal{V}M$. We shall, in particular, need the complete lift Υ^{c} and the vertical lift Υ^{v} of the fundamental vector field Υ, and also the 'squared volume' $\mathrm{d}\Theta^2$; the latter is a genuine $(2n+2)$-form on the orientable manifold $T\mathcal{V}M$.

Let (x^i) be local coordinates on M, and (x^i, u^i) the induced coordinates on TM. Define a coordinate x^0 in $\mathcal{V}M$ (depending on the weight p) by setting

$$x^0([\omega]) = s\,, \qquad [\omega] = \mu^p_s([\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n])\,.$$

We shall write $(x^a) = (x^0, x^i)$ for these local coordinates on $\mathcal{V}M$, and $(u^a) = (u^0, u^i)$ for the induced fibre coordinates on $T\mathcal{V}M$. In these coordinates the fundamental vector field Υ on $\mathcal{V}M$ appears as $x^0\,\partial/\partial x^0$. We may also define fibre coordinates (w, u^i) on $\mathcal{W}M$ by setting

$$w \circ \chi = \frac{u^0}{x^0}\,, \qquad u^i \circ \chi = u^i$$

so that $w \circ e = 1$ and $u^i \circ e = 0$. A local basis for the sections of $\mathcal{W}M \to M$ is given by (e, e_i) where e_i is the projection under (χ, ν) of the vector field $\partial/\partial x^i$ on $\mathcal{V}M$.

Finally, we shall write $T^\circ M \to M$ for the slit tangent bundle, obtained by deleting the zero section from TM, and we shall put $\mathcal{W}^\circ M = \rho^{-1}(T^\circ M)$ and $T^\circ\mathcal{V}M = (T\nu)^{-1}(T^\circ M)$, so that the latter are proper submanifolds of the corresponding slit vector bundles.

3 Lagrangian sections

Let $\mathcal{L}\colon T^\circ M \to \mathcal{W}^\circ M$ be a section of $\rho\colon \mathcal{W}^\circ M \to T^\circ M$ satisfying the positive homogeneity condition that $\mathcal{L}(\lambda v) = \lambda\mathcal{L}(v)$ for all $\lambda \in \mathbb{R}_+$ and all $v \in T^\circ M$. We shall say that $\mathcal{L}$ is a *Lagrangian section* on $T^\circ M$.

If $\sigma\colon M \to \mathcal{V}M$ is a section of the volume bundle then σ gives rise to a Lagrangian section $\hat\sigma$ by $\hat\sigma = \chi \circ T\sigma|_{T^\circ M}$. Certainly $\hat\sigma$ is positively homogeneous, and indeed it is linear (it is the composition of two linear maps). We shall call such a section a *trivial Lagrangian section*.

Suppose that $\hat\sigma$ is a fixed trivial Lagrangian section. Any other Lagrangian section $\mathcal{L}$ then gives rise to a function $F_{\mathcal{L},\sigma}$ on $T^\circ M$ in the following way. For any $v \in T^\circ M$, the difference $\mathcal{L}(v) - \hat\sigma(v)$ is a multiple of the global section e of $\mathcal{W}M \to M$; so set $F_{\mathcal{L},\sigma}(v)e = \mathcal{L}(v) - \hat\sigma(v)$. The function $F_{\mathcal{L},\sigma}$ is positively homogeneous because both $\mathcal{L}$ and $\hat\sigma$ are positively homogeneous sections; we shall say that $F_{\mathcal{L},\sigma}$ is a *pre-Finsler function* on $T^\circ M$.

Now suppose that $\hat\varsigma$ is another trivial Lagrangian section. This gives rise to another pre-Finsler function $F_{\mathcal{L},\varsigma}$; but the difference $F_{\mathcal{L},\varsigma} - F_{\mathcal{L},\sigma}$ is a total derivative, so that $F_{\mathcal{L},\sigma}$ and $F_{\mathcal{L},\varsigma}$ are 'gauge-equivalent' and give rise to the same Euler-Lagrange equations. Indeed, let f be the function defined on M by $\mu(\varsigma(x), f(x)) = \sigma(x)$; then

$$F_{\mathcal{L},\varsigma} - F_{\mathcal{L},\sigma} = \frac{\mathrm{d}(\log f)}{\mathrm{d}t} \,.$$

To see this in coordinates, put $\mathcal{L}^0 = w \circ \mathcal{L}$. If $v \in T^\circ_x M$, $v = v^i \partial/\partial x^i$, then

$$T\sigma(v) = v^i \left(\frac{\partial \sigma^0}{\partial x^i} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^i} \right)_{\sigma(x)}$$

so that

$$\chi(T\sigma(v)) = v^i \left(\frac{1}{\sigma^0} \frac{\partial \sigma^0}{\partial x^i} e + e_i \right)_x$$

and hence

$$w \circ \hat{\sigma} = w \circ \chi \circ T\sigma = \frac{u^i}{\sigma^0} \frac{\partial \sigma^0}{\partial x^i} = \frac{1}{\sigma^0} \frac{\mathrm{d}\sigma^0}{\mathrm{d}t} = \frac{\mathrm{d}(\log \sigma^0)}{\mathrm{d}t} \,;$$

thus

$$F_{\mathcal{L},\sigma} = w \circ (\mathcal{L} - \hat{\sigma}) = \mathcal{L}^0 - \frac{\mathrm{d}(\log \sigma^0)}{\mathrm{d}t} \,.$$

Similarly

$$F_{\mathcal{L},\varsigma} = w \circ (\mathcal{L} - \hat{\varsigma}) = \mathcal{L}^0 - \frac{\mathrm{d}(\log \varsigma^0)}{\mathrm{d}t}$$

so that

$$F_{\mathcal{L},\varsigma} - F_{\mathcal{L},\sigma} = \frac{\mathrm{d}(\log \sigma^0)}{\mathrm{d}t} - \frac{\mathrm{d}(\log \varsigma^0)}{\mathrm{d}t} = \frac{\mathrm{d}(\log \sigma^0/\varsigma^0)}{\mathrm{d}t} = \frac{\mathrm{d}(\log f)}{\mathrm{d}t} \,.$$

Similar arguments show that two pre-Finsler functions on M differing by a total derivative $\mathrm{d}f/\mathrm{d}t$ will correspond to the same Lagrangian section by taking sections of $\mathcal{V}M \to M$ related by f using the action μ. In this respect, therefore, the use of a section to correspond to an equivalence class of pre-Finsler functions (modulo total derivatives) is similar to the result obtained by Massa *et al.* in the affine case. In general, though, one cannot obtain a unique Lagrangian section from a pre-Finsler function, and the most appropriate formulation of the result is as follows.

Proposition 1. *There is a bijection, given by the procedure outlined above, between equivalence classes of pre-Finsler functions, differing by total derivatives, and equivalence classes of Lagrangian sections, differing by multiples (by total derivatives) of the canonical global section* $e\colon M \to \mathcal{W}^\circ M$.

There is, however, a special case where specific Lagrangian section may be chosen to correspond to a (genuine) Finsler function F on $T^\circ M$, and that is when F is the Finsler function of a Riemannian metric g on M, so that $F(v) = \sqrt{g(v,v)}$. In such a case we can make a specific choice of volume bundle section $\sigma_g\colon M \to \mathcal{V}M$ and therefore a specific choice of Lagrangian section $\mathcal{L}$.

Proposition 2. *If g is a Riemannian metric M with corresponding Finsler function F, then the section $\mathcal{L}\colon T^\circ M \to \mathcal{W}^\circ M$ defined by*

$$\mathcal{L}(v) = \hat{\sigma}_g(v) + F(v)e$$

is a canonical choice of section in the equivalence class corresponding to F.

4 Sprays and geodesics

In this section we consider pre-Finsler functions, such as those obtained from Lagrangian sections, satisfying a regularity condition. For any pre-Finsler function F on $T^\circ M$ we put $L = \frac{1}{2}F^2$.

The function L obtained from a Riemannian metric is necessarily quadratic in the velocity variables, whereas in general Finsler geometry is, famously, 'just Riemannian geometry without the quadratic restriction' [1]. Indeed, each pre--Finsler function F on $T^\circ M$ defines a symmetric type $(0,2)$ tensor field g_L along the projection $T^\circ M \to M$ (rather than on the manifold M) by setting

$$g_L(X,Y) = X^{\mathsf{v}}\big(Y^{\mathsf{v}}(L)\big) = \frac{1}{2}X^{\mathsf{v}}\big(Y^{\mathsf{v}}(F^2)\big)$$

where X, Y are vector fields on M and V^{v}, Y^{v} are their vertical lifts as vector fields on $T^\circ M$. It is evident that g_L is well-defined on tangent vectors, and so is indeed tensorial; in Finsler geometry it is known as the *fundamental tensor* of F. In coordinates, if $X = X^i\,\partial/\partial x^i$ and $Y = Y^j\,\partial/\partial x^j$ then $X^{\mathsf{v}} = X^i\,\partial/\partial u^i$ and $Y^{\mathsf{v}} = Y^j\,\partial/\partial u^j$ so that

$$g_L(X,Y) = g_{ij}X^iY^j\,, \qquad g_{ij} = \frac{\partial^2 L}{\partial u^i \partial u^j} = \frac{1}{2}\frac{\partial^2 (F^2)}{\partial u^i \partial u^j}\,.$$

If L is quadratic then the functions g_{ij} will be projectable to M, but in general they are defined locally on $T^\circ M$. The pre-Finsler function F will be a Finsler function if it is positive (so that $F(v) > 0$ for all $v \in T^\circ M$) and strongly convex (so that g_L is positive definite at each point of $T^\circ M$).

We may, of course, apply the same construction to F itself, rather than to L, giving another tensor field g_F along $T^\circ M \to M$. It is, however, a consequence of the homogeneity of F that g_F can never be non-degenerate: indeed $g_F|_v(v,v) = 0$ for any $v \in T^\circ_x M$ with $x \in M$. So we shall say that g_F is *positive quasi-definite* at x if this is the worst that can happen, and that $g_F|_v(w,w) \geq 0$ for all $v \in T^\circ_x M$, $w \in T_x M$ and all $x \in M$, with equality only when w is a scalar multiple of v. We shall say that F is a *pseudo-Finsler function* if g_F is positive quasi-definite at each $x \in M$.

Proposition 3 (see [3], Theorem 1). *Let F be a pseudo-Finsler function on $T^\circ M$. Then for each $x \in M$ there is a neighbourhood U of x and a function $\hat{F}$ defined on $T^\circ U$ such that $\hat{F}$ is a Finsler function and $\hat{F} - F$ is a total derivative df/dt for some function f defined on U.*

We shall say that a Lagrangian section $\mathcal{L}\colon T^\circ M \to \mathcal{W}^\circ M$ is *regular* if its corresponding equivalence class of pre-Finsler functions, differing by total derivatives, contains a pseudo-Finsler function. It is immediate that if the pre-Finsler functions F_1, F_2 differ by a total derivative then $g_{F_1} = g_{F_2}$, so that every pre-Finsler function in the equivalence class of a regular Lagrangian section must be a pseudo-Finsler function.

Corollary 1. *If $\mathcal{L}\colon T^\circ M \to \mathcal{W}^\circ M$ is a regular Lagrangian section then for each $x \in M$ there is a neighbourhood U of x such that the equivalence class of pseudo-Finsler functions corresponding to the restriction $\mathcal{L}|_{T^\circ U}$ contains a Finsler function.*

We may use this observation to associate geodesics with regular Lagrangian sections.

Every (genuine) Finsler function F is associated with a vector field Γ on $T^\circ M$, its *geodesic spray*, satisfying the condition

$$i_\Gamma \mathrm{d}\theta_L = -\mathrm{d}L$$

where θ_L, the Poincaré-Cartan form of L, is defined by $\theta_L = S(\mathrm{d}L)$ where S is the 'almost tangent structure' on TM; θ_L is expressed in coordinates as

$$\theta_L = \frac{\partial L}{\partial u^i}\mathrm{d}x^i\,.$$

One may check, using the fact that g_L is positive definite and that $\mathrm{d}\theta_L(X^{\mathsf{v}}, Y^{\mathsf{c}}) = g_L(X, Y)$ and $\mathrm{d}\theta_L(X^{\mathsf{v}}, Y^{\mathsf{v}}) = 0$ for any vector fields X, Y on M, that $\mathrm{d}\theta_L$ is a symplectic form; thus Γ exists and is unique. It is called a 'spray' because it is

- *second-order*, so that $S(\Gamma) = \Delta$, where Δ is the dilation vector field defined on TM but restricted to $T^\circ M$; and
- *homogeneous*, so that $[\Delta, \Gamma] = \Gamma$.

The second-order property of Γ implies that each of its integral curves is the natural lift of a curve in M. We shall call these curves in M the *geodesics* of Γ, and they are the extremals of L when it is regarded as the Lagrangian of a variational problem. It is, though, a consequence of the homogeneity of F (and hence of L) that the extremals of the variational problem are invariant under sense-preserving reparametrization, and so may be regarded as oriented geometric curves in M, whereas the geodesics of Γ come with a specific parametrization. Indeed, any spray of the form $\Gamma + \alpha\Delta$, where α is a function on $T^\circ M$ satisfying $\Delta(\alpha) = \alpha$, will be another spray whose geodesics are reparametrizations of those of Γ: the sprays are said to be *projectively equivalent*. Such projectively equivalent sprays span the kernel of $\mathrm{d}\theta_F$ where θ_F, the *Hilbert form* of F, is constructed from F in the same way as the Poincaré-Cartan form θ_L is constructed from L. The distinguishing feature of the geodesic spray Γ is that $\Gamma(F) = 0$; we say that the geodesics γ of this particular spray have constant speed, because $\mathrm{d}/\mathrm{d}t(F \circ \dot\gamma) = 0$. It is easy to see that if $\hat F$ is a pseudo-Finsler function differing from F by a total derivative then there is a spray $\hat\Gamma = \Gamma + \alpha\Delta$ in the same projective class (and so with the same geometric geodesics) satisfying $\hat\Gamma(\hat F) = 0$.

Now let $\mathcal{L}$ be a regular Lagrangian section, so that locally we may find Finsler functions corresponding to $\mathcal{L}$ and hence construct well-defined geometric geodesics. There is, however, no guarantee that a global Finsler function can be found (the positivity condition needs to be satisfied, and even then there is no guarantee of uniqueness) so we need to consider other approaches to the construction of canonically parametrized geodesics. We shall adopt the approach of looking for a single spray on $T^\circ\mathcal{V}M$ (rather than on $T^\circ M$) which incorporates all the information about a projective class of sprays on $T^\circ M$. Following [4] we define a 'BTW-spray' to be a spray $\tilde{\Gamma}$ on $T^\circ\mathcal{V}M$ satisfying the conditions

- $[\Upsilon^{\mathsf{c}}, \tilde{\Gamma}] = 0$
- $[\Upsilon^{\mathsf{v}}, \tilde{\Gamma}] = \Upsilon^{\mathsf{c}} - 2\tilde{\Delta}$
- $\mathcal{L}_{\tilde{\Gamma}}(\mathrm{d}\Theta^2) = 0$
- $\tilde{R} = 0$

where we adopt the weight $p = n+1$ for the fundamental vector field Υ, and where $\tilde{R}$ is the trace of the Jacobi endomorphism of $\tilde{\Gamma}$; the initials 'BTW' acknowledge the pioneering work of L. Berwald, J. Douglas and J.H.C. Whitehead. The coordinate expression of a spray on $T^\circ\mathcal{V}M$ satisfying these conditions is

$$u^0 \frac{\partial}{\partial x^0} + u^i \frac{\partial}{\partial x^i} - 2x^0\Gamma^0 \frac{\partial}{\partial u^0} - 2(\Gamma^i + (x^0)^{-1}u^0 u^i)\frac{\partial}{\partial u^i}$$

where the functions Γ^0, Γ^i are pulled back by $T\nu$ from functions defined locally on $T^\circ M$ and which also satisfy the conditions

$$\frac{\partial \Gamma^i}{\partial u^i} = 0\,, \qquad \Gamma^0 = -\frac{1}{2(n-1)}\left(2\frac{\partial \Gamma^i}{\partial x^i} - \frac{\partial \Gamma^i}{\partial u^j}\frac{\partial \Gamma^j}{\partial u^i}\right)$$

[4, Section 3.5]. We also note that $[\Upsilon^{\mathsf{c}}, \tilde{\Gamma}] = 0$ implies the projectability of any such spray to a vector field on $\mathcal{W}^\circ M$ with coordinate expression

$$u^i \frac{\partial}{\partial x^i} - (w^2 + 2\Gamma^0)\frac{\partial}{\partial w} - 2(\Gamma^i + wu^i)\frac{\partial}{\partial u^i}\,;$$

thus we may restrict this latter vector field to the image of the Lagrangian section $\mathcal{L}$ and hence obtain a vector field $\Gamma_{\mathcal{L}}$ on $T^\circ M$.

The global existence of BTW-sprays is a consequence of the following result.

Proposition 4 (see [4]). *Each projective equivalence class $[\Gamma]$ of sprays on $T^\circ M$ determines a unique BTW-spray $\tilde{\Gamma}$ on $T^\circ\mathcal{V}M$ such that the vector field $\Gamma_{\mathcal{L}}$ on $T^\circ M$ obtained by the method described above is a spray in that equivalence class.*

We may therefore use the following procedure to define parametrized geodesics of a regular Lagrangian section $\mathcal{L}$.

- The section $\mathcal{L}$ defines an equivalence class of pseudo-Finsler functions $[F]$.

- Each $x \in M$ has a neighbourhood U such that the restriction of $[F]$ to $T^\circ U$ contains a Finsler function F_U.
- Each such Finsler function F_U defines a spray Γ_U on $T^\circ U$, and hence a projective class of sprays $[\Gamma_U]$ on $T^\circ U$. Any other Finsler function in the same class determines the same projective class of sprays.
- The projective class $[\Gamma_U]$ determines a unique BTW-spray $\tilde{\Gamma}_U$ on $T^\circ \mathcal{V}U$. If U' is the neighbourhood of $x' \in M$ with Finsler function $F_{U'}$ and projective class $[\Gamma_{U'}]$, and if $U \cap U' \neq \emptyset$, then by uniqueness $\tilde{\Gamma}_U = \tilde{\Gamma}_{U'}$ on $T^\circ \mathcal{V}U \cap T^\circ \mathcal{V}U'$, so that there is a unique global BTW-spray $\tilde{\Gamma}$ on $T^\circ \mathcal{V}M$ whose restriction to each $T^\circ \mathcal{V}U$ is $\tilde{\Gamma}_U$.
- The global spray $\Gamma_{\mathcal{L}}$ on $T^\circ M$ constructed from $\tilde{\Gamma}$ and $\mathcal{L}$ has the property that its restriction to each $T^\circ M$ is in the projective class $[\Gamma_U]$.
- The geodesics of $\Gamma_{\mathcal{L}}$, restricted to U, are parametrized geodesics of the local Finsler function F_U. Thus the images of the geodesics of $\Gamma_{\mathcal{L}}$ are the geometric geodesics of the regular Lagrangian section $\mathcal{L}$, so that we may use the parametrization of the former to define a canonical parametrization of the latter.

Thus we obtain our main result.

Theorem 1. *Each regular Lagrangian section $\mathcal{L}\colon T^\circ M \to \mathcal{W}^\circ M$ determines a canonical family of geodesics on M, such that the geodesics of any local Finsler function associated with $\mathcal{L}$ are reparametrizations of the canonical geodesics.*

Acknowledgement

Research supported by grant no 14-02476S 'Variations, Geometry and Physics' of the Czech Science Foundation.

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Received: 25 October, 2016
Accepted for publication: 4 December, 2016
Communicated by: Olga Rossi