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Variations by generalized symmetries of local Noether strong currents equivalent to global canonical Noether currents

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Abstract. We will pose the inverse problem question within the Krupka variational sequence framework. In particular, the interplay of inverse problems with symmetry and invariance properties will be exploited considering that the cohomology class of the variational Lie derivative of an equivalence class of forms, closed in the variational sequence, is trivial. We will focalize on the case of symmetries of globally defined field equations which are only locally variational and prove that variations of local Noether strong currents are variationally equivalent to global canonical Noether currents. Variations, taken to be generalized symmetries and also belonging to the kernel of the second variational derivative of the local problem, generate canonical Noether currents – associated with variations of local Lagrangians – which in particular turn out to be conserved *along any section*. We also characterize the variation of the canonical Noether currents associated with a local variational problem.

1 Introduction

In a series of papers [31], [5], [32], [19], [3], [4], [33] Gennady Sardanashvily dedicated to the study and the extension of Noether Theorems, in particular of the Second Noether Theorem and of Noether identities in quite general geometric contexts involved in classical and quantum physics, particularly within (variational) differential bicomplexes.

As well known, the paper *Invariante Variationsprobleme* by Amalie (Emmy) Noether [24] has been a cornerstone in the history of the calculus of variation. The relevance of her work is testified by the widespread literature dedicated to the applications in Physics, Statistics and Engineering. However, the very importance of

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Noether Theorems principally consists of the new ideas concerning the structure itself of the calculus of variations. Those aspects have been deeply studied in a recent work by Y. Kosmann-Schwarzbach, who, besides providing a fine French and English translation of the celebrated 1918 Emmy Noether paper, thoroughly analyzed the inception and the influence in Physics as well as the historical developments of Noether Theorems during the XXth Century [20].

Physical theories of the fundamental interactions can be all formulated as gauge-natural field theories in terms of invariant Lagrangians with respect to symmetries (gauge-natural lifts of infinitesimal principal automorphisms) satisfying the hypotheses of the Second Noether Theorem. The epistemological base [28] of the Noether Theorems became therefore a basic requirement for the meaningfulness of a physical theory and admissible symmetries.

The first important novelty introduced by Noether is the relevance of boundary terms versus Euler-Lagrange field equations. The second main novelty is Noether's concept of a variation field, i.e. of what 'virtual displacements' (i.e. variation fields) should be in field theory. She takes as variation field the vertical parts of infinitesimal generator of invariant transformations of the Lagrangian.

In particular, she concentrates on the term obtained by contraction of the Lagrange expressions with the generators of invariant transformations and formulate about properties of variation fields such that this terms could be put in the form of a divergence.

She first splits the term above in a summand containing vertical parts of generators of the invariance transformation contracted with Lagrange expressions and a summand going under a divergence which contains the horizontal part of generators of the invariance transformation contracted with the Lagrangian. By applying the standard variation calculus by *variation fields which are generated by the invariance transformation* gives a 'work' term (as it is called by physicists) plus a further divergence term (the momentum term) which sum up with the contribution due to the horizontal part of the symmetry.

The meaning of the First Noether Theorem is in relating, by invariance properties of the Lagrangian, Lagrangian expressions (thus 'equations'), more precisely the 'work' term, with conservation laws. Equations obtained from invariant Lagrangians can be related to conserved quantities if and only if they derive variationally from an *invariant* Lagrangian.

The Second Theorem is a further investigation of the 'work term' containing the Lagrange expressions. Once realized that such a term can be related to conserved quantities, under which conditions can we further transform it in such a way that it becomes a divergence itself (*independently from the invariance of the Lagrangian*)? The answer found by Noether is: it is possible to make the 'work term' become a divergence – independently from the invariance of the Lagrangian – if and only if the group of symmetry transformations is an infinite continuous group (i.e. depends on a given number of functions rather than just parameters) satisfying a few other requirements. In this case the existence of identities among Lagrange expressions and their derivatives guarantees that the 'work term' reduces to a further divergence. At a further step, if invariance is required for the Lagrangian a divergence vanishes identically (along any section, not necessarily critical) and it thus generates an 'improper' conservation law.

Taking as variations the generators of transformations of independent and dependent coordinates of a specific type, always guarantees that contractions of Lagrangian expressions with vertical parts of transformations become divergences and *vice versa*. If, and only if, the Lagrangian is invariant under such transformations we can further characterize such divergences as conserved quantities also off shell (so-called 'strong' conserved quantities). Notice that this means that when such symmetries are given, due to Noether identities we can always transform the dynamical content in a component of the strongly conserved quantity.

Equations obtained from Lagrangians invariant with respect to symmetries according to the Second Theorem *always* can be put in the form of an 'improper' conservation law; more specifically what changes 'is' part of what remains unchanged if and only if it derives variationally from an invariant Lagrangian with respect to an infinite group of transformation.

The two statements are in fact determinations of *conditions on invariant transformations* in order that a specific (variational) form – the 'work' form – be (locally) exact.

In this paper we first review some cohomological aspects concerned with local variational problems (Section 2), investigate when such local problems are equivalent to global ones (Section 3), then study currents associated with symmetries of locally variational dynamical forms (Section 4); in particular we concentrate on the case of symmetries of globally defined field equations which are only locally variational and prove that the variations of local Noether strong currents are variationally equivalent to global canonical Noether currents (Proposition 2). Variations, taken to be generalized symmetries and also belonging to the kernel of the second variational derivative of the local problem, generate canonical Noether currents – associated with variations of local Lagrangians – which in particular turn out to be conserved *along any section*. We also characterize the variation of the canonical Noether currents associated with a local variational problem (Remark 3).

2 Local variational problems and cohomology

In this paper we shall concentrate on the issues of the Second Noether Theorem, which are independent of the invariance of the Lagrangian. In fact we shall be interested in generalized symmetries, i.e. symmetries of global dynamical form defining non trivial cohomology classes (i.e. only locally variational). Note that even if the Lagrangian is not invariant a (local) strong Noether current is always defined provided that the equations are invariant. Such a strong current however is not conserved since generated by a generalized symmetry.

The modern geometrical formulations of the calculus of variations on fibered manifolds include a large class of theories for which the Euler-Lagrange operator is a morphism of an exact sequence [2], [22], [34], [35], [36]. The module in degree $(n+1)$, contains dynamical forms; a given equation is globally an Euler-Lagrange equation if its dynamical form is the differential of a Lagrangian and this is equivalent to the dynamical form being closed in the complex, i.e. Helmholtz conditions hold true, and its cohomology class being trivial.

Dynamical forms which are only *locally variational*, i.e. which are closed in the complex and define a non trivial cohomology class, admit a system of local

Lagrangians. We shall consider global projectable vector field on a jet fiber manifold which are symmetries of dynamical forms, in particular of locally variational dynamical forms. It is clear the relevant role played by the *variational Lie derivative*, a differential operator acting on equivalence classes of variational forms in the variational sequence [9], [25], by which Noether theorems can be formulated. In particular, variations of currents can be recognized in this approach.

We consider the cohomology defined by a system of local Lagrangian and investigate under which conditions the variational Lie derivative of associated local strong Noether currents is a system of global *conserved* currents.

We shall consider the variational sequence [22] defined on a fibered manifold $\pi\colon \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$. For $r \geq 0$ we have the r-jet space $J_r\boldsymbol{Y}$ of jet prolongations of sections of the fibered manifold π. We have also the natural fiberings $\pi^r_s\colon J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, and $\pi^r\colon J_r\boldsymbol{Y} \to \boldsymbol{X}$; among these the fiberings π^r_{r-1} are *affine bundles* which induce the natural fibered splitting

$$J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} T^*J_{r-1}\boldsymbol{Y} \simeq J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} (T^*\boldsymbol{X} \oplus V^*J_{r-1}\boldsymbol{Y})\,,$$

which, in turn, induces also a decomposition of the exterior differential on $\boldsymbol{Y}$ in the *horizontal* and *vertical differential*, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$. By $(j_r\Xi, \xi)$ we denote the jet prolongation of a *projectable vector field* (Ξ, ξ) on $\boldsymbol{Y}$, and by $j_r\Xi_H$ and $j_r\Xi_V$ the horizontal and the vertical part of $j_r\Xi$, respectively.

Let ρ be a q-form on $J^r\boldsymbol{Y}$; in particular we obtain a natural decomposition of the pull-back by the affine projections of ρ, as

$$(\pi^{r+1}_r)^*\rho = \sum_{i=0}^{q} p_i\rho\,,$$

where $p_i\rho$ is the i-contact component of ρ (by definition a contact form is zero along any holonomic section of $J^r\boldsymbol{Y}$).

Starting from this splitting one can define sheaves of contact forms Θ^*_r, suitably characterized by the kernel of p_i [22]; the sheaves Θ^*_r form an exact subsequence of the de Rham sequence on $J^r\boldsymbol{Y}$ and one can define the quotient sequence

$$0 \longrightarrow \mathbb{R}_{\boldsymbol{Y}} \longrightarrow \dots \xrightarrow{\mathcal{E}_{n-1}} \Lambda^n_r/\Theta^n_r \xrightarrow{\mathcal{E}_n} \Lambda^{n+1}_r/\Theta^{n+1}_r \xrightarrow{\mathcal{E}_{n+1}} \Lambda^{n+2}_r/\Theta^{n+2}_r \xrightarrow{\mathcal{E}_{n+2}} \dots \longrightarrow 0$$

the *r-th order variational sequence* over the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$. It turns out that it is a soft sheaf resolution of the constant sheaf $\mathbb{R}_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$.

The quotient sheaves (the sections of which are classes of forms modulo contact forms) in the variational sequence can be represented as sheaves $\mathcal{V}^k_r$ of k-forms on jet spaces of the higher order; see e.g. [25] for a review.

Currents are sheaf sections ϵ of $\mathcal{V}^{n-1}_r$ and the quotient morphism $\mathcal{E}_{n-1} = d_H$ is represented by a total divergence; Lagrangians are sections λ of $\mathcal{V}^n_r$, while $\mathcal{E}_n$ is called the Euler-Lagrange morphism; sections η of $\mathcal{V}^{n+1}_r$ are called *source forms* or also *dynamical forms*, while $\mathcal{E}_{n+1}$ is called the Helmholtz morphism.

The cohomology groups of the corresponding complex of global sections

$$0 \to \mathbb{R}_{\boldsymbol{Y}} \to \dots \xrightarrow{\mathcal{E}_{n-1}} (\Lambda^n_r/\Theta^n_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_n} (\Lambda^{n+1}_r/\Theta^{n+1}_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_{n+1}} (\Lambda^{n+2}_r/\Theta^{n+2}_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0$$

will be denoted by $H^*_{\mathrm{VS}}(\boldsymbol{Y})$.

Since the variational sequence is a soft sheaf resolution of the constant sheaf $\mathbb{R}_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$, the cohomology of the complex of global sections is naturally isomorphic to both the Čech cohomology of $\boldsymbol{Y}$ with coefficients in the constant sheaf $\mathbb{R}$ and the de Rham cohomology $H^k_{\mathrm{dR}}\boldsymbol{Y}$ [22].

Let now $\boldsymbol{K}^p_r := \ker \mathcal{E}_p$. We have the short exact sequence of sheaves

$$0 \longrightarrow \boldsymbol{K}^p_r \xrightarrow{i} \mathcal{V}^p_r \xrightarrow{\mathcal{E}_p} \mathcal{E}_p(\mathcal{V}^p_r) \longrightarrow 0\,.$$

In particular $\mathcal{E}_n(\mathcal{V}^n_r)$ is the sheaf of Euler-Lagrange morphisms: for a global section $\eta \in (\mathcal{V}^{n+1}_r)_{\boldsymbol{Y}}$ we have $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ if and only if $\mathcal{E}_{n+1}(\eta) = 0$, which are the Helmholtz conditions of local variationality.

The above exact sequence gives rise to the long exact sequence in Čech cohomology

$$0 \longrightarrow (\boldsymbol{K}^p_r)_{\boldsymbol{Y}} \longrightarrow (\mathcal{V}^p_r)_{\boldsymbol{Y}} \longrightarrow (\mathcal{E}_p(\mathcal{V}^p_r))_{\boldsymbol{Y}} \xrightarrow{\delta_p} H^1(\boldsymbol{Y}, \boldsymbol{K}^p_r) \longrightarrow 0\,.$$

Hence, every $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ (i.e. locally variational) defines a cohomology class

$$\delta_n(\eta) \in H^1(\boldsymbol{Y}, \boldsymbol{K}^n_r) \simeq H^{n+1}_{VS}(\boldsymbol{Y}) \simeq H^{n+1}_{dR}(\boldsymbol{Y})\,.$$

Furthermore, every $\mu \in (d_H(\mathcal{V}^{n-1}_r))_{\boldsymbol{Y}}$ (i.e. variationally trivial) defines a cohomology class $\delta_{n-1}(\mu) \in H^1(\boldsymbol{Y}, \boldsymbol{K}^{n-1}_r) \simeq H^n_{VS}(\boldsymbol{Y}) \simeq H^n_{dR}(\boldsymbol{Y})$.

The above gives rise to a well known diagram of cochain complexes:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(Y, K^p_r) & \xrightarrow{i} & C^0(Y, \mathcal{V}^p_r) & \xrightarrow{\mathcal{E}_p} & C^0(Y, \mathcal{E}_p(\mathcal{V}^p_r)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathfrak{d}} & & \downarrow{\scriptstyle \mathfrak{d}} & & \downarrow{\scriptstyle \mathfrak{d}} & & \\
0 & \longrightarrow & C^1(Y, K^p_r) & \xrightarrow{i} & C^1(Y, \mathcal{V}^p_r) & \xrightarrow{\mathcal{E}_p} & C^1(Y, \mathcal{E}_p(\mathcal{V}^p_r)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

whereby we recognize the *connecting homomorphism* $\delta_p = i^{-1} \circ \mathfrak{d} \circ \mathcal{E}_p^{-1}$ as a mapping of cohomologies (here $\mathfrak{d}$ is the *coboundary operator*).

Note that η is globally variational if and only if $\delta_n(\eta) = 0$.

We are therefore interested to the non trivial case where

$$\begin{array}{rcl}
\mathcal{E}_{n+1}(\eta) = 0 & & \mathfrak{d}\eta \neq 0 \\
\delta_n(\eta) \neq 0 & \Longleftrightarrow & \mathfrak{d}\lambda_i \neq 0\,,
\end{array}$$

whereby $\eta = \mathcal{E}_n(\lambda)$ can be solved only locally, i.e. for any countable good covering of $\boldsymbol{Y}$ there exists a local Lagrangian λ_i over each subset $\boldsymbol{U}_i \subset \boldsymbol{Y}$ such that $\eta_i = \mathcal{E}_n(\lambda_i)$.

A system of local sections λ_i of $(\mathcal{V}^n_r)_{U_i}$ such that $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i \cap U_j}) = 0$, is what we call a *local variational problem* [13]; every nontrivial cohomology class gives rise to local variational problems.

For any countable open covering of $\boldsymbol{Y}$, $\lambda = \{\lambda_i\}_{i \in I}$ is then a 0-cochain of Lagrangians in Čech cohomology with values in the sheaf $\mathcal{V}^n_r$. By an abuse of notation we shall denote by η_λ the 0-cochain formed by the restrictions $\eta_i = \mathcal{E}_n(\lambda_i)$ (and so will do at any degree of forms). Of course, $\mathfrak{d}\lambda \equiv \{\lambda_{ij}\} \equiv (\lambda_i - \lambda_j)|_{U_i \cap U_j} = 0$ if and only if λ is globally defined on $\boldsymbol{Y}$; analogously $\mathfrak{d}\eta = 0$ if and only if η is global. Note that $\mathfrak{d}\lambda = 0$ implies $\mathfrak{d}\eta_\lambda = 0$, while by $\mathbb{R}$-linearity we only have $\mathfrak{d}\eta_\lambda = \eta_{\mathfrak{d}\lambda} = 0$ i.e. $\mathfrak{d}\lambda$ is 1-cochain of Lagrangians. Two systems of local Lagrangians represent the same local variational problem if they differ by a Čech-cochain of trivial Lagrangians on a common refinement.

Definition 1. Two local variational problems of degree p are equivalent if and only if their difference is in the kernel of the corresponding morphism $\mathcal{E}_p$ in the variational sequence.

3 Local variational problems equivalent to global ones

As well known Noether Theorems relate symmetries of a variational problem to conserved quantities. In [9], [25] we formulated the Noether Theorems in terms of variational Lie derivatives of classes of forms modulo the contact structure.

- The case $q \leq n-1$

  In [9] formulae for the Lie derivative of classes of q-forms have been obtained.

  Theorem 1. *Let $\alpha \in \mathcal{V}^q_r$, $0 \leq q \leq n-1$, and let Ξ be a π-projectable vector field on $\boldsymbol{Y}$; the following holds locally*

  $$\mathcal{L}_\Xi \alpha = \Xi_H \rfloor \mathcal{E}_q(\alpha) + \mathcal{E}_{q-1}(\Xi_V \rfloor \tilde{p}_{d_V \alpha} + \Xi_H \rfloor \alpha)\,.$$

  Here $\tilde{p}$ is a generalized momentum associated with a horizontal p-form.

- The case $q = n$

  Theorem 2. *(Noether's Theorem I)*

  Let $\alpha \in \mathcal{V}^n_r$ and Ξ be a π-projectable vector field on $\boldsymbol{Y}$; the following holds (locally):

  $$\mathcal{L}_\Xi \alpha = \Xi_V \rfloor \mathcal{E}_n(\alpha) + \mathcal{E}_{n-1}(\Xi_V \rfloor p_{d_V \alpha} + \Xi_H \rfloor \alpha)\,.$$

  Here p is a momentum associated with the Lagrangian.

- The case $q \geq n+1$

  In [25] it was proved the following variational Cartan formula for classes of forms of degree $q \geq n+1$.

  The case $q = n+1$ for locally variational dynamical forms encompasses Noether's Theorem II, or so-called Bessel-Hagen symmetries.

Theorem 3. *Let $q = n+k$, with $k \geq 1$ and $\alpha \in \mathcal{V}^q_r$. Let Ξ be a π-projectable vector field on $\boldsymbol{Y}$; we have*

$$\mathcal{L}_\Xi \alpha = \Xi_V \rfloor \mathcal{E}_q(\alpha) + \mathcal{E}_{q-1}(\Xi_V \rfloor \alpha)\,.$$

Let now η_λ be the Euler-Lagrange morphism of a local variational problem and let $j_r\Xi$ be a generalized symmetry; the Noether Theorems imply that

$$0 = \Xi_V \rfloor \eta_\lambda + d_H(\epsilon_i - \beta_i)\,,$$

where $\epsilon_i = j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i$ is the usual *canonical* Noether current; the Noether-Bessel-Hagen current $\epsilon(\lambda_i, \Xi) - \beta_i$ is a *local* object and it is conserved along the solutions of Euler-Lagrange equations (*local conservation law*) [9], [17], [18], [30].

Note that, since $0 = \mathcal{L}_{j_r\Xi}\eta = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$, the horizontal n-form $\Xi_V \rfloor \eta_\lambda$ defines a cohomology class and we have that the local currents are the restrictions of a global conserved current if and only if the cohomology class $\delta_{n-1}(\Xi_V \rfloor \eta_\lambda) \in H^n_{dR}(\boldsymbol{Y})$ vanishes.

As a consequence of linearity and naturality properties, independently from the fact that Ξ be a generalized symmetry or not, the variational Lie derivative trivializes cohomology classes [16], [17], [18], [29]. We could write this fact defining an operator acting trivially on the de Rham cohomology as $\hat{\mathcal{L}}_{j^r\Xi}([\rho]_{dR}) = [\mathcal{L}_{j^r\Xi}\rho]_{dR} \equiv 0$. Since the variational Lie derivative with respect to projectable vector fields defines variations, therefore variations trivialize cohomology classes.

In particular, we have $\eta_{\mathcal{L}_\Xi\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda) = \mathcal{L}_\Xi\eta_\lambda$; this implies that

$$\delta(\mathcal{L}_\Xi\eta_\lambda) = \delta(\eta_{\mathcal{L}_\Xi\lambda_i}) = 0$$

although $\delta(\eta_\lambda) \neq 0$ and thus we get that Euler-Lagrange equations of the local problem defined by $\mathcal{L}_\Xi\lambda_i$ are equal to Euler-Lagrange equations of the global problem defined by $\Xi_V \rfloor \eta_\lambda$.

An analogous result holds true also at any degree $k \leq n$ in the variational sequence; specifically for local potentials of variationally trivial classes of horizontal forms. Let μ be a variationally trivial Lagrangian, i.e. such that $\mathcal{E}_n(\mu) = 0$, this means that we have a 0-cocycle of currents ν_i such that $\mu = d_H\nu_i$ and $\mathfrak{d}\mu_\nu = 0$ but we suppose $\delta_{n-1}(\mu_\nu) \neq 0$. We can consider the Lie derivative $\mathcal{L}_\Xi\nu_i$ and the corresponding $\mu_{\mathcal{L}_\Xi\nu_i}$. We have

$$\mu_{\mathcal{L}_\Xi\nu_i} = d_H(\Xi_H \rfloor \mu_\nu + \Xi_V \rfloor p_{d_V\mu_\nu}) = \mathcal{L}_\Xi\mu_\nu\,,$$

so that $\delta_{n-1}(\mathcal{L}_\Xi\mu_\nu) = 0$, although $\delta_{n-1}(\mu_\nu) \neq 0$.

In particular, the local problem defined by $\mathcal{L}_\Xi\nu_i$ is variationally equivalent to the global problem defined by $\Xi_H \rfloor \mu_\nu + \Xi_V \rfloor p_{d_V\mu_\nu}$. Recall that $\mu_\nu = d_H\nu_i$ is assumed to satisfy $\mathfrak{d}\mu_\nu = 0$, i.e. it is a global object.

Remark 1. We note that, as a consequence of Theorem 1, we get the interesting relation $\Xi_V \rfloor \tilde{p}_{d_V\nu_i} = \Xi_V \rfloor p_{d_V\mu_\nu} + d_H\phi_i$, with ϕ_i a 0-cocycle.

Example 1. Let us assume Ξ be a symmetry of dynamical forms: we have $\mathcal{L}_{\Xi}\eta_{\lambda} = 0$ then, in particular, $\delta(\mathcal{L}_{\Xi}\eta_{\lambda}) \equiv 0$; furthermore, under the same assumption, we have $\mathcal{E}_n(\Xi_V \rfloor \eta_{\lambda}) = 0$ then there exists a 0-cocycle ν_i as above, defined by $\mu_{\nu} = \Xi_V \rfloor \eta_{\lambda} := d_H\nu_i$. In this case, divergence expressions of the local problem defined by $\mathcal{L}_{\Xi}\nu_i$ coincide with divergence expressions for the global current

$$\Xi_H \rfloor \Xi_V \rfloor \eta_{\lambda} + \Xi_V \rfloor p_{d_V(\Xi_V \rfloor \eta_{\lambda})}\,;$$

see [16].

4 Currents associated with locally variational dynamical forms

As we saw, each ρ such that $\mathcal{E}_{p+1}(\rho) = 0$, i.e. $\rho =_{\text{loc.}} \mathcal{E}_p(\tau)$ defines a cohomology class $\delta_p(\rho)$. Let us then consider symmetries of global dynamical forms locally variational.

$$\mathcal{L}_{j^r\Xi}\eta = 0, \qquad \mathcal{E}_{n+1}\eta = 0 \implies \eta = \eta_{\lambda_i}\,.$$

Noether Th. (I) implies $\mathcal{L}_{j^r\Xi}\lambda_i = \Xi_V \rfloor \eta_{\lambda_i} + d_H\epsilon_{\lambda_i}$. On the other hand $\mathcal{L}_{j^r\Xi}\eta = 0 \implies \mathcal{E}_n(\Xi_V \rfloor \eta_{\lambda_i}) = 0$, and we saw that if $\delta_{n-1}(\Xi_V \rfloor \eta) \neq 0$ then $\Xi_V \rfloor \eta = d_H\nu_i$, with $\mathfrak{d}\nu_i \neq 0$, therefore Noether Th. (I) implies

$$\mathcal{L}_{j^r\Xi}\lambda_i = d_H(\nu_i + \epsilon_i)\,,$$

and *under the conditions* $\delta_n(\eta) \neq 0$ and $\delta_{n-1}(\Xi_V \rfloor \eta_{\lambda_i}) \neq 0$ we have necessarily

$$\mathfrak{d}(\nu_i + \epsilon_i) \neq 0\,.$$

If $\mathcal{L}_{j^r\Xi}\lambda_i = 0$ would hold true, then we would have a Noether strong (in the words of Noether 'improper') conservation law $d_H(\nu_i + \epsilon_i) = 0$.

4.1 Variations of strong Noether currents

We are here particularly concerned with the case when $\mathcal{L}_{j^r\Xi}\lambda_i \neq 0$ i.e. $j^r\Xi$ is a generalized symmetry, that means $\mathcal{L}_{j^r\Xi}\lambda_i =_{\text{loc.}} d_H\beta_i$. In fact, since $\mathcal{E}_n(\mathcal{L}_{j^r\Xi}\lambda_i) = 0$ then $\mathcal{L}_{j^r\Xi}\lambda_i$ defines a cohomology class and we have $\delta_{n-1}(\mathcal{L}_{j^r\Xi}\lambda_i) \neq 0$ iff and only if $\mathfrak{d}\beta_i \neq 0$. Notice that if $\mathfrak{d}\mathcal{L}_{j^r\Xi}\lambda_i = 0$ then $0 = \mathfrak{d}d_H\beta_i = d_H\mathfrak{d}\beta_i$.

Remark 2. It turns out that $\mathcal{L}_{j^r\Xi}\lambda_i$ is *closed*; therefore, since the variational Lie derivative of closed classes trivializes cohomology classes,

$$\delta_{n-1}(\mathcal{L}_{j^r\Xi}\mathcal{L}_{j^r\Xi}\lambda_i) = 0 \implies \mathcal{L}_{j^r\Xi}\mathcal{L}_{j^r\Xi}\lambda_i = d_H\psi_i\,,$$

therefore

$$d_H\mathcal{L}_{j^r\Xi}(\nu_i + \epsilon_i) = d_H\psi_i\,,$$

with $\mathfrak{d}\psi_i = 0$, Thus the *variation of the strong Noether current* $\mathcal{L}_{j^r\Xi}(\nu_i + \epsilon_i)$ is necessarily variationally equivalent to a global current as a consequence of the invariance of field equations.

In the following we characterize more precisely this fact and we find that a global representative is given by the *canonical* Noether current associated with the Lagrangian $\mathcal{L}_{j^r\Xi}\lambda_i$.

As a preliminary fact, we first show that a condition assumed in [29] is in fact always satisfied for generalized symmetries of global dynamical forms.

Proposition 1. *Let $j^r\Xi$ be a generalized symmetry. The coboundary of the strong Noether currents is locally exact, i.e.* $d_H(\mathfrak{d}(\nu_i + \epsilon_i)) = 0$.

Proof. To get the assertion it is enough to prove that $\mathfrak{d}\mathcal{L}_{j^r\Xi}\lambda_i = 0$. This is equivalent to prove that

$$\delta_{n+1}\mathcal{E}_n(\mathfrak{d}\mathcal{L}_{j^r\Xi}\lambda_i) = 0 \,.$$

By linearity

$$\delta_{n+1}\mathcal{E}_n(\mathfrak{d}\mathcal{L}_{j^r\Xi}\lambda_i) = \delta_{n+1}\mathfrak{d}\mathcal{E}_n(\mathcal{L}_{j^r\Xi}\lambda_i) = \delta_{n+1}\mathfrak{d}\mathcal{L}_{j^r\Xi}\mathcal{E}_n(\lambda_i)$$

and being $j^r\Xi$ a generalized symmetry $\mathcal{L}_{j^r\Xi}\mathcal{E}_n(\lambda_i) = 0$, thus we get immediately the result. □

Proposition 2. *The variation of a local strong Noether current by a generalized symmetry is variationally equivalent to the canonical Noether current associated to the variation of the local Lagrangians by the same symmetry, i.e. we have*

$$\mathcal{L}_{j^r\Xi}(\nu_i + \epsilon_i) \simeq \epsilon_{\mathcal{L}_{j^r\Xi}\lambda_i} \,.$$

Proof. Let $\gamma_{\phi_i} =_{loc.} d_H\phi_i$ be a locally variationally trivial Lagrangian. As a consequence of Theorem 1 and Theorem 2 and by naturality we have

$$d_H\epsilon_{\gamma_{\phi_i}} = \mathcal{L}_{j^r\Xi}\gamma_{\phi_i} \doteq \gamma_{\mathcal{L}_{j^r\Xi}\phi_i} = d_H(\mathcal{L}_{j^r\Xi}\phi_i) \,.$$

We have $0 = \delta_{n-1}(\mathcal{L}_{j^r\Xi}\gamma_{\phi_i}) = \delta_{n-1}(\gamma_{\mathcal{L}_{j^r\Xi}\phi_i})$, i.e. $\mathcal{L}_{j^r\Xi}\phi_i$ is equivalent to a global current and in particular $\mathcal{L}_{j^r\Xi}\phi_i \simeq \epsilon_\gamma$, i.e. the (local) current $\mathcal{L}_{j^r\Xi}\phi_i$ is variationally equivalent to the global current ϵ_γ.

Since, as a consequence of the global invariance of field equations, the coboundary of the strong Noether currents is locally exact, by applying the above to $\gamma = d_H(\nu_i + \epsilon_i)$ we get the statement. □

Let Ξ be a symmetry of the Euler-Lagrange form η_λ. If we then require that the current be a current variationally associated [16] to the generalized symmetry (i.e. if the second variational derivative by a generalized symmetry of the local problems is vanishing) then we have the conservation law $d_H\mathcal{L}_\Xi(\nu_i + \epsilon_i) = 0$. This means that the current $\epsilon_{\mathcal{L}_{j^r\Xi}\lambda_i}$ is also conserved and the corresponding conservation laws reads $d_H(\epsilon_{\mathcal{L}_{j^r\Xi}\lambda_i}) = 0$.

Notice that this canonical Noether current is conserved along *any section.*

Remark 3. Comparing the latter Proposition with Example 1 we characterize the Lie derivative of the local canonical Noether current $\epsilon_i \doteq \epsilon_{\lambda_i}$. Specifically we have

$$\mathcal{L}_{j^r\Xi}\epsilon_{\lambda_i} \simeq \epsilon_{d_H\epsilon_{\lambda_i}} \,.$$

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