ON NUMERICAL METHODS FOR A BOUNDARY LAYER ON A BODY OF REVOLUTION

BAYEZID HOSSAIN AND ALI R. ANSARI

Department of Mathematics & Statistics, University of Limerick,
Limerick, Ireland

GRIGORII I. SHISHKIN

Institute of Mathematics and Mechanics, Russian Academy of Sciences,
Ekaterinburg, Russia

Dedicated to John J.H. Miller on the occasion of his 65th birthday.

Abstract — The flow of a viscous incompressible fluid past a body of revolution with a parabolic profile when the stream is parallel to its axis falls into a class of problems that exhibit boundary layers. This problem does not have solutions in closed form, and is modeled by boundary-layer equations. Using a self-similar approach based on a Blasius series expansion (up to two terms), the boundary-layer equations can be reduced to a Blasius-type problem consisting of a system of three 3rd order ordinary differential equations on a semi-infinite interval. Numerical methods need to be employed to attain the solutions of these equations and their derivatives, which are required for the computation of the velocity components, on a finite domain with accuracy independent of the viscosity $\nu$, which can take arbitrary values from the interval $(0, 1]$. Numerical methods for which the accuracy of the velocity components depend on the number of mesh points $N$, used to solve the Blasius-type problem, and do not depend on the viscosity $\nu$, are referred to as robust methods. To construct a robust numerical method we reduce the original problem on a semi-infinite axis to a problem on the finite interval $[0, K]$, where $K = K(N) = \ln N$. Employing numerical experiments, we justify that the constructed numerical method is parameter robust.

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Keywords: body of revolution, parameter robust, boundary layers.

1. Introduction

The flow of an incompressible fluid past a body of revolution when the stream is parallel to its axis, exhibits boundary layers on the surface of the body. Some linear and nonlinear two-dimensional problems with boundary layers [4] have been treated using piecewise uniform meshes [1–3, 5, 6].

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For boundary layers on a body of revolution we have to deal with more complicated geometries and singularities of the solution related to the stagnation point. Assuming a curvilinear coordinate system, we denote the length along a meridian from the stagnation point by $x$, and the coordinate at right angles to the surface by $y$. The contour of the body of revolution is represented by radii $r(x)$ which essentially represent the sections taken at right angles to the axis of the body of revolution. We will assume that there are no sharp corners, i.e., $\frac{d^2 r}{dx^2}$ does not become large. We further assume that the minimum radius of the body of revolution is much larger than the thickness of the boundary layer. Furthermore, we denote the velocity components as $u$ parallel to the surface, and $v$ normal to the surface. The potential flow is denoted by $U(x)$. The steady state boundary-layer equations for this problem can now be written as [7]

\begin{align}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \\
\frac{\partial (ur)}{\partial x} + \frac{\partial (vr)}{\partial y} &= 0
\end{align}

with the boundary conditions

\begin{align}
u(x,0) = v(x,0) &= 0, \quad \lim_{y \to \infty} u(x,y) = U(x),
\end{align}

where $\rho$ is the density, $\nu$ is the viscosity of the fluid, and $p$ is the pressure. Standard magnitude analysis shows that the pressure gradient in the $y$ direction is approximately of order one [7]. Consequently, it is possible to assume that the pressure gradient of the potential stream $U(x)$ is acting on the boundary layer thus

\begin{equation}
U \frac{dU}{dx} = -\frac{1}{\rho} \frac{\partial p}{\partial x}.
\end{equation}

For the purpose of our preliminary analysis here we will consider the body of revolution to be the sphere, the body contour is therefore given as

\begin{equation}
r(x) = R \sin \left( \frac{x}{R} \right),
\end{equation}
where \( R \) is the radius of the section of the surface. The velocity distribution at the surface of the body is

\[
U(x) = \frac{3}{2} U_\infty \sin \left( \frac{x}{R} \right),
\]

where \( U_\infty \) is the free stream velocity parallel to the \( x \) axis and \( x/R \) denotes the central angle measured from the stagnation point.

It is known that as the viscosity \( \nu \) of the fluid changes, the thickness of the boundary layer is affected. It is desirable to have numerical methods for which the associated errors for the velocity components \( u, v \) are independent of the viscosity \( \nu \). From a physical point of view \( \nu \) does not have to approach zero, so we will adopt this nonphysical range for the purpose of testing our method. The work that will follow in the succeeding sections concerns a self-similar approach to finding the velocity components using a Blasius series expansion up to two terms, \( i.e., \) a body of revolution with a parabolic profile is considered. This results in a system of three 3rd order ordinary differential equations which need to be solved numerically. The approach here is similar to that introduced in [2] on a semi-infinite axis for the boundary layer over a flat plate. However, there the problem was reduced to just one ordinary differential equation that was required to solve numerically. Here the problem is more complicated as simultaneous numerical solutions of three ordinary differential equations are required and the results then have to be combined to compute the velocity components. We will employ monotone methods [2] to solve these differential equations.

2. Problem formulation

It is worth noting that no explicit solutions are known for problem (1) – (6). Following the analysis of [7] employing a Blasius series approach, it is possible to attain a semi-analytical self-similar solution \( \mathbf{u}_B = (u_B, v_B) \) which can be written as

\[
u_B(x, y) = \frac{\partial \psi}{\partial y},
\]

\[
v_B(x, y) = -\frac{\partial \psi}{\partial x} - \frac{1}{r} \frac{dr}{dx} \psi,
\]

where \( \psi \) is the stream function. Using (4), (7), and (8) reduces equation (1) to

\[
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \left( \frac{\partial \psi}{\partial x} + \frac{1}{r} \frac{dr}{dx} \psi \right) \frac{\partial^2 \psi}{\partial y^2} = U' \frac{dU}{dx} + \nu \frac{\partial^3 \psi}{\partial y^3}
\]

with the boundary conditions

\[
\psi(x, 0) = \frac{\partial \psi}{\partial y}(x, 0) = 0, \quad \lim_{y \to \infty} \frac{\partial \psi(x, y)}{\partial y} = U(x).
\]

Note that as we are interested in only considering the first two terms of the Blasius series expansion, thus from (5) we have the parabolic body of revolution given by [7]:

\[
r(x) = R \left\{ \frac{x}{R} - \frac{1}{3!} \frac{x^3}{R^3} \right\}
\]

\[
U(x) = \frac{3}{2} U_\infty \sin \left( \frac{x}{R} \right),
\]
with the velocity distribution

$$U(x) = \frac{3}{2} U_\infty \left\{ \frac{x}{R} - \frac{1}{3!} \frac{x^3}{R^3} \right\},$$

(10)

which leads to the stream function

$$\psi(x, y) = \sqrt{\frac{\nu U_\infty R}{12}} \left[ 3 \frac{x}{R} f_1(\eta) - \left( \frac{x}{R} \right)^3 f_3(\eta) \right]$$

with

$$\eta = \frac{y}{R} \sqrt{\frac{3 U_\infty R}{\nu}}.$$

(11)

We are interested in finding $u_B$, i.e., the self-similar solution of problem (1) – (4), (9), (10) in the rectangular domain $\Omega = (0, 1) \times (0, 1)$; in accordance with (7), (8) the velocity components are given by

$$u_B(x, y) = \frac{3 U_\infty}{2R} x f_1'(\eta) - \frac{U_\infty}{2 R^3} x^3 f_3'(\eta)$$

(12)

and

$$v_B(x, y) = \sqrt{\frac{3 \nu U_\infty}{R}} \left\{ \frac{6 R^2 - 2 x^2}{6 R^2 - x^2} f_1(\eta) + \frac{4 x^2 R^2 - x^4}{R^2 (6 R^2 - x^2)} f_3(\eta) \right\},$$

(13)

where $\eta$ is given by (11) and the viscosity $\nu$ can take arbitrary values in the half interval $(0, 1]$. Thus, both components $u(x, y)$ and $v(x, y)$, $(x, y) \in \Omega$ depend on the parameter $\nu$.

Here $f_1(\eta)$ is the solution of the problem

$$f'''_1(\eta) = -f_1(\eta) f''_1(\eta) + \frac{1}{2} \left( f''_1(\eta) - 1 \right), \quad 0 < \eta < \infty$$

(14)

with the boundary conditions

$$f_1(0) = f'_1(0) = 0, \quad \lim_{\eta \to \infty} f'_1(\eta) = 1.$$  

(15)

The function $f_3(\eta)$ can be represented by

$$f_3(\eta) = g_3(\eta) + h_3(\eta),$$

(16)

where the function $g_3(\eta)$ is the solution of the problem

$$g'''_3(\eta) = -f_1(\eta) g''_3(\eta) + 2 f'_1(\eta) g'_3(\eta) - 2 f''_1(\eta) g_3(\eta) - 1, \quad 0 < \eta < \infty$$

(17)

with the boundary conditions

$$g_3(0) = g'_3(0) = 0, \quad \lim_{\eta \to \infty} g'_3(\eta) = \frac{1}{2}$$

(18)

and the function $h_3(\eta)$ is the solution of the problem

$$h'''_3(\eta) = -f_1(\eta) h''_3(\eta) + 2 f'_1(\eta) h'_3(\eta) - 2 f''_1(\eta) h_3(\eta) - \frac{1}{2} f_1(\eta) f''_3(\eta), \quad 0 < \eta < \infty$$

(19)
with the boundary conditions

\[ h_3(0) = h'_3(0) = 0, \quad \lim_{\eta \to \infty} h'_3(\eta) = 0. \] (20)

Our aim is to construct a robust numerical method for singular problems (14) – (20), i.e., the method for which the accuracy of the components \(u_B(x, y)\) and \(v_B(x, y)\) given by (12) and (13), respectively, is independent of the parameter \(\nu\) and is defined only by the number of mesh points used to solve problem (14) – (20) on \([0, \infty)\). To construct such a robust method in accordance with (12) and (13), it is sufficient to have an accurate method to approximate \(f_1(\eta), f'_1(\eta)\) and \(f_3(\eta), f'_3(\eta)\) (or \(g_3(\eta), g'_3(\eta), h_3(\eta), h'_3(\eta)\)) uniformly on the semi-infinite axis \([0, \infty)\).

Note that when the parameter \(\nu\) is bounded from below, i.e.,

\[ \nu \geq \nu_0 > 0, \]

then \(\eta\) varies on the bounded interval \([0, \eta_0]\), for all \((x, y) \in \Omega\), where

\[ \eta_0 = \frac{1}{R} \sqrt{\frac{3U_\infty R}{\nu_0}}. \]

It is worth noting that the computation of the components \(u_B(x, y)\) and \(v_B(x, y)\) for \((x, y) \in \Omega, \nu \in [\nu_0, 1]\), if \(f'_1(\eta_0), g'_3(\eta_0)\) and \(h'_3(\eta_0)\) are given, is a simple problem, i.e., no specific robust method is required.

3. Numerical solution

In the succeeding subsections we will first construct numerical methods to solve problem (14) – (15), which will give us the approximation to \(f_1\). We will next consider numerical methods for problems (17) – (20), which will give us the approximations to \(g_3\) and \(h_3\), which in turn will give us \(f_3\). Finally, using these numerical solutions, we will compute the required velocity components. In Section 4, we will show with the aid of numerical experiments that the constructed numerical method is robust.

3.1. Numerical solution of problem (14) – (15)) for \(f_1\)

To solve problem (14) – (15) we use a technique similar to that introduced in [2]. Instead of solving this problem on a semi-infinite domain \([0, \infty)\), we solve this problem on a finite domain \([0, K]\), where \(K\) is an auxiliary parameter giving the length of the interval on which we are now going to solve the approximate problems. Thus, we have

\[ f''''_{1K}(\eta) + f'_{1K}(\eta)f''_{1K}(\eta) - \frac{1}{2} \left( f'_{1K}(\eta) \right)^2 = -\frac{1}{2}, \] (21)

\[ f_{1K}(0) = f'_{1K}(0) = 0; \quad f'_{1K}(K) = 1. \] (22)

To determine the approximate solution on the domain \([K, \infty)\), we extend the function \(f_{1K}\) using the extrapolation

\[ f_{1K}(\eta) = (\eta - K) + f_{1K}(K) \quad \forall \eta \geq K. \] (23)
And from (23) we have
\[ f'_{1K}(\eta) = 1, \quad f''_{1K}(\eta) = 0 \quad \forall \eta \geq K. \] (24)
Thus, the auxiliary continual problem (21) – (24) is defined by the parameter \( K \). The solution \( f_1(\eta) \) of problem (14) – (15) is the limiting solution of problem (21) – (24) for \( K \to \infty \).

To solve problem (21) – (22) we use a finite difference approximation on a uniform mesh on the interval \([0, K_N]\) with \( N \) number of mesh nodes [2]
\[ \bar{I}^N = \{ \eta_i \mid \eta_i = iN^{-1}K_N, \quad 0 \leq i \leq N \}. \] (25)
Problem (21) – (22) is approximated by the nonlinear finite difference scheme
\[ \delta^2(D^-F_1)(\eta_i) + F_1(\eta_i)D^+(D^-F_1)(\eta_i) - \frac{1}{2}(D^-F_1)(\eta_i)(D^-F_1)(\eta_i) = -\frac{1}{2} \] (26)
with
\[ F_1(0) = D^+F_1(0) = 0 \quad \text{and} \quad D^0F_1(\eta_{N-1}) = 1, \] (27)
where
\[ D^-F_1(\eta_i) = \frac{F_1(\eta_i) - F_1(\eta_{i-1})}{h}, \quad D^+F_1(\eta_i) = \frac{F_1(\eta_{i+1}) - F_1(\eta_i)}{h}, \] (28)
\[ \delta^2F_1(\eta_i) = \frac{D^+F_1(\eta_i) - D^-F_1(\eta_i)}{h}, \quad D^0F_1(\eta_i) = \frac{F_1(\eta_{i+1}) - F_1(\eta_{i-1})}{2h} \] (29)
and the mesh spacing is defined as
\[ h = \frac{\ln N}{N}. \] (30)

For \( \eta \in [K_N, \infty) \) we use
\[ \bar{F}_{1K}(\eta) = (\eta - K_N) + \bar{F}_{1K}(K_N) \quad \forall \eta \geq K_N. \] (31)
The solution of problem (26) – (31) depends on two parameters \( K \) and \( N \). Similar to [2], one can justify that under the condition
\[ K_N = \ln N, \] (32)
the solution \( F_1 \) (i.e., solution of problem (26) – (32)) depends on the parameter \( N \) only and converges to the function \( f_1(\eta) \) uniformly on \([0, \infty)\) (see numerical results in Tables 1–3 in Section 4).

Since problem (26) – (32) is nonlinear, we use a continuation algorithm with an upwind finite difference scheme resulting in the linearized difference equation [2]
\[ \delta^2(D^-F_1^m)(\eta_i) + F_1^{m-1}(\eta_i)D^+(D^-F_1^m)(\eta_i) - D^-(F_1^m - F_1^{m-1})(\eta_i) \]
\[ - \frac{1}{2}(D^-F_1^{m-1})(\eta_i)(D^-F_1^m)(\eta_i) = -\frac{1}{2} \] (33)
with
\[ F_1^m(0) = D^+F_1^m(0) = 0 \quad \text{and} \quad D^0F_1^m(\eta_{N-1}) = 1. \] (34)
Here \( m \) is the continuation parameter associated with the continuation algorithm (33), (34). The algorithm generates the solution of a sequence of \( m \) linear problems. In addition, we note that the term \( D^-((F_1^m - F_1^{m-1})(\eta_i)) \) is used to preserve the monotonicity of the scheme. Iterations are continued until the required tolerance is achieved. We note that following the analysis of [2], it is possible to show that problem (21), (22) reflects a behavior similar to singularly perturbed problems.
3.2. Numerical solution of problem (17) – (20) for \( g_3 \) and \( h_3 \)

The equations and boundary conditions for problem (17) – (20) have been defined earlier. Both equations (17) and (19) are linear. As done earlier, to solve problems (17) – (18) and (19) – (20), instead of solving these problems on a semi-infinite domain \([0, \infty)\), we solve these problems on a finite domain \([0, K]\), where \( K \) is an auxiliary parameter. Thus, we can express (17) – (18) as

\[
g''_3(\eta) + f'_{1K}(\eta)g'_{3K}(\eta) - 2f_{1K}(\eta)g''_{3K}(\eta) + 2f'_{1K}(\eta)g_{3K}(\eta) = -1,
\]

(35)

\[
g_{3K}(0) = g'_{3K}(0) = 0, \quad g_{3K}(K) = \frac{1}{2}.
\]

(36)

To determine approximate solutions on the domain \([K, \infty)\), we extend the function \( g_{3K} \) by using the extrapolation

\[
g_{3K}(\eta) = \frac{1}{2}(\eta - K) + g_{3K}(K) \quad \forall \eta \geq K.
\]

(37)

And from (37), we have

\[
g'_{3K}(\eta) = \frac{1}{2}, \quad g''_{3K}(\eta) = 0 \quad \forall \eta \geq K.
\]

(38)

Thus, the auxiliary problems (35) – (38) is defined by the parameter \( K \). The solution \( g_3(\eta) \) of problems (17) – (18) is the limiting solution of problems (35) – (38) for \( K \to \infty \).

To solve problem (35) – (36) we use the finite difference approximation

\[
\delta^2(D^-G_3)(\eta_h) + F_{1K}(\eta_h)D^+(D^-G_3)(\eta_h) - 2(D^+F_{1K})(\eta_h)(D^-G_3)(\eta_h)
\]

\[
+ 2D^-(D^+G_{1K})(\eta_h)G_3(\eta_h) = -1
\]

(39)

with

\[
G_3(0) = D^+G_3(0) = 0 \quad \text{and} \quad D^0G_3(\eta_{N-1}) = \frac{1}{2},
\]

(40)

where \( D^- \), \( D^+ \), \( D^0 \), and \( \delta^2 \) are defined in (29), on a uniform mesh (25), on the interval \([0, K_N]\) with \( N \) number of mesh nodes.

In addition, for \( \eta \in [K_N, \infty) \) we use

\[
\overline{G}_{3K}(\eta_h) = \frac{1}{2}(\eta_h - K_N) + \overline{G}_{3K}(K_N) \quad \forall \eta \geq K_N.
\]

(41)

Similar to the case for the function \( f_1 \), the solution of problem (39) – (41) depends on two parameters \( K \) and \( N \), also one can justify that under condition (32) the solution \( G_3 \) (i.e., solution of problem (39) – (41), (32)) depends on the parameter \( N \) only and converges to the function \( g_3(\eta) \) uniformly on \([0, \infty)\).

To solve problems (19) – (20) on the same domain, i.e., on \([0, K]\), we follow the same procedure as for \( f_1 \) and \( g_3 \), i.e., we express (19) – (20) as

\[
h''_3(\eta) + f_{1K}(\eta)h'_{3K}(\eta) - 2f'_{1K}(\eta)h''_{3K}(\eta) + 2f''_{1K}(\eta)h_{3K}(\eta) = -\frac{1}{2}f_{1K}(\eta)f'_{1K}(\eta),
\]

(42)

\[
h_{3K}(0) = h'_{3K}(0) = 0, \quad h'_{3K}(K) = 0.
\]

(43)
To determine approximate solutions on the domain \([K, \infty)\), we extend the function \(h_{3K}\) by using the extrapolation

\[ h_{3K}(\eta) = h_{3K}(K) \quad \forall \eta \geq K. \]  

(44)

From (44) we have

\[ h_{3K}'(\eta) = 0, \quad h_{3K}''(\eta) = 0 \quad \forall \eta \geq K. \]  

(45)

The solution \(h_3(\eta)\) of the problem (19) – (20) is the limiting solution of the problems (42) – (45) for \(K \to \infty\).

We employ a finite difference approximation to solve problem (42) – (43)

\[
\delta^2(D^-H_3)(\eta_i) + F_{1K}(\eta_i)D^+(D^-H_3)(\eta_i) - 2(D^+F_{1K})(\eta_i)(D^-H_3)(\eta_i) \\
+ 2D^-(D^+F_{1K})(\eta_i)H_3(\eta_i) = -\frac{1}{2}F_{1K}D^-(D^+F_{1K})(\eta_i)
\]  

(46)

with

\[ H_3(0) = D^+H_3(0) = 0 \quad \text{and} \quad D^0H_3(\eta_{N-1}) = 0 \]  

(47)

on a uniform mesh (25) on the interval \([0, K_N]\). In addition, for \(\eta \in [K_N, \infty)\), we use

\[ \bar{F}_{3K}(\eta_i) = \bar{F}_{3K}(K_N) \quad \forall \eta \geq K_N. \]  

(48)

The solution \(H_{3K}(\eta)\) of problem (46) – (48), (32) depends on the parameter \(N\) only and converges to the function \(h_3(\eta)\) uniformly on \([0, \infty)\).

As mentioned earlier, \(f_3\) is composed of \(g_3\) and \(h_3\) as defined in (16). Since we already have the approximate solutions of \(g_3\) and \(h_3\), (16) can be approximated as

\[ f_{3K}(\eta) = g_{3K}(\eta) + h_{3K}(\eta) \]

and the function \(f_{3K}\) can be extended by employing the extrapolations for \(g_3\) and \(h_3\) given in (37) – (38) and (44) – (45). We approximate the function \(f_3(\eta)\) by

\[ F_3(\eta) = G_3(\eta) + H_3(\eta). \]

The function \(F_3\) is convergent uniformly on \([0, \infty)\) (see numerical results in Tables 4 and 5 in Section 4). To compute the velocity components \(u_B(x, y)\) and \(v_B(x, y)\) we will use the functions \(F_1(\eta)\) and \(F_3(\eta)\).

4. Numerical experiments

In this section we present the results of computations performed using the numerical method outlined in the previous section. We consider the two problems for \(f_1\) and \(f_3\) separately. To proceed to the error analysis we first introduce some notions related to the errors and orders of convergence that we will employ here. For any mesh function \(W^N\), we denote by \(W^N\) its piecewise linear interpolant. We introduce the computed global two-mesh differences \(\bar{D}^N\), which have to be computed over three subintervals \([0, K_N]\), \([K_N, K_{2N}]\), and \([K_{2N}, \infty)\) separately. For \(\eta \in [0, K_N]\), the two-mesh difference for \(\bar{W}\) is \(\bar{W}^N - W^{2N}\) and for \(\bar{D}^+\bar{W}\) it is \(\bar{D}^+\bar{W}^N - \bar{D}^+W^{2N}\). For \(\eta \in [K_N, K_{2N}]\), the two-mesh difference at \(\eta\) for \(\bar{W}\) is \(\bar{W}^{2N}(\eta) - W^N(\eta) - L(\eta - K_N)\) and for \(\bar{D}^+\bar{W}\) is \(\bar{D}^+\bar{W}^{2N} - L\). For \(f_1\) we consider \(L = 1\) and for \(f_3\) we consider \(L = 1/2\). The two-mesh difference in the sub-interval \([K_{2N}, \infty)\) at \(\eta\) for \(\bar{W}\) is...
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$W^{2N} - W^N(K_N) - \ln N$ and for $D^+W$ it is zero. In this manner we compute the global two mesh differences $D^N$ over $[0, \infty]$ for various values of $N$.

We finally define the orders of convergence

$$R^N = \log_2 \frac{D^N}{D^{2N}},$$

which are based on the computed global two mesh differences.

We will now compute $D^N$ and $R^N$ for the numerical solution of the problems for $f_1$ and $f_3$ in the following subsections.

4.1. Computed solution and error analysis for the $f_1$ problem

The solution of problem (14) – (15) using numerical method (26) – (27) is shown in Fig. 2.

![Figure 2. Plot of the numerical solution for $N = 8192$](chart)

All the tables are presented in a similar fashion and show the computed global two mesh differences $D^N$ and the corresponding orders of convergence $R^N$. In Table 1 we see $D^N$ and $R^N$ for the numerical solution $F_1$. In Table 2 we see similar results for the approximation to the derivative $D^+F_1$. We note that all the tables reflect uniform convergence on $[0, \infty)$, as $N$ increases. The order of convergence of the numerical solution $F_1$ is better than 0.8 for all $N \geq 256$. We observe that the order of convergence of the discrete derivative $D^+F_1$ is better than 0.8 for all $N \geq 128$. The computed global two-mesh difference $D^N$ of the numerical solution $F_1$ and of the discrete derivative $D^+F_1$ reflects uniform convergent behavior on $[0, \infty)$.

**Table 1.** Computed two-mesh differences $D^N$, and computed orders of convergence $R^N$ for $F_1$ on $[0, \infty)$ for various values of $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
<th>2048</th>
<th>4096</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^N$</td>
<td>0.71</td>
<td>0.78</td>
<td>0.81</td>
<td>0.83</td>
<td>0.85</td>
<td>0.86</td>
<td></td>
</tr>
</tbody>
</table>
Table 2. Computed two-mesh differences $D^N$, and computed orders of convergence $R^N$ for $D^+ F_1$ on $[0, \infty)$ for various values of $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>64</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$R^N$</td>
<td>0.64</td>
<td>0.82</td>
<td>0.84</td>
<td>0.85</td>
<td>0.86</td>
<td>0.87</td>
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</tr>
</tbody>
</table>

4.2. Computed results for the $f_3$ problem

The solution of problem (16) using numerical method (39) – (48), (32) is shown in Fig. 3.

![Plot of the numerical solution for $N = 8192$](image)

Figure 3. Plot of the numerical solution for $N = 8192$

The global two mesh differences given in Table 3 correspond to the function $F_3$, and in Table 4 the global two mesh differences correspond to the derivative $D^+ F_3$. The function $F_3$ and derivative $D^+ F_3$ are convergent uniformly on $[0, \infty)$, and the order of convergence for $F_3$ is similar to $F_1$.

Table 3. Computed two-mesh differences $D^N$, and computed orders of convergence $R^N$ for $F_3$ on $[0, \infty)$ for various values of $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
<th>2048</th>
<th>4096</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^N$</td>
<td>0.84</td>
<td>0.77</td>
<td>0.80</td>
<td>0.83</td>
<td>0.85</td>
<td>0.86</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Computed two-mesh differences $D^N$, and computed orders of convergence $R^N$ for $D^+ F_3$ on $[0, \infty)$ for various values of $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
<th>2048</th>
<th>4096</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D^N$</td>
<td>5.753D-03</td>
<td>3.397D-03</td>
<td>1.988D-03</td>
<td>1.139D-03</td>
<td>6.421D-04</td>
<td>3.571D-04</td>
<td>1.966D-04</td>
</tr>
<tr>
<td>$R^N$</td>
<td>0.76</td>
<td>0.77</td>
<td>0.80</td>
<td>0.83</td>
<td>0.85</td>
<td>0.86</td>
<td></td>
</tr>
</tbody>
</table>

4.3. Computation of the semianalytical solution

The computation of the self-similar semianalytical solution $u_B$ is now done easily using the numerical approximations $F_1$ and $D^+ F_1$, which are substituted for the exact solution $f_1$ and
For the discrete derivative \( f'_1 \), respectively. In an analogous way \( \mathcal{T}_3 \) and \( \mathcal{D}^+F_3 \) are substituted for \( f_3 \) and \( f'_3 \), respectively. We use relations (7) and (8) for each \((x, y)\) in the rectangular domain \( \Omega = (0, 1) \times (0, 1) \). We define

\[
U_B(x, y) = \frac{3U_\infty}{2R} x \mathcal{T}_1(\eta) - \frac{U_\infty}{2R^3} x^3 \mathcal{D}_3 F_3(\eta)
\]

and

\[
V_B(x, y) = \sqrt{\frac{3\nu U_\infty}{R}} \left( -\frac{6R^2 - 2x^2}{6R^2 - x^2} F_1(\eta) + \frac{4x^2 R^2 - x^4}{R^2 (6R^2 - x^2)} F_3(\eta) \right),
\]

where \( \eta = (0, \infty) \) is given by (11).

Graphs of the resulting approximate solution \( u_B \) are given in Fig. 4 for \( \nu = 2^{-5} \). Note that we set \( U_\infty = 1 \) and \( R = 1 \) for computations. These graphs are constructed using the data of the Blasius solution for \( F_1 \) and \( F_3 \) corresponding to \( N = 8192 \).

![Figure 4](image)

**Figure 4.** Graphs of the semianalytical solution \((U_B, V_B)\) for \( \nu = 2^{-5} \) generated from the numerical solution \( F_1 \& F_3 \) with \( N = 32 \)

### 4.4. Summary

We have constructed a robust numerical method to obtain a semianalytical solution to problem (1) – (4), (9), (10) on the domain \( \Omega \) based on the approach given in [7]. This approach of obtaining a semianalytical solution in \( \Omega \) reduces the problem to solving on the semi-infinite domain three ordinary differential equations in \( f_1 \) and \( g_3, h_3 \), which generate \( f_3 \). The numerical method for solving the problem on a semi-infinite axis is reduced to solving the problem on a large interval \([0, K]\), \( K = K(N) = \ln N \), where \( N \) is the number of mesh points used to solve the problem on \([0, K]\). By this choice of \( K(N) \), the discrete solution depends on \( N \) only and converges uniformly on \([0, \infty)\) to the continuous problem. For the body of revolution with a parabolic profile, the approximations of \( f_1 \) and \( f_3 \) were used to compute \( u_B \) and \( v_B \) employing a Blasius series approach (up to two terms). To show that the new method for computing the velocity components is parameter robust (with respect to the parameter \( \nu \)), it is sufficient to show that the method of finding the functions \( f_1 \) and \( f_3 \) along with their derivatives, is uniformly convergent on \([0, \infty)\). Thus, by numerical experiments we have demonstrated that the method constructed is robust.

It is worth noting that such a robust method can be employed to compute a reference solution for the velocity components to find errors for the direct numerical method employed to solve problem (1) – (4), (9), (10) in the domain \( \Omega \).
References


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