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On a class of (p,q)-Laplacian problems involving the critical Sobolev-Hardy exponents in starshaped domain

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded *starshaped* domain and consider the (p,q)-Laplacian problem

$$-\Delta_p u - \Delta_q u = \lambda(\mathbf{x})|u|^{p^\star-2}u + \mu|u|^{r-2}u$$

where μ is a positive parameter, $1 < q \le p < n$, $r \ge p^\star$ and $p^\star := \frac{np}{n-p}$ is the critical Sobolev exponent. In this short note we address the question of *non-existence* for *non-trivial* solutions to the (p,q)-Laplacian problem. In particular we show the non-existence of non-trivial solutions to the problem by using a method based on *Pohozaev* identity.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded *starshaped* domain with smooth boundary. In this short paper we consider the *quasi-linear* elliptic problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda(\mathbf{x})|u|^{p^\star-2}u + \mu|u|^{r-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $\mu \ge 0$, $1 < q \le p < n$, $r \ge p^\star$, $\lambda \in \mathrm{C}^1(\Omega)$ and $p^\star := \frac{np}{n-p}$ is the critical Sobolev exponent. This kind of elliptic problems involving the (p,q)-Laplacian operator

$$\Delta_p u + \Delta_q u := \operatorname{div}\Big[|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\Big]$$

which appears in a variety of areas from reaction-diffusion equations [5] and models of elementary particles [2], [6] counterparts to more applied branches in chemical reaction design, plasma physics [20], [15] and biophysics [7]. For a comprehensive treatment of the quasi-linear elliptic boundary value problems involving (p,q)-Laplacian operator we refer the interested reader to [1], [3], [17], [19] and the references therein.

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Here we are primarily concerned with the question of *existence* of *non-trivial* solutions to the quasi-linear elliptic problem (1) in a starshaped domain.[1] Since this type of quasi-linear elliptic problems involves critical Sobolev exponent that causes the main difficulty, as the lack of *compactness*. This turns out to impose the absence of a direct sum decomposition suitable for applying the *Linking* theorem for obtaining the existence and regularity results. However due to both the intrinsic mathematical interest and its applications to sciences many results have been settled on this kind of problems in recent years. (See e.g., [4], [8], [13], [14]).

In this paper in *contrast* to what seen before we introduce a new *non-existence* results of non-trivial solution to the problem (1) subject to Ω being strictly starshaped domain and $\langle \nabla\lambda, \mathbf{x}\rangle \le 0$ for a.e. $\mathbf{x} \in \Omega$. This approach has the advantage of *Pohozaev* identity (see Theorem 1 below). Consequently this conclusion extends to quasi-linear elliptic problem with multiple critical Sobolev-Hardy terms in Theorem 2. Indeed we show that the quasi-linear partial differential equation

$$\begin{cases} -\Delta_p u - \Delta_q u = \frac{1}{|\mathbf{x}|^s}\lambda(\mathbf{x})|u|^{p^\star(s)-2}u + \mu|u|^{r-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

which involves the critical Sobolev-Hardy exponents does not admit a non-trivial solution. One can conclude that the effect of domain topology and geometry on multiplicity versus uniqueness of solutions plays a significant role. Without *further* restriction on the domain Ω non-existence result of non-trivial solutions in general may *fail*. Indeed one can construct domains Ω such that this type of quasi-linear problem admits multiple *infinitely* many non-trivial solutions. (See [9], [10], [15], [19].)

2 The main results

Before presenting the paper's main results we pause briefly to state the following propositions, main ideas and tools which will turn useful during the proofs.

Proposition 1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with boundary $\partial\Omega$. Then subject to $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\bar{\Omega})$ and $u|_{\partial\Omega} = 0$ we have that*

$$\textbf{(1)}\ \int_\Omega \langle \nabla u, \mathbf{x}\rangle \Delta_p u\, d\mathbf{x} = \frac{p-1}{p}\int_{\partial\Omega} |\nabla u|^p \langle \mathbf{x}, \nu\rangle\, d\sigma + \frac{n-p}{p}\int_\Omega |\nabla u|^p\, \mathrm{d}\mathbf{x},$$

$$\textbf{(2)}\ \int_\Omega u\ \Delta_p u\, \mathrm{d}\mathbf{x} = -\int_\Omega |\nabla u|^p\, \mathrm{d}\mathbf{x},$$

where ν is the unit outwards normal to the boundary $\partial\Omega$.

[1]Note that the weak solutions of the problem (1) coincide with the critical points of the energy functional

$$\mathbb{E}[u;\Omega] := \int_\Omega \Big[\frac{1}{p}|\nabla u|^p + \frac{1}{q}|\nabla u|^q - \frac{1}{p^*}\lambda(\mathbf{x})|u|^{p^\star} - \frac{1}{r}\mu|u|^r\Big]\, \mathrm{d}\mathbf{x}$$

over the Sobolev space $W_0^{1,p}(\Omega)$.

Proof. The proof of these assertions is based on the *divergence* theorem along with a direct verification. Indeed for (**1**) by direct *differentiation* we can write

$$\begin{aligned}\operatorname{div}\Big[\langle \nabla u, \mathbf{x}\rangle |\nabla u|^{p-2}\nabla u\Big] &= \langle \nabla u, \mathbf{x}\rangle \operatorname{div}\Big[|\nabla u|^{p-2}\nabla u\Big] + \langle \nabla(\langle \nabla u, \mathbf{x}\rangle), |\nabla u|^{p-2}\nabla u\rangle \\ &= \langle \nabla u, \mathbf{x}\rangle \Delta_p u + |\nabla u|^{p-2}\langle \nabla(\langle \nabla u, \mathbf{x}\rangle), \nabla u\rangle \\ &= \langle \nabla u, \mathbf{x}\rangle \Delta_p u + |\nabla u|^p + \frac{1}{2}|\nabla u|^{p-2}\langle \nabla |\nabla u|^2, \mathbf{x}\rangle \end{aligned} \tag{2}$$

where in obtaining the last identity we have used the fact that

$$\langle \nabla\big(\langle \nabla u, \mathbf{x}\rangle\big), \nabla u\rangle = |\nabla u|^2 + \frac{1}{2}\langle \nabla |\nabla u|^2, \mathbf{x}\rangle\,.$$

In order to further simplify the identity (2), we notice that

$$\begin{aligned}\frac{1}{2}|\nabla u|^{p-2}\langle \nabla |\nabla u|^2, \mathbf{x}\rangle &= \frac{1}{p}\langle \nabla |\nabla u|^p, \mathbf{x}\rangle \\ &= \frac{1}{p}\Big[\operatorname{div}(|\nabla u|^p \mathbf{x}) - n|\nabla u|^p\Big].\end{aligned}$$

Consequently substituting this into (2), upon simplification, results in

$$\operatorname{div}\Big[\langle \nabla u, \mathbf{x}\rangle |\nabla u|^{p-2}\nabla u - \frac{1}{p}|\nabla u|^p \mathbf{x}\Big] = \langle \nabla u, \mathbf{x}\rangle \Delta_p u + \frac{p-n}{p}|\nabla u|^p.$$

Thus by utilising of the divergence theorem in conjunction with the above identity we conclude that

$$\begin{aligned}\int_\Omega \Big[\langle \nabla u, \mathbf{x}\rangle \Delta_p u + \frac{p-n}{p}|\nabla u|^p\Big]\,\mathrm{d}\mathbf{x} &= \int_\Omega \operatorname{div}\Big[\langle \nabla u, \mathbf{x}\rangle |\nabla u|^{p-2}\nabla u - \frac{1}{p}|\nabla u|^p \mathbf{x}\Big]\,\mathrm{d}\mathbf{x} \\ &= \int_{\partial\Omega} \Big[\langle \nabla u, \mathbf{x}\rangle |\nabla u|^{p-2}\langle \nabla u, \nu\rangle - \frac{1}{p}|\nabla u|^p \langle \mathbf{x}, \nu\rangle\Big]\,\mathrm{d}\sigma.\end{aligned}$$

In view of u being zero on $\partial\Omega$ we conclude that $\nabla u = u_\nu \nu$ on $\partial\Omega$. This fact together with the above identity gives (**1**). To prove (**2**) the argument here is in a similar way. First we note that

$$\begin{aligned}\operatorname{div}\Big[u|\nabla u|^{p-2}\nabla u\Big] &= u \operatorname{div}\Big[|\nabla u|^{p-2}\nabla u\Big] + \langle \nabla u, |\nabla u|^{p-2}\nabla u\rangle \\ &= u\ \Delta_p u + |\nabla u|^p.\end{aligned}$$

Alternatively an application of divergence theorem together with the fact that $u|_{\partial\Omega} = 0$ implies that

$$\begin{aligned}\int_\Omega \Big[u\ \Delta_p u + |\nabla u|^p\Big]\,\mathrm{d}\mathbf{x} &= \int_\Omega \operatorname{div}\Big[u|\nabla u|^{p-2}\nabla u\Big]\,\mathrm{d}\mathbf{x} \\ &= \int_{\partial\Omega} u|\nabla u|^{p-2}\langle \nabla u, \nu\rangle\,\mathrm{d}\sigma = 0\end{aligned}$$

which is the required conclusion. □

Proposition 2. *With the aid of similar assumptions on u and Ω used in the previous proposition together with $f(\mathbf{x},u) := \lambda(\mathbf{x})|u|^{p^\star-2}u + \mu|u|^{r-2}u$, we have also that*

$$\int_\Omega \langle \nabla u, \mathbf{x}\rangle f(\mathbf{x},u)\,\mathrm{d}\mathbf{x} = -\frac{1}{p^\star}\int_\Omega \Big[n\lambda(\mathbf{x}) + \langle \nabla\lambda, \mathbf{x}\rangle\Big]|u|^{p^\star}\,\mathrm{d}\mathbf{x} - \frac{n}{r}\int_\Omega \mu|u|^r\,\mathrm{d}\mathbf{x}.$$

Proof. For the sake of convenience and reasons that will become clear shortly we introduce the function

$$\mathrm{F}(\mathbf{x},u) : = \int_0^u f(\mathbf{x},v)\,\mathrm{d}v = \frac{1}{p^\star}\lambda(\mathbf{x})|u|^{p^\star} + \frac{1}{r}\mu|u|^r.$$

Evidently with the aid of this we can write

$$\begin{aligned}\frac{\partial}{\partial x_i}(\mathrm{F}(\mathbf{x},u)) &= \frac{\partial \mathrm{F}}{\partial x_i}(\mathbf{x},u) + f(\mathbf{x},u)\frac{\partial u}{\partial x_i}\\ &= \frac{1}{p^\star}|u|^{p^\star}\frac{\partial\lambda}{\partial x_i}(\mathbf{x}) + f(\mathbf{x},u)\frac{\partial u}{\partial x_i}.\end{aligned}$$

As this is true for every $1 \le i \le n$ using vector notation we can express this as

$$f(\mathbf{x},u)\nabla u = \nabla\mathrm{F} - \frac{1}{p^\star}|u|^{p^\star}\nabla\lambda.$$

To this end, by using the above equality we have

$$\begin{aligned}\int_\Omega f(\mathbf{x},u)\langle\nabla u,\mathbf{x}\rangle\,\mathrm{d}\mathbf{x} &= \int_\Omega\Big[\langle\nabla\mathrm{F},\mathbf{x}\rangle - \frac{1}{p^\star}|u|^{p^\star}\langle\nabla\lambda,\mathbf{x}\rangle\Big]\,\mathrm{d}\mathbf{x}\\ &= \int_{\partial\Omega}\mathrm{F}(\mathbf{x},u)\langle\mathbf{x},\nu\rangle\,\mathrm{d}\sigma - n\int_\Omega \mathrm{F}(\mathbf{x},u)\,\mathrm{d}\mathbf{x} - \int_\Omega\frac{1}{p^\star}|u|^{p^\star}\langle\nabla\lambda,\mathbf{x}\rangle\,\mathrm{d}\mathbf{x}\\ &= -n\int_\Omega\Big[\frac{1}{p^\star}\lambda(\mathbf{x})|u|^{p^\star} + \frac{1}{r}\mu|u|^r\Big]\,\mathrm{d}\mathbf{x} - \int_\Omega\frac{1}{p^\star}|u|^{p^\star}\langle\nabla\lambda,\mathbf{x}\rangle\,\mathrm{d}\mathbf{x},\end{aligned}$$

where in the *second* and *third* lines we have used the divergence theorem and the fact that F vanishing on $\partial\Omega$ respectively. The proof of the proposition is thus complete. □

Motivated by the above propositions, we are now in a position to state the following *non-existence* results on (p,q)-Laplacian problem.

Theorem 1 (Non-existence I). *Let $\Omega \subset \mathbb{R}^n$ be a C^1 bounded starshaped domain with respect to the origin. Then subject to $\langle\nabla\lambda,\mathbf{x}\rangle \le 0$ for a.e. $\mathbf{x}\in\Omega$, the (p,q)-problem described in* (1) *admits no non-trivial solution in $W_0^{1,p}(\Omega)$.*

Proof. Assume the contrary. Then there exists *non-trivial* solution u, satisfies in stated (p,q)-Laplacian problem (1). In what follows we assume without loss of generality that u has the necessary regularity, otherwise we can use an approximation argument as in [11]. Hence, with the notation used in the Proposition 2 we have

$$
-\Delta_p u - \Delta_q u = f(\mathbf{x},u) \Longrightarrow \left\{\begin{array}{l} -\langle \mathbf{x}, \nabla u\rangle \Big[\Delta_p u + \Delta_q u\Big] = \langle \mathbf{x}, \nabla u\rangle f(\mathbf{x},u) \\ \\ -u\Big[\Delta_p u + \Delta_q u\Big] = u f(\mathbf{x},u) \end{array}\right\}
$$

$$
\Longrightarrow \left\{\begin{array}{ll} \displaystyle\int_\Omega \langle \mathbf{x}, \nabla u\rangle \Big[\Delta_p u + \Delta_q u\Big] = -\int_\Omega \langle \mathbf{x}, \nabla u\rangle f(\mathbf{x},u) & (\mathbf{I}) \\ \\ \displaystyle\int_\Omega u\Big[\Delta_p u + \Delta_q u\Big] = -\int_\Omega u f(\mathbf{x},u) & (\mathbf{II}) \end{array}\right\}
$$

We now proceed by simplifying of each term separately. Regarding the first identity, using (**1**) in Proposition 1 together with Proposition 2 we arrive at

$$
\begin{aligned}
(\mathbf{I}) \Longleftrightarrow \frac{p-1}{p}\int_{\partial\Omega} |\nabla u|^p \langle \mathbf{x}, \nu\rangle \,\mathrm{d}\sigma + \frac{n-p}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}\mathbf{x} \\
+ \frac{q-1}{q}\int_{\partial\Omega} |\nabla u|^q \langle \mathbf{x}, \nu\rangle \,\mathrm{d}\sigma + \frac{n-q}{q}\int_\Omega |\nabla u|^q \,\mathrm{d}\mathbf{x} \\
= \frac{1}{p^\star}\int_\Omega \Big[n\lambda(\mathbf{x}) + \langle \nabla\lambda, \mathbf{x}\rangle\Big] |u|^{p^\star} \,\mathrm{d}\mathbf{x} + \frac{n}{r}\int_\Omega \mu |u|^r \,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3}
$$

Consequently an application of Proposition 1 part (**2**) gives

$$
\begin{aligned}
(\mathbf{II}) &\Longleftrightarrow \int_\Omega |\nabla u|^p \,\mathrm{d}\mathbf{x} + \int_\Omega |\nabla u|^q \,\mathrm{d}\mathbf{x} = \int_\Omega \lambda(\mathbf{x}) |u|^{p^\star} \,\mathrm{d}\mathbf{x} + \int_\Omega \mu |u|^r \,\mathrm{d}\mathbf{x} \\
&\Longleftrightarrow \int_\Omega |\nabla u|^p \,\mathrm{d}\mathbf{x} = -\int_\Omega |\nabla u|^q \,\mathrm{d}\mathbf{x} + \int_\Omega \lambda(\mathbf{x}) |u|^{p^\star} \,\mathrm{d}\mathbf{x} + \int_\Omega \mu |u|^r \,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Motivated by the above identity and substituting it into (3), upon re-arranging, it follows that

$$
\begin{aligned}
\int_{\partial\Omega} \Big[\frac{p-1}{p}|\nabla u|^p + \frac{q-1}{q}|\nabla u|^q\Big] \langle \mathbf{x}, \nu\rangle \,\mathrm{d}\sigma &= n\Big(\frac{1}{p} - \frac{1}{q}\Big)\int_\Omega |\nabla u|^q \,\mathrm{d}\mathbf{x} + \frac{1}{p^\star}\int_\Omega \langle \nabla\lambda, \mathbf{x}\rangle |u|^{p^\star} \,\mathrm{d}\mathbf{x} \\
&\quad + \Big(\frac{p-n}{p} + \frac{n}{p^\star}\Big)\int_\Omega \lambda(\mathbf{x}) |u|^{p^\star} \,\mathrm{d}\mathbf{x} \\
&\quad + \Big(\frac{p-n}{p} + \frac{n}{r}\Big)\int_\Omega \mu |u|^r \,\mathrm{d}\mathbf{x} \\
&=: \int_\Omega \Big[\alpha |\nabla u|^q + \frac{1}{p^\star}\langle \nabla\lambda, \mathbf{x}\rangle |u|^{p^\star} + \beta |u|^r\Big] \,\mathrm{d}\mathbf{x}
\end{aligned}
$$

where in the last line we have used the fact that $\frac{p-n}{p} + \frac{n}{p^\star} = 0$ and for the sake of brevity we have also set

$$
\begin{cases}
\alpha = \alpha(n,p,q) := n\Big(\dfrac{1}{p} - \dfrac{1}{q}\Big), \\
\\
\beta = \beta(n,p,r,\mu) := \mu\Big(\dfrac{p-n}{p} + \dfrac{n}{r}\Big).
\end{cases}
$$

Since $\mu \geq 0$, $q \leq p$ and $p^\star \leq r$ we conclude that $\alpha, \beta < 0$ and therefore the *right-hand* side of earlier estimate is *non-positive*. This however is a contradiction since $\langle \mathbf{x}, \nu \rangle > 0$ on $\partial\Omega$ and $u \neq 0$ which make the *left-hand* side to be *positive* and hence the proof is complete.[2] □

On passing we point out an immediate consequence of the above theorem which is the following non-existence result for p-Laplacian problem. It is instructive to compare this corollary with the results in [10].

Corollary 1. *Let $\Omega \subset \mathbb{R}^n$ be a C^1 bounded starshaped domain with respect to the origin and consider*

$$\begin{cases} -\Delta_p u = \lambda(\mathbf{x})|u|^{p^\star-2}u + \mu|u|^{r-2}u(\mathbf{x}) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $1 < p < n$, $\frac{np}{n-p} =: p^\star \leq r$ and $\mu \geq 0$. Then subject to $\langle \nabla\lambda, \mathbf{x} \rangle \leq 0$ for a.e. $\mathbf{x} \in \Omega$, the described problem admits no non-trivial solution in $W_0^{1,p}(\Omega)$.

Proof. The conclusion follows from the previous theorem with the particular choice of $q = p$. □

We end the section by a very quick outline of some further results and generalisation of Theorem 1 when we replace the critical Sobolev exponent with critical Sobolev-Hardy exponents.

Theorem 2 (Non-existence II). *Let $\Omega \subset \mathbb{R}^n$ be a C^1 bounded starshaped domain with respect to the origin and consider the quasi-linear elliptic problem*

$$\begin{cases} -\Delta_p u - \Delta_q u = \dfrac{1}{|\mathbf{x}|^s}\lambda(\mathbf{x})|u|^{p^\star(s)-2}u + \mu|u|^{r-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{4}$$

where $1 < q \leq p < n$, $0 \leq s < p$, $\frac{p(n-s)}{n-p} =: p^\star(s) \leq r$ and $\mu \geq 0$. Then subject to $\langle \nabla\lambda, \mathbf{x} \rangle \leq 0$ for a.e. $\mathbf{x} \in \Omega$, the described problem admits no non-trivial solution in $W_0^{1,p}(\Omega)$.

Proof. The argument here is based upon suitably modifying of the technique from Theorem 1. To this end we put

$$g(\mathbf{x}, u) := \frac{1}{|\mathbf{x}|^s}\lambda(\mathbf{x})|u|^{p^\star(s)-2}u + \mu|u|^{r-2}u.$$

Then a similar proof to that in Proposition 2 shows

$$\int_\Omega \langle \nabla u, \mathbf{x} \rangle g(\mathbf{x}, u) = -\frac{1}{p^\star(s)} \int_\Omega \Big[(n-s)\lambda(\mathbf{x}) + \langle \nabla\lambda, \mathbf{x} \rangle \Big] \frac{|u|^{p^\star(s)}}{|\mathbf{x}|^s}\, \mathrm{d}\mathbf{x} - \frac{n}{r}\int_\Omega \mu|u|^r\, \mathrm{d}\mathbf{x}.$$

The remainder of the proof is exactly the same to that in Theorem 1 by utilising the above identity and therefore will be abbreviated. □

[2] It can easy to see that for *starshaped* domain, Ω, with respect to the origin we have $\langle \mathbf{x}, \nu \rangle > 0$ on $\partial\Omega$ (See [16]).

Remark 1. We end this paper by giving a direct and natural generalisation of the previous theorem to the case of (p,q)-Laplacian equations involving *multiple* critical Sobolev-Hardy terms. Indeed for $\Omega \subset \mathbb{R}^n$ bounded and smooth starshaped domain with respect to the origin, the following (p,q)-Laplacian problem subject to $\langle \nabla\lambda_i, \mathbf{x}\rangle \le 0$ for a.e. $\mathbf{x} \in \Omega$ and $1 \le i \le k$ admits no non-trivial solution in $W_0^{1,p}(\Omega)$.

$$\begin{cases} -\Delta_p u - \Delta_q u = \displaystyle\sum_{i=1}^{k} \frac{1}{|\mathbf{x}|^{s_i}} \lambda_i(\mathbf{x})|u|^{p^\star(s_i)-2}u + \mu|u|^{r-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $1 < q \le p < n$, $0 \le s_i < p$, $\frac{p(n-s_i)}{n-p} =: p^\star(s_i) \le r$ and $\mu \ge 0$.

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