

# A Note on Transcendental Power Series Mapping the Set of Rational Numbers into Itself

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**Abstract.** In this note, we prove that there is no transcendental entire function  $f(z) \in \mathbb{Q}[[z]]$  such that  $f(\mathbb{Q}) \subseteq \mathbb{Q}$  and  $\text{den } f(p/q) = F(q)$ , for all sufficiently large  $q$ , where  $F(z) \in \mathbb{Z}[z]$ .

## 1 Introduction

A real number  $\xi$  is called a *Liouville number*, if for any real number  $\omega > 0$  there exists a rational number  $p/q$ , with  $q > 1$ , such that

$$0 < \left| \xi - \frac{p}{q} \right| < q^{-\omega}.$$

In his pioneer book, Maillet [3, Chapter III] proved that the set of the Liouville numbers is preserved under rational functions with rational coefficients. Based on this result, in 1984, Mahler [2] posed the following question

**Question 1.** *Are there transcendental entire functions  $f(z)$  such that if  $\xi$  is any Liouville number, then so is  $f(\xi)$ ?*

Recently, some authors (see [4], [5], [7]) constructed classes of Liouville numbers which are mapped into Liouville numbers by transcendental entire functions. For example, to prove this, Marques and Moreira [4] showed the existence of transcendental entire functions  $f$ , such that  $f(\mathbb{Q}) \subseteq \mathbb{Q}$  and  $\text{den } f(p/q) < q^{8q^2}$ , for all  $p/q \in \mathbb{Q}$ , with  $q > 1$  (where  $\text{den } z$  denotes the denominator of the rational number  $z$ ). Moreover, their proof implies that the Mahler's question has an affirmative answer if the answer to the below question is also 'yes' (see also [7, Theorem 2.1]).

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**Question 2.** Are there transcendental entire functions  $f(z)$  such that  $f(\mathbb{Q}) \subseteq \mathbb{Q}$  and

$$\text{den } f(p/q) \leq F(q),$$

for some fixed polynomial  $F(z) \in \mathbb{Z}[z]$  and for all sufficiently large  $q$ ?

In 2015, Marques, Ramirez and Silva [6] proved that the answer for the previous question is ‘no’ for lacunary power series in  $\mathbb{Q}[[z]]$  (see [1] for the definition of lacunary power series as well as some results related to their arithmetic properties). Moreover, their proof also implies that there is no transcendental entire function  $f(z) \in \mathbb{Q}[[z]]$  such that  $f(\mathbb{Q}) \subseteq \mathbb{Q}$  and

$$\text{den } f(p/q) = o(q).$$

In an attempt of answering the previous question, a natural question arises: Could  $\text{den } f(p/q)$  be a polynomial in  $q$  for all sufficiently large  $q$ ?

In this paper, we shall answer this previous question by proving that

**Theorem 1.** There is no transcendental entire function  $f(z) \in \mathbb{Q}[[z]]$  such that  $f(\mathbb{Q}) \subseteq \mathbb{Q}$  and

$$\text{den } f(p/q) = F(q),$$

for all sufficiently large  $q$ , where  $F(z) \in \mathbb{Z}[z]$ .

## 2 The proof

Suppose, towards a contradiction, that for some  $F(z) \in \mathbb{Z}[z]$  with degree  $m \geq 1$  (the case  $m = 0$  was solved in Remark 2.1 of [6]), there exists such a function, say  $f(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{Q}[[z]]$ .

Thus, for all sufficiently large  $q$ , we have that  $f(1/q) = n(q)/F(q)$ , where  $n(q)$  is an integer. Then

$$f\left(\frac{1}{q}\right) - \left(a_0 + \frac{a_1}{q} + \cdots + \frac{a_{m-1}}{q^{m-1}}\right) = \sum_{k \geq m} \frac{a_k}{q^k}.$$

By setting  $A = \prod_{i=0}^{m-1} \text{den}(a_i)$ , we have

$$A \frac{n(q)}{F(q)} - \left(b_0 + \frac{b_1}{q} + \cdots + \frac{b_{m-1}}{q^{m-1}}\right) = \frac{Aa_m}{q^m} + A \sum_{k \geq m+1} \frac{a_k}{q^k},$$

where  $b_i = Aa_i \in \mathbb{Z}$ . Therefore,

$$A \frac{n(q)}{F(q)} - \frac{C(q)}{q^{m-1}} = \frac{Aa_m}{q^m} + A \sum_{k \geq m+1} \frac{a_k}{q^k},$$

where  $C(z) \in \mathbb{Z}[z]$  has degree  $\leq m - 1$ . After multiplying by  $F(q)$ , we obtain

$$An(q) - \frac{D(q)}{q^{m-1}} = \frac{Aa_m F(q)}{q^m} + A \sum_{k \geq m+1} \frac{a_k F(q)}{q^k},$$

where  $D(z) \in \mathbb{Z}[z]$  has degree  $\leq 2m - 1$ . However, we can write  $D(q)/q^{m-1} = E(q)/q^{m-1} + G(q)$ , where  $E, G \in \mathbb{Z}[z]$ ,  $\deg E \leq m - 2$  and  $\deg G \leq m$ . Now, write

$$An(q) - G(q) = \frac{Aa_m F(q)}{q^m} + A \sum_{k \geq m+1} \frac{a_k F(q)}{q^k} + \frac{E(q)}{q^{m-1}}. \tag{1}$$

Now, we want to evaluate the limit in the right-hand side above when  $q \rightarrow \infty$ . Let  $\epsilon$  be the leading coefficient of  $F(z)$  (which we can assume to be  $\geq 1$ ). Note that  $\lim_{q \rightarrow \infty} F(q)/q^m = \epsilon$  and  $\lim_{q \rightarrow \infty} E(q)/q^{m-1} = 0$ . Now, we need to calculate the limit of the summatory. For that, take a real number  $\delta$  such that  $0 < \delta < 1/\epsilon \leq 1$ . Then  $\delta^k < 1/\epsilon$ , for all  $k \geq m + 1$ . Thus,

$$\frac{q^k}{\delta^k} \geq \frac{q^{m+1}}{\delta^k} > qF(q)$$

for all sufficiently large  $q$  (since the degree of  $zF(z)$  is  $m + 1$  and its leading coefficient is  $\epsilon$ ). Hence,

$$\frac{|F(q)|}{q^k} = \frac{F(q)}{q^k} < \frac{1}{q\delta^k}$$

(for all sufficiently large  $q$  and for all  $k \geq m + 1$ ) and so

$$\left| \sum_{k \geq m+1} \frac{a_k F(q)}{q^k} \right| \leq \frac{1}{q} \sum_{k \geq m+1} \frac{|a_k|}{\delta^k}.$$

Since  $\sum_{k \geq m+1} |a_k|/\delta^k < \infty$  (by the absolute convergence of  $\sum_{k \geq 0} a_k z^k$  in  $\mathbb{C}$ ), then

$$\lim_{q \rightarrow \infty} \sum_{k \geq m+1} \frac{a_k F(q)}{q^k} = 0.$$

In conclusion, the right-hand side of (1) tends to  $Aa_m \epsilon$  as  $q \rightarrow \infty$ . Therefore, for all sufficiently large  $q$ , it holds that

$$0 \leq |An(q) - G(q)| \leq Aa_m \epsilon + 1.$$

Since  $An(q) - G(q)$  is an integer, then there exist  $t \in \mathbb{Z}$  and an infinite set  $S \subseteq \mathbb{N}$  such that  $An(q) - G(q) = t$  for all  $q \in S$ . Thus

$$f\left(\frac{1}{q}\right) = \frac{n(q)}{F(q)} = \frac{G(q) + t}{AF(q)} = \frac{e_0 + e_1 q + \dots + e_m q^m}{d_0 + d_1 q + \dots + d_m q^m} = \frac{P(1/q)}{Q(1/q)}, \tag{2}$$

where  $P(z) = \sum_{i=0}^m e_i z^{m-i}$  and  $Q(z) = \sum_{i=0}^m d_i z^{m-i}$ .

Let  $r$  be a positive real number such that  $r < \min\{|z| : Q(z) = 0\}$  (observe that  $Q(0) = d_m = A\epsilon \neq 0$ ). Then the function  $h(z)$  given by

$$h(z) = \frac{P(z)}{Q(z)}$$

is analytic on the interval  $(-r, r)$ . Moreover, by (2), we have that the analytic functions  $f(z)$  and  $h(z)$  coincide on the set  $\{1/q : q \in S \cap (1/r, \infty)\} \subseteq (-r, r)$  which has a limit point in  $(-r, r)$ . Thus, by the identity principle for analytic functions, we have that  $f(z) = h(z)$  on  $(-r, r)$ . In particular, the entire functions  $Q(z)f(z)$  and  $P(z)$  coincide on  $(-r, r)$  yielding, by the same principle, that they have equal values for all  $z \in \mathbb{C}$ . Hence the function  $f(z)$  satisfies  $P(z, f(z)) = 0$ , for all  $z \in \mathbb{C}$ , where  $P(x, y) = Q(x)y - P(x)$  (which is a nonzero polynomial). However, this contradicts the transcendence of  $f$ . The proof is then complete.  $\square$

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