A CLASS OF SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS WITH A MOVING INTERIOR LAYER. AN A POSTERIORI ADAPTIVE MESH TECHNIQUE\(^1\)

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Abstract — We study numerical approximations for a class of singularly perturbed convection-diffusion type problems with a moving interior layer. In a domain (segment) with a moving interface between two subdomains, we consider an initial boundary value problem for a singularly perturbed parabolic convection-diffusion equation. Convection fluxes on the subdomains are directed towards the interface. The solution of this problem has a moving transition layer in the neighbourhood of the interface. Unlike problems with a stationary layer, the solution exhibits singular behaviour also with respect to the time variable. Well-known upwind finite difference schemes for such problems do not converge ε-uniformly in the uniform norm, even under the condition \(N^{-1} + N_0^{-1} \approx \varepsilon\), where \(\varepsilon\) is the perturbation parameter and \(N\) and \(N_0\) denote the number of mesh points with respect to \(x\) and \(t\). In the case of rectangular meshes which are (a priori or a posteriori) locally condensed in the transition layer, there are no schemes that converge uniformly in \(\varepsilon\) even under the very restrictive condition \(N^{-2} + N_0^{-2} \approx \varepsilon\). However, the condition for convergence can be considerably weakened if we take the geometry of the layer into account, i.e., if we introduce a new coordinate system which captures the interface. For the problem in such a coordinate system, one can use either an a priori, or an a posteriori adaptive mesh technique. Here we construct a scheme on an adaptive meshes (based on the solution gradient), whose solution converges ‘almost \(\varepsilon\)-uniformly’, viz., under the condition \(N^{-1} = o(\varepsilon^{\nu})\), where \(\nu > 0\) is an arbitrary number from the half-open interval \((0, 1]\).

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1. Introduction

In this paper we study a problem with discontinuous coefficients. In particular, we consider the case of a parabolic problem where the convection coefficient is discontinuous and has opposite signs at both sides of an interface. Such boundary value problems for singularly perturbed equations with partial derivatives arise (see, e.g., [8, 14] and bibliography therein) in studying heat/mass transfer processes in composite materials with small heat-conduction/diffusion in the case of stationary interfaces. The terms with the highest derivatives in those equations are multiplied by the small parameter \( \varepsilon \) that gives rise to boundary and transition layers in the solution. Because the coefficients in the equations (and the source terms) are discontinuous, the derivatives of the solution have discontinuities at the interface even for fixed values of the parameter \( \varepsilon \). Note that the errors of standard numerical methods developed for regular boundary value problems strongly depend on the value of the parameter \( \varepsilon \). For example, the solution of a finite difference scheme in the case of a problem with a stationary interface converges when the step size in the space mesh is much smaller than \( \varepsilon \) (see remarks on Theorems 3.1, 3.2 in Section 3). Because of the restrictive convergence condition for the classical finite difference schemes, interest arises to construct special schemes for which errors in the solution weakly depend on the parameter \( \varepsilon \). In particular, we are interested in methods where the error behaviour is independent of \( \varepsilon \) (we say that the numerical methods converge \( \varepsilon \)-uniformly).

In the case of nonstationary interfaces, moving transition layers appear. Unlike problems with stationary singularities, in the case of moving transition layers, the solution exhibits singular behaviour also with respect to the time variable (see, e.g., estimates (9.5) in Section 9). The considerably complicated character of arising singularities (as compared to problems with stationary singularities for which numerical methods are treated, e.g., in [2, 5, 6, 9]) forces us to use a more complicated discrete construction.

The use of a technique developed to improve the accuracy of the solutions in the case of regular boundary value problems (e.g., the technique of \textit{a priori}/\textit{a posteriori} adaptive meshes; see [1, 3, 6] and bibliography therein), turns out to be ineffective in the case of singularly perturbed problems (see [10] and the statement of Theorem 4.1 in Section 4). Therefore, the quest for conditions necessary (and for specific discrete methods also sufficient) for the \( \varepsilon \)-uniform convergence of numerical methods, for problems with moving transition layers is relevant.

In this paper we consider discrete approximations of an initial boundary value problem for the singularly perturbed parabolic convection-diffusion equation in a domain with a moving interface boundary between two subdomains where convective fluxes on the subdomains are directed towards the interface. The solution of this problem has a singularity near the transition layer that moves in time.

We study finite difference schemes based on classical discrete approximations of the problem. When rectangular uniform meshes are used, the order of magnitude of the error is not smaller than the exact solution when \( \varepsilon = \mathcal{O}(N^{-1} + N_0^{-1}) \), where \( N, N_0 \) denote the number of nodes in the space and time variables respectively. Finite difference schemes are considered on meshes that are locally refined in the neighbourhood of the moving interface. It turns out that in the class of finite difference schemes on rectangular meshes locally condensing in \( x \) and \( t \) in the neighbourhood of the trajectory of the moving interface, \( \varepsilon \)-uniformly convergent schemes do not exist even under the condition \( \varepsilon \approx N^{-2} + N_0^{-2} \). However, if a new coordinate system for which the interface becomes a meshline, is introduced, then one can construct a
scheme whose solution converges ‘almost’ \( \varepsilon \)-uniformly, i.e., it converges under the condition \( N^{-1} = \mathcal{O}(\varepsilon^n) \), where \( \nu > 0 \) is an arbitrary small number. For the problem in the new coordinate system, it is possible to use either \textit{a priori} or \textit{a posteriori} mesh refinement techniques to obtain an almost \( \varepsilon \)-uniformly accurate result. As \textit{a posteriori} adaptive technique, we use adaptive mesh refinement on the basis of the gradient of the solution.

This approach can be used to construct effective numerical methods for representative classes of boundary value problems with the dominant convection with known, moving transition layers.

\textbf{Organization of the paper}. The problem formulation and the aim of the research are given in Section 2. The classical schemes and auxiliary problems related to the construction of \( \varepsilon \)-uniformly and almost \( \varepsilon \)-uniformly convergent schemes are considered in Sections 3 and 4. Schemes on \textit{a posteriori} adaptive meshes are constructed and studied in Sections 5–8 (in Sections 5–7 for a problems with stationary and in Section 8 with moving interface boundaries). \textit{The priori} estimates are given in Section 9.

The technique for constructing \textit{a posteriori} adaptive meshes based on the solution gradient was used in [12] to construct almost \( \varepsilon \)-uniformly convergent schemes for the initial boundary value problem for a parabolic convection-diffusion equation with a “standard” singularity (a “stationary” boundary layer). Note that in [12] a scheme on \textit{a posteriori} adaptive meshes for a quasilinear equation was written and the results of the numerical experiments for a stationary linear problem confirming the theoretical results were presented. In [10], for the initial value problem for a parabolic reaction-diffusion equation, a special scheme was considered on meshes that are \textit{a priori} refined in the transition layer caused by a moving point source. To construct the scheme, special coordinates for which the location of the point source was fixed were used.

2. Problem formulation. Aim of research

1. In a bounded domain with a moving interface between two subdomains we consider the initial boundary value problem for a singularly perturbed parabolic convection-diffusion equation.

Let the domain \( \Omega \) with the boundary \( S = \partial \Omega \setminus G \), where \( G = D \times (0, T] \), \( D = (-d, d) \), be decomposed into non-overlapping subdomains

\[
\Omega = \Omega^1 \cup \Omega^2, \quad G^1 \cap G^2 = \emptyset,
\]

in each of which we consider the equation

\[
L^r u(x, t) = \left\{ \varepsilon a^r(x, t) \frac{\partial^2}{\partial x^2} \right\} + (-1)^r b^r(x, t) \frac{\partial}{\partial x} - c^r(x, t) - p^r(x, t) \frac{\partial}{\partial t} \bigg\} u(x, t) = f^r(x, t), \quad (x, t) \in G^r, \quad r = 1, 2,
\]

where

\[
G^1 = \{(x, t) : x < \beta(t), \quad t \in (0, T] \}, \quad G^2 = \{(x, t) : x > \beta(t), \quad t \in (0, T] \}.
\]

The curve \( \gamma = \{(x, t) : x = \beta(t), \quad |\beta(t)| < d, \quad t \in (0, T] \} \), i.e., the interface between the subdomains is given by a sufficiently smooth function. On the set \( S \) the function \( u(x, t) \) takes the prescribed values

\[
u(x, t) = \varphi(x, t), \quad (x, t) \in S,
\]

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and on the interface $\gamma$ it obeys the conjugation condition, i.e., the solution and the diffusion flux are continuous

$$[u(x, t)] = 0, \quad lu(x, t) \equiv \varepsilon \left[a(x, t) \frac{\partial}{\partial x} u(x, t)\right] = 0, \quad (x, t) \in \gamma. \quad (2.2c)$$

The symbol $[v(x, t)]$ denotes the jump of the function $v(x, t)$ when passing through $\gamma$ from the set $G^1$ to the set $G^2$:

$$[v(x^*, t)] = \lim_{x \to x^*+0} v(x, t) - \lim_{x \to x^*-0} v(x, t),$$

$$a(x^*, t) \frac{\partial}{\partial x} v(x^*, t) = \lim_{x \to x^*+0} a(x, t) \frac{\partial}{\partial x} v(x, t) - \lim_{x \to x^*-0} a(x, t) \frac{\partial}{\partial x} v(x, t), \quad (x^*, t) \in \gamma.$$

In (2.2a) $a^r(x, t), \ldots, f^r(x, t), (x, t) \in \overline{G}^r, \ r = 1, 2$ and $\varphi(x, t), (x, t) \in S$ are sufficiently smooth functions on $\overline{G}^r$ and $S_0, \overline{S}^L$, respectively, $\varphi(x, t) \in C(\overline{S})$, and also

$$a_0 \leq a^r(x, t) \leq a^0, \quad b_0 \leq b^r(x, t) \leq b^0, \quad p_0 \leq p^r(x, t) \leq p^0, \quad (2.4)$$

$$0 \leq c^r(x, t) \leq c^0, \quad (x, t) \in \overline{G}^r, \quad a_0, b_0, p_0 > 0;$$

$$-\beta_0 \leq \beta(t) \leq \beta^0, \quad |\beta(t)| \leq \beta^1, \quad t \in [0, T], \quad \beta_0, \beta^0 < d, \quad 0 < \beta^1 < \beta_0 (p_0)^{-1};$$

$$|f^r(x, t)| \leq M, \quad (x, t) \in \overline{G}^r, \quad |\varphi(x, t)| \leq M, \quad (x, t) \in S, \quad r = 1, 2;$$

$S = S_0 \cup S^L, S^L$ and $S_0$ are the lateral and lower parts of the boundary $S$, $S_0 = \overline{S}_0$; let $\beta(0) = 0; \varepsilon$ is a parameter taking arbitrary values from the half-open interval $(0, 1]$.

For equation (2.2a), we also use the following notation

$$L_{(2.2a)}u(x, t) \equiv \left\{ \varepsilon a(x, t) \frac{\partial^2}{\partial x^2} + b(x, t) \frac{\partial}{\partial x} - c(x, t) - p(x, t) \frac{\partial}{\partial t}\right\}u(x, t) = f(x, t), \quad (x, t) \in G^{(s)},$$

where $G^{(s)} = G \setminus \gamma, b(x, t) = b^2(x, t), (x, t) \in \overline{G}^2, b(x, t) = -b^1(x, t), (x, t) \in \overline{G}^1$, the functions $a(x, t), c(x, t), p(x, t), f(x, t)$ are defined by the relation $v(x, t) = v^r(x, t), (x, t) \in \overline{G}^r, \ r = 1, 2$.

For simplicity, we assume that the compatibility conditions are fulfilled on the sets $S^c = S_0 \cap \overline{S}^L$ and $\gamma^0 = \{(\beta(0), 0)\}$ to ensure sufficient smoothness of the solution of the problem on each of the subsets $\overline{G}^r$ (for fixed values of the parameter $\varepsilon$); we suppose $S^r = \overline{G}^r \setminus G^r, \ r = 1, 2$.

As $\varepsilon \to 0$, in the neighbourhood of the set $\gamma$ there appears a transition layer decreasing exponentially away (in the $x$-direction) from the set $\gamma$ (the typical “width” of the layer is of order $\varepsilon$). Hence, in the case of the moving interface, the transition layer decreases exponentially away from $\gamma$ for a fixed value of $x$. Under the condition

$$\beta(t) = \text{const} \ , \ t \in [0, T] \quad (2.5)$$

(steady interface) the $t$-derivatives of the singular part of the solution to the problem are bounded on $\overline{G}$ $\varepsilon$-uniformly (see estimates (9.4), (9.5) from Section 9).

2 Here and below $M, M_i$ (or $m$) denote sufficiently large (small) positive constants independent of $\varepsilon$ and discretization parameters.

3 Throughout the paper, the notation $L_{(1.3)}(M_{(1.3)}, G_{h(1.3)})$ means that these operators (constants, grids) are introduced in equation (j.k).
The solution of the reduced problem is a function which is sufficiently smooth on each of the sets $G^r$ and has a discontinuity of the first kind on $\gamma$.

Note that the boundary layer does not appear in the neighbourhood of the lateral boundary $S^L$ (because of the condition imposed on the coefficient $b(x,t)$ so that the convection flow is directed to the interface $\gamma$). In the case of the condition $\beta^1 < b_0(p^0)^{-1}$ (see (2.4)) the interface $\gamma$ between the subdomains $G^r$ is noncharacteristic (the characteristics of the reduced equation are not tangent to the set $\gamma$).

For definiteness, we consider that the functions $a(x,t), \ldots, f(x,t)$ on the set $\gamma$ are equal to the half-sum of the values (mean value) of their limits from the sets $G^1$ and $G^2$.

2. The errors in the solutions of finite difference schemes based on classical difference approximations to problem (2.2), (2.1) depend on the parameter $\varepsilon$ and become small only for those values of $\varepsilon$ that considerably exceed the “effective” mesh steps with respect to $x$ and $t$. So, by virtue of estimates (3.7), (3.13) (see Section 3), the classical difference schemes (3.4), (3.3) and (3.11), (3.10), (3.3) converge under the condition

$$\varepsilon >> N^{-1} + N_0^{-1},$$

(2.6)

where $N, N_0$ denote the number of mesh points with respect to $x$ and $t$, respectively. If this condition fails, the solutions of the difference schemes do not generally converge to the solution of problem (2.2), (2.1).

By this argument, we are interested in constructing special difference schemes whose errors do not depend on the value of the parameter $\varepsilon$. In particular, it is of interest to develop such schemes that converge under a weaker condition than condition (2.6), which is the convergence condition for the discrete problems (3.4), (3.3) and (3.11), (3.10), (3.3).

**Definition 2.1.** Let $z(x,t), (x,t) \in G_h$ be a solution of some difference scheme. This scheme converges uniformly with respect to the parameter $\varepsilon$ (or $\varepsilon$-uniformly) if the function $z(x,t)$ satisfies the estimate

$$|u(x,t) - z(x,t)| \leq M \mu (N^{-1}, N_0^{-1}), \quad (x,t) \in G_h,$$

where $\mu(N^{-1}, N_0^{-1})$ tends to zero for $N, N_0 \to \infty$ uniformly with respect to the parameter. We say that the solution of the scheme converges almost $\varepsilon$-uniformly if for any arbitrarily small number $\nu > 0$ there exists a function $\mu(\varepsilon^{-\nu}N^{-1}, \varepsilon^{-\nu}N_0^{-1})$ such that the following estimate holds for the mesh function $z(x,t)$:

$$|u(x,t) - z(x,t)| \leq M \mu (\varepsilon^{-\nu}N^{-1}, \varepsilon^{-\nu}N_0^{-1}), \quad (x,t) \in G_h.$$  

(2.7)

where $\mu(N^{-1}, N_0^{-1}) \to 0$ for $N, N_0 \to \infty$ uniformly with respect to the parameter $\varepsilon$. If estimate (2.7) is satisfied, then we say that the scheme converges almost $\varepsilon$-uniformly with a defect $\nu$. (If estimate (2.7) is satisfied for $\nu = 0$, then the convergence is $\varepsilon$-uniform.)

**The aim of the research.** The defect (see Definition 4.1) of schemes (3.4), (3.3) and (3.11), (3.10), (3.3) is equal to 1. Thus, in the case of problem (2.2), (2.1) we arrive at the following question of theoretical (and practical) interest: how to construct schemes whose defect is less than 1, and in particular, how to construct almost $\varepsilon$-uniformly convergent schemes.

3. Classical difference schemes

Let us give classical difference schemes for problem (2.2), (2.1) and show estimates for their solutions.
1. We give a difference scheme for problem (2.2), (2.1) considering the problem in a week formulation and not focusing on the conjugation condition (2.2c) in the approximation of the problem.

On the set $\mathcal{G}$ we introduce the rectangular mesh

$$
\mathcal{G}_h = \mathcal{G}_1 \times \mathcal{G}_0,
$$

where $\mathcal{G}_1$ and $\mathcal{G}_0$ are meshes on the segments $\mathcal{D}$ and $[0, T]$, respectively; $\mathcal{G}_1$ and $\mathcal{G}_0$ are meshes with any distribution of the nodes satisfying only the condition $h \leq MN^{-1}$, $h_t \leq MN_0^{-1}$, where $h = \max_i h_i$, $h_i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \mathcal{G}_1$, $h_t = \max_j h^j$, $h^j = t^{j+1} - t^j$, $t^i, t^{j+1} \in \mathcal{G}_0$. Here $N + 1$ and $N_0 + 1$ are the number of nodes in the meshes $\mathcal{G}_1$ and $\mathcal{G}_0$, respectively. It is of great interest to consider meshes that are uniform with respect to $x$

$$
\mathcal{G}_h = \mathcal{G}_h(3.1),
$$

where $\mathcal{G}_1$ is a uniform mesh; such meshes for $b(x, t) \equiv 0$ allow us to obtain the second order of the approximation with respect to $x$ for sufficiently smooth solutions. It is also interesting to consider difference schemes on the simplest meshes that are uniform with respect to both $x$ and $t$:

$$
\mathcal{G}_h = \mathcal{G}_h(3.3),
$$

where both $\mathcal{G}_1$ and $\mathcal{G}_0$ are uniform meshes.

We approximate problem (2.2), (2.1) by the implicit finite difference scheme [7]

$$
\Delta z(x, t) \equiv \{ \varepsilon a(x, t)\delta_{xx} + b^+(x, t)\delta_x + b^-(x, t)\delta_x - c(x, t)
$$

$$
- p(x, t)\delta_t \} z(x, t) = f(x, t), \quad (x, t) \in G_h,
$$

$$
z(x, t) = \varphi(x, t), \quad (x, t) \in S_h.
$$

Here $G_h = G \cap \mathcal{G}_h$, $S_h = S \cap \mathcal{G}_h$; $\delta_{xx} z(x, t)$, $\delta_x z(x, t)$, $\delta_{xx} z(x, t)$, $\delta_t z(x, t)$ are the second and first difference derivatives; $\delta_{xx} z(x, t) = 2(h^i + h^{i-1})^{-1} \{ \delta_x - \delta_{xx} \} z(x, t)$, $\delta_x z(x, t) = (h^i)^{-1} (z(x^{i+1}, t) - z(x, t))$, $\delta_{xx} z(x, t) = (h^{i-1})^{-1} (z(x, t) - z(x^{i-1}, t))$, $x = x^i$, $h^{i-1}$ and $h^i$ are the left and right “arms” of the three-point stencil (for the operator $\delta_{xx}$) on $G_h$ with the center at the node $(x^i, t) \in G_h$;

$$
b^+(x, t) = 2^{-1} |b(x, t) + |b(x, t)||, \quad b^-(x, t) = 2^{-1} |b(x, t) - |b(x, t)||.
$$

For the difference scheme (3.4), (3.1) the maximum principle is valid. By using the majorant function technique, we find the estimate

$$
|z(x, t)| \leq M, \quad (x, t) \in \mathcal{G}_h.
$$

Taking into account the a priori estimates for the solution of the initial boundary value problem, we find the following estimate in the case of mesh (3.1):

$$
|u(x, t) - z(x, t)| \leq M \varepsilon^{-2} [N^{-1} + N_0^{-1}], \quad (x, t) \in \mathcal{G}_h.
$$

(3.6)

For mesh (3.3) we have

$$
|u(x, t) - z(x, t)| \leq M \varepsilon^{-1} [N^{-1} + N_0^{-1}], \quad (x, t) \in \mathcal{G}_h.
$$

(3.7)

We summarize this in the following.
**Theorem 3.1.** Let the components of the solution of the initial boundary value problem (2.2), (2.1) in the representation of (9.1) satisfy the a priori estimates (9.4), (9.5). Then the solution of the difference scheme (3.4), (3.1) converges for fixed values of the parameter $\varepsilon$. For the discrete solutions estimates (3.5), (3.6), (3.7) are valid.

**Remark 3.1.** In the case of condition (2.5) for scheme (3.4), (3.3) we have the estimate

$$ |u(x, t) - z(x, t)| \leq M [\varepsilon^{-1} N^{-1} + N_0^{-1}], \quad (x, t) \in \mathcal{G}_h. $$

The condition

$$ \varepsilon^{-1} = o(N) \quad (3.8) $$

is necessary and sufficient for the convergence of the scheme.

2. We now consider a difference scheme for problem (2.2), (2.1) based on “direct” approximation of the conjugation condition (2.2c), i.e. the interface condition. To this end, we need meshes containing nodes on the set $\gamma$ at each time level $t = t^j$ of this difference scheme which is an alternative to scheme (3.4), (3.1). Let us construct such meshes.

To the construct these meshes, we use the regular mesh (3.1) (or (3.2), (3.3)) as the basic mesh; we construct the mesh $\mathcal{G}_h = \mathcal{G}_h(\mathcal{G}_h(3.1))$. Let

$$ G^*_h = \{(x^i, t^j) : (x^i, t^j) \in G_h, \ x^i \neq \beta(t), \ t \in [t^{j-1}, t^j], \ t^{j-1}, t^j \in \mathcal{W}_0 \}. \quad (3.9) $$

On the time level $t = t^n$ we construct the sets $G^{1n}_h$, $G^{2n}_h$: $G^{rn}_h = G^*_h \cap \{t = t^n\}$, $t^n \in \mathcal{W}_0$, $r = 1, 2$. The corresponding nodes $(x^i, t^{j-1})$ generate the sets $S^{1n}_0$ and $S^{2n}_0$, which are the lower mesh boundaries for sets $G^{1n}_h$, $G^{2n}_h$. The nodes $(x^i, t^j)$ from $S^{L}_h$ form the set $S^{rn}_h$. We assume $S^{1n}_h = \gamma^{1n}_h \cup S^{1n}_0 \cup \{S^{L}_h \cap \mathcal{G}^1\}$, $S^{2n}_h = \gamma^{2n}_h \cup S^{2n}_0 \cup \{S^{L}_h \cap \mathcal{G}^2\}$, $G^{1n}_h = C^{1n}_h \cup S^{1n}_h$, $G^{2n}_h = C^{2n}_h \cup S^{2n}_h$, where $\gamma^{n}_h = \{(\beta(t^n), t^n)\}$. We introduce the sets $G^{rn}_h = C^{rn}_h \cup G^{hn}_h$, $G^{hn}_h = G^{1n}_h \cup \gamma^{n}_h$, $G^{hn}_h = G^{hn}_h \cup G^{2n}_h$, $G^{rn}_h = G^{rn}_h \cup S^{hn}_h$. The mesh $\mathcal{G}^{*}_h$ is defined by the relation

$$ \mathcal{G}^{*}_h = \bigcup_{n=1}^{N_0} G^{rn}_h. \quad (3.10) $$

We approximate problem (2.2), (2.1) by the difference scheme

\begin{align*}
\Lambda z(x, t) &= f(x, t), \quad (x, t) \in G^{(n)}_h, \quad (3.11a) \\
b^h z(x, t) &\equiv \varepsilon \{a^2(x, t)\delta z(x, t) - a^1(x, t)\delta z(x, t)\} = 0, \quad (x, t) \in \gamma^{n}_h, \quad (3.11b) \\
z(x, t) &= \begin{cases} \\
\mathcal{G}^{-1}_n(x, t), \quad (x, t) \in S^{rn}_h \setminus S \\
\varphi(x, t), \quad (x, t) \in S^{rn}_h \cap S 
\end{cases}, \quad (x, t) \in \mathcal{G}^{rn}_h, \ n = 1, \ldots, N_0. \quad (3.11c)
\end{align*}

Here $z^n(x, t) = z(x, t)$ for $(x, t) \in \mathcal{G}^{rn}_h$, $t = t^n$, $\mathcal{G}^{n}(x, t) = \mathcal{G}$, $t = t^n \in \mathcal{W}_0$ is the linear interpolant constructed from the values of $z^n(x, t)$, $(x, t) \in \mathcal{G}^{rn}_h$, $t = t^n$. The function

$$ z(x, t) = \begin{cases} \\
z^n(x, t), \quad (x, t) \in \mathcal{G}^{rn}_h, \ t = t^n, \\
z^{n-1}(x, t), \quad (x, t) \in \mathcal{G}^{rn}_h, \ t = t^{n-1} 
\end{cases}, \quad (x, t) \in \mathcal{G}^{rn}_h, \ n = 1, \ldots, N_0, \ (x, t) \in \mathcal{G}^{*}_h.$$
will be called the solution of the difference scheme (3.11), (3.10).

For the difference scheme (3.11), (3.10) the maximum principle is valid.

Taking into account the a priori estimates for the solutions of the differential problem, we establish estimates similar to (3.6), (3.7)

\[ |u(x,t) - z(x,t)| \leq M \varepsilon^{-2} \left[ N^{-1} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h, \quad G_h = G_{h(3.1)}; \]  
\[ |u(x,t) - z(x,t)| \leq M \varepsilon^{-1} \left[ N^{-1} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h, \quad G_h = G_{h(3.3)}. \]  

**Theorem 3.2.** Let the condition of Theorem 3.1 be fulfilled. Then the solution of the difference scheme (3.11), (3.10) converges to the solution of problem (2.2), (2.1) for fixed values of the parameter \( \varepsilon \). For the discrete solutions estimates (3.12), (3.13) are valid.

**Definition 3.1.** Let the function \( z(x,t) \), \( (x,t) \in \overline{G}_h \) be the solution of some difference scheme. An estimate of the following form

\[ |u(x,t) - z(x,t)| \leq M \left[ \varepsilon^{-\nu_i} N^{-\mu_1} + \varepsilon^{-\nu_0} N_0^{-\nu_0} \right], \quad (x,t) \in \overline{G}_h, \]

where \( \nu_i, \nu_1 \geq 0 \), is said to be unimprovable with respect to the values of \( N, N_0, \varepsilon \) if the estimate

\[ |u(x,t) - z(x,t)| \leq M \left[ \varepsilon^{-\alpha_i} N^{-\alpha_1} + \varepsilon^{-\alpha_0} N_0^{-\alpha_0} \right], \quad (x,t) \in \overline{G}_h, \]

in general, fails under the conditions \( \alpha_i \geq \nu_i, \alpha_1 \leq \nu_1 \) and also \( \alpha_1 + \alpha_0 - \alpha_1 - \alpha_0 > \alpha_1 + \nu_0 - \nu_1 - \nu_0 \).

**Remark 3.2.** In the case of condition (2.5), for the solutions of the difference scheme (3.11), (3.10), (3.3) we have the unimprovable estimate

\[ |u(x,t) - z(x,t)| \leq M \left[ \varepsilon^{-1} N^{-1} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h; \]

the scheme converges under the unimprovable condition (3.8).

3. For problem (2.2), (2.1), we discuss the conditions under which the solution of the difference scheme (3.4) converges for \( N, N_0 \to \infty \) and \( \varepsilon \to 0 \), where \( \varepsilon \to 0 \) for \( N, N_0 \to \infty \).

It follows from estimates (3.6) and (3.7) that scheme (3.4), (3.1) converges under the condition \( N^{-1}, N_0^{-1} \ll \varepsilon^2 \), and scheme (3.4), (3.3) converges under the condition \( N^{-1}, N_0^{-1} \ll \varepsilon \), i.e., for

\[ \varepsilon^{-1} = o(N), \quad \varepsilon^{-1} = o(N_0). \]  

Estimate (3.7) (as well as estimate (3.13)) is unimprovable with respect to the values of \( N, N_0, \varepsilon \). The defect of schemes (3.4), (3.1) and (3.4), (3.3) (schemes (3.11), (3.10), (3.1) and (3.11), (3.10), (3.3)), according to estimates (3.6) and (3.7) (estimates (3.12) and (3.13)), is not less than 2 and 1, respectively. It follows from the unimprovability of estimate (3.7) (estimate (3.13)) that the unimprovable defect of scheme (3.4), (3.3) (scheme (3.11), (3.10), (3.3)) is equal to 1, moreover, the unimprovable defect of scheme (3.4), (3.1) (scheme (3.11), (3.10), (3.1)) is not less than 1.

Thus, the estimates of the discrete solutions strongly depend on the value of the parameter \( \varepsilon \); if condition (3.14) fails then schemes (3.4) and (3.11), (3.10), generally speaking, do not converge.
Theorem 3.3. Let the hypothesis of Theorem 3.1 be fulfilled. In the case of schemes (3.4), (3.3) and (3.11), (3.10), (3.3) (schemes (3.4), (3.1) and (3.11), (3.10), (3.1)), condition (3.14) is necessary and sufficient (is necessary) for the convergence of the discrete solutions to the solution of problem (2.2), (2.1) for $N, N_0 \to \infty$ and $\varepsilon \to 0$; the defect of schemes (3.4), (3.3) and (3.11), (3.10), (3.3) (schemes (3.4), (3.1) and (3.11), (3.10), (3.1)) is equal to 1 (not less than 1). Estimates (3.7) and (3.13) are unimprovable with respect to the values of $N, N_0, \varepsilon$.

Remark 3.3. Taking into account the above considerations of the classical difference schemes, in the case of problem (2.2), (2.1) one comes to the problem of construction of special schemes which converge under a weaker condition than condition (3.14) (i.e., schemes whose defect is less than 1), in particular, $\varepsilon$-uniformly convergent schemes.

Remark 3.4. We construct a triangulation of the domain $\overline{G}$ on the basis of the mesh $\mathcal{T}_{h(3.1)}$; triangular elements obtained by dividing the elementary quadrangles in two by a diagonal have vertices at the nodes from $\mathcal{T}_{h}$; see, e.g., [4]. In the case of the difference scheme (3.4), (3.3), the function $\pi(x, t)$, $(x, t) \in \overline{G}$, i.e., the linear interpolant of $z(x, t)$ on triangular elements satisfies the error estimate

$$|u(x, t) - \pi(x, t)| \leq M\varepsilon^{-1}[N^{-1} + N_0^{-1}], \quad (x, t) \in \overline{G},$$

which is unimprovable with respect to the values of $N, N_0, \varepsilon$. A similar estimate is valid for scheme (3.11), (3.10), (3.3); the triangulation of the domain $\overline{G}$ is performed on the basis of the mesh $\mathcal{T}_{h}^*$.

4. On the construction of $\varepsilon$-uniformly convergent schemes on locally condensing meshes

Note that the singularity in the solution of problem (2.2), (2.1) exponentially decreases away from the set $\gamma$ (see estimates (9.4), (9.5)). The singular component for $|x - \beta(t)| \geq \sigma$ does not exceed $M\delta$, where $\delta$ is a sufficiently small number, when $\sigma = m_1^{-1} \varepsilon \ln \delta^{-1}$, $m_1 = m_1(9.4)$. The local truncation error of the classical scheme on the solution of the problem is large, but only in this neighbourhood, which is sufficiently narrow for small values of the parameter $\varepsilon$.

1. Bearing in mind the possible use of schemes on sufficiently arbitrary locally condensing meshes for solving problem (2.2), (2.1), it would be convenient to measure the “amount” of computational work in order to evaluate the efficiency of the difference schemes. The amount of computational work (denoted by $P$) is defined by the number of the mesh points at which it is necessary to find the solution of the discrete problem. The “quality” of the solution to the difference scheme is defined by the distance, in the maximum norm, between the solution of the problem and the interpolant constructed from the solution of the difference scheme. In the case of schemes (3.4), (3.3) and (3.11), (3.10), (3.3) and optimal meshes (with respect to the order of convergence of the scheme), we have the following relation for such meshes

$$N \sim N_0,$$

and the unimprovable error estimate (with respect to $P$ and $\varepsilon$)

$$|u(x, t) - \pi(x, t)| \leq M\varepsilon^{-1}P^{-1/2}, \quad (x, t) \in \overline{G}.$$

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Definition 4.1. We say that the scheme converges with defect \( \nu \) for \( P \rightarrow \infty \) if there exists a function \( \mu(\eta^{-1}) \), \( \mu(\eta^{-1}) \rightarrow 0 \) for \( \eta \rightarrow \infty \) \( \varepsilon \)-uniformly such that the following estimate holds:

\[
|u(x,t) - \bar{z}(x,t)| \leq M \mu(\varepsilon^{-\nu} P^{-1/2}), \quad (x,t) \in \overline{G}; \quad \nu > 0,
\]

where \( \bar{z}(x,t) \) is the interpolant of the mesh function \( z(x,t) \).

Thus, for difference schemes (3.4), (3.3) and (3.11), (3.10), (3.3) the unimprovable convergence defect (for \( P \rightarrow \infty \)) is equal to 1.

2. To construct schemes with an improved convergence defect, it is tempting to use a technique based on locally condensing rectangular meshes. So, in the case of regular boundary value problems whose solutions have singularities, an improvement of the accuracy of a discrete solution can be achieved by means of \textit{a priori} and/or \textit{a posteriori} local refinement of a rectangular mesh in those subdomains where the errors of the discrete solution are larger (see, e.g., [1,3,13]).

For problem (2.2), (2.1), it is required to redistribute the given number of nodes \( P \) in the domain so as to weaken the condition

\[
P^{-1/2} = o(\varepsilon),
\]

i.e., the condition of convergence of schemes (3.4), (3.3) and (3.11), (3.10), (3.3).

Note that the derivatives \( \left( \partial^{k_1+k_0} / \partial x^{k_1} \partial t^{k_0} \right) u(x,t) \) in the \( M\varepsilon \)-neighbourhood of the set \( \gamma \) are of the order \( \varepsilon^{-\left( k_1+k_0 \right)} \) if the interface boundary \( \gamma \) between the subdomains is moving at a rate distinct from zero in some time interval (let \( \beta'(t) = m, \ t \in [0,t_0] \)). The derivatives are \( \varepsilon \)-uniformly bounded outside the sufficiently large (compared to \( \varepsilon \) ) neighbourhood of the set \( \gamma \).

A similar behaviour of the derivatives of the solution is observed in the case of a problem with a moving concentrated source [10]. In papers [10, 11], it is shown that in the class of schemes on adaptive meshes based on meshes rectangular in the \( M\varepsilon \)-neighbourhood of the moving source, there are no schemes with a convergence defect (for \( P \rightarrow \infty \)) less than \( 2^{-1} \).

In the case of problem (2.2), (2.1), by considering the lower errors for the solutions of equations (3.4a), (3.11a) on “piecewise-uniform” meshes (i.e., meshes that are uniform in the nearest \( M\varepsilon \)-neighbourhood of the boundary \( \gamma \) for \( t \in [0,t_0] \), as well as outside some larger neighbourhood), we verify, similarly to the constructions from [10], that there are no meshes on which the solution of the discrete problem converges under the condition

\[
P^{-1} \geq \varepsilon.
\]

Thus, in the case of schemes (3.4) and (3.11), (3.10) there are no meshes in the given class of meshes on which the schemes converge with a defect not exceeding \( 2^{-1} \).

Theorem 4.1. For problem (2.2), (2.1), in the class of schemes constructed on the basis of approximations (3.4a) and (3.11a) (classical difference approximations of the initial boundary value problem) on locally condensing meshes that are uniform in the \( M\varepsilon \)-neighbourhood of the set \( \gamma \), there are no schemes convergent under condition (4.2).

Remark 4.1. Direct use of adaptive mesh refinement techniques without taking account the orientation of the transition layer is not effective enough to solve numerically problems from this class of singularly perturbed problems with a moving transition layer. In order to construct schemes on adaptive meshes with convergence defect less than \( 2^{-1} \), it is necessary to use meshes condensing (in the transition layer) along the normal to the interface boundary \( \gamma \).
Remark 4.2. To construct special schemes for problem (2.2), (2.1), we introduce new variables (connected with the moving interface boundary $\gamma$) in which the interface boundary is already stationary. For the problem in these new variables it is possible to construct a difference scheme on rectangular meshes (in particular, a scheme on adaptive meshes) and then return to the old variables. It is convenient to use the variables $\xi, t$, $\xi = \xi_{(2)}(x, t)$ as new variables.

5. Grid approximations on locally refined meshes.

Problem (2.2), (2.1), (2.5)

We now give an algorithm for constructing a locally refined (in the transition layer) mesh. On domains subjected to refinement, this algorithm uses uniform meshes in space and time (the time mesh is not refined).

1. At first, we describe the formal iterative algorithm used by us to construct difference schemes in order to find a numerical solution of problem (2.2), (2.1), (2.5). Suppose that the function $u_1(x, t)$ is found on the set $G$, i.e., the approximation to the solution of the boundary value problem, moreover,

$$|u(x, t) - u_1(x, t)| \leq M\delta, \quad (x, t) \in G, \quad x \notin (d_1, d^1).$$  (5.1a)

where $\delta > 0$ is an arbitrary small number, the constant $M$ does not depend on $\delta; d_1, d^1 \in D$.

By $u_{(2)}(x, t), (x, t) \in \overline{G}_{(2)}$, where $G_{(2)} = D_{(2)} \times (0, T], D_{(2)} = (d_1, d^1)$, we denote the solution of the problem

$$L_{(2,2)} u_{(2)}(x, t) = f(x, t), \quad (x, t) \in G_{(2)},
\begin{align*}
u_{(2)}(x, t) &= \begin{cases} u_1(x, t), & (x, t) \in S_{(2)} \setminus S, \\ \varphi(x, t), & (x, t) \in S_{(2)} \cap S. \end{cases} \tag{5.1b} 
\end{align*}
$$

Here $S_{(2)} = \overline{G}_{(2)} \setminus G_{(2)}$. Let $\overline{u}_{(2)}^i(x, t), (x, t) \in \overline{G}_{(2)}$ be the approximation of the solution $u_{(2)}(x, t)$, moreover,

$$|u_{(2)}(x, t) - \overline{u}_{(2)}^i(x, t)| \leq M\delta, \quad (x, t) \in \overline{G}_{(2)}, \quad x \notin (d_2, d^2).$$

Assume

$$u_2(x, t) = \begin{cases} \overline{u}_{(2)}^i(x, t), & (x, t) \in \overline{G}_{(2)}, \\ u_1(x, t), & (x, t) \in \overline{G} \setminus \overline{G}_{(2)}. \end{cases}$$

Then we have:

$$|u(x, t) - u_2(x, t)| \leq M\delta, \quad (x, t) \in \overline{G}, \quad x \notin (d_2, d^2).$$

Let the function $u_{k-1}(x, t), (x, t) \in \overline{G}$ be constructed for $k \geq 3$, and this function has a convenient representation in order to compute it for $x \notin (d_{k-1}, d^{k-1}), d_{k-1}, d^{k-1} \in D$, and also

$$|u(x, t) - u_{k-1}(x, t)| \leq M\delta, \quad (x, t) \in \overline{G}, \quad x \notin (d_{k-1}, d^{k-1}).$$  (5.1c)

Here $M = M(k)$. The function $u_{(k)}(x, t), (x, t) \in \overline{G}_{(k)}$, where $G_{(k)} = D_{(k)} \times (0, T], D_{(k)} = (d_{k-1}, d^{k-1})$, denotes the solution of the problem

$$L_{(2,2)} u_{(k)}(x, t) = f(x, t), \quad (x, t) \in G_{(k)},
\begin{align*}
u_{(k)}(x, t) &= \begin{cases} u_{k-1}(x, t), & (x, t) \in S_{(k)} \setminus S, \\ \varphi(x, t), & (x, t) \in S_{(k)} \cap S. \end{cases} \tag{5.1d} 
\end{align*}$$
where $S_{(k)} = \overline{G}_{(k)} \setminus G_{(k)}$. Let $\tilde{u}_k^{(i)}(x, t), \ (x, t) \in \overline{G}_{(k)}$ be the approximation of the function $u_{(k)}(x, t)$, and

$$|u_{(k)}(x, t) - \tilde{u}_k^{(i)}(x, t)| \leq M\delta, \ (x, t) \in \overline{G}_{(k)}, \ x \notin (d_k, d_k^k).$$

Assume

$$u_k(x, t) = \begin{cases} \tilde{u}_k^{(i)}(x, t), & (x, t) \in \overline{G}_{(k)}, \\ u_{k-1}(x, t), & (x, t) \in \overline{G} \setminus \overline{G}_{(k)}. \end{cases}$$

The function $u_k(x, t), \ (x, t) \in \overline{G}$ satisfies the estimate:

$$|u(x, t) - u_k(x, t)| \leq M\delta, \ (x, t) \in \overline{G}, \ x \notin (d_k, d_k^k).$$

If for some value of $k = K_0$ it occurs that $|u_{(k)}(x, t) - \tilde{u}_k^{(i)}(x, t)| \leq M\delta, \ (x, t) \in \overline{G}_{(k)}$ for all $x \in \overline{D}_{(k)}$, then for $k \geq K_0 + 1$ we consider that the sets $\overline{G}_{(k)}$ are empty, and further the functions $u_{(k)}(x, t)$ are not computed. For example, for $k \geq K_0$ we have $u_k(x, t) = u_{K_0}(x, t), \ (x, t) \in \overline{G}$.

For $k = K$, where $K$ is a given fixed number, $K \geq 1$, we assume

$$u^K(x, t) = u_K(x, t), \ (x, t) \in \overline{G}.$$  

(5.1e)

The functions $u^K(x, t)$ and $u_k(x, t), k = 1, \ldots, K, \ (x, t) \in \overline{G}$ denote the solution and the components of the solution to the iterative process (5.1).

The functions $u^K(x, t)$ have suitable representations for computing them on the subdomains, which are extending as $K$ grows. The functions $u_k(x, t)$ and $u^K(x, t)$ satisfy the estimates

$$|u(x, t) - u^K(x, t)| \leq M\delta, \ (x, t) \in \overline{G}, \ x \notin (d_k, d_k^k),$$

$$|u(x, t) - u_k(x, t)| \leq M\delta, \ (x, t) \in \overline{G}, \ x \notin (d_k, d_k^k), \ k = 1, \ldots, K.$$  

(5.2)

**Lemma 5.1.** The functions $u^K(x, t)$ and $u_k(x, t), (x, t) \in \overline{G}, k = 1, \ldots, K$, i.e., the solution of the iterative process (5.1) and its components, satisfy estimate (5.2).

2. We now give the grid construction approximating the iterative process (5.1). On the set $\overline{G}$ we introduce the coarse (primary) mesh

$$\overline{G}_{1h} = \overline{\omega}_1 \times \overline{\omega}_0,$$  

(5.3a)

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are uniform meshes, $\overline{\omega}_0 = \overline{\omega}_{0(3,3)}$; the step size in the mesh $\overline{\omega}_1$ is equal to $h_1 = 2dN^{-1}$. We denote by $z_1(x, t), (x, t) \in \overline{G}_{1h}$, where $\overline{G}_{1h} = \overline{G}_{1h(5,3)} = \overline{G}_{h(3,3)}$, the solution of problem (3.4), (5.3a).

Let the values $d_1, d_k^1 \in \overline{\omega}_1$ be obtained in some a way so that for $x \notin (d_1, d_k^1)$ the discrete solution $z_1(x, t), (x, t) \in \overline{G}_{1h}$ well approximates the solution of problem (2.2), (2.1), (2.5). If it occurs that $d_1 - d_1 > 0$, then we define the subdomain on which the mesh will be refined:

$$G_{(2)} = G_{(2)}(d_1, d_1^1), \ G_{(2)} = D_{(2)} \times (0, T], \ D_{(2)} = (d_1, d_1^1).$$  

(5.3b)

On the subdomain $\overline{G}_{(2)}$ we introduce the mesh $\overline{G}_{(2)h} = \overline{\omega}_{(2)} \times \overline{\omega}_0$, where $\overline{\omega}_{(2)}$ is a uniform mesh with the number of nodes $N + 1$. On the set $\overline{G}_{(2)h}$ we find the solution $z_{(2)}(x, t)$ of the discrete problem

$$\Lambda_{(3,4)} z_{(2)}(x, t) = f(x, t), \quad (x, t) \in G_{(2)h},$$

$$z_{(2)}(x, t) = \begin{cases} z_1(x, t), & (x, t) \in S_{(2)} \setminus S, \\ \varphi(x, t), & (x, t) \in S_{(2)} \cap S, \end{cases}$$

(5.3c)

where $S_{(2)} = \overline{G}_{(2)} \setminus G_{(2)}$, $S_{(2)} = \overline{G} \setminus G_{(2)}$.
where \( G_{(2)h} = G_{(2)h} \cap \overline{G}_{(2)h}, S_{(2)h} = S_{(2)} \cap \overline{G}_{(2)h}, S_{(2)} = \overline{G}_{(2)} \setminus G_{(2)}. \) The mesh set \( \overline{G}_{2h} \) and the function \( z_2(x, t), (x, t) \in \overline{G}_{2h} \) are defined by the relations:

\[
\overline{G}_{2h} = \overline{G}_{(2)h} \cup \{ \overline{G}_{1h} \setminus \overline{G}_{(2)} \}, \quad z_2(x, t) = \begin{cases} 
(z_{(2)}(x, t), & (x, t) \in \overline{G}_{(2)h}, \\
(z_1(x, t), & (x, t) \in \overline{G}_{1h} \setminus \overline{G}_{(2)}. 
\end{cases}
\]

Let the mesh set \( \overline{G}_{k-1,h} \) and the mesh function \( z_{k-1}(x, t) \) on this set have been constructed for \( k \geq 3 \). Let also \( d_{k-1}, d_k \in \omega_{k-1} \) be found so that for \( x \not\in (d_{k-1}, d_k) \) the discrete solution \( z_{k-1}(x, t), (x, t) \in \overline{G}_{k-1,h} \) well approximates the solution of problem (2.2), (2.1), (2.5).

Here \( \omega_{k-1} \) is a mesh that generates the mesh \( \overline{G}_{k-1,h} = \overline{G}_{k-1} \times \omega_0; N_k + 1 \) is the number of nodes in the mesh \( \omega_k, k \geq 1; N_1 = N \). If it occurs that \( d_k - d_{k-1} > 0 \), then we define the domain

\[
G_{(k)} = G_{(k)}(d_{k-1}, d_k), \quad G_{(k)} = D_{(k)} \times (0, T], \quad D_{(k)} = (d_{k-1}, d_k).
\]

On the set \( \overline{G}_{(k)} \) we introduce the mesh

\[
\overline{G}_{(k)h} = \omega_{(k)} \times \omega_0,
\]

where \( \omega_{(k)} \) is a uniform mesh with the number of nodes \( N + 1 \); \( h_{(k)} \) is the step size of the mesh \( \omega_{(k)} \). Let \( z_{(k)}(x, t), (x, t) \in \overline{G}_{(k)h} \) be the solution of the discrete problem

\[
\Lambda_{(3,4)} z_{(k)}(x, t) = f(x, t), \quad (x, t) \in G_{(k)h},
\]

\[
z_{(k)}(x, t) = \begin{cases}
(z_{k-1}(x, t), & (x, t) \in S_{(k)h} \setminus S, \\
\varphi(x, t), & (x, t) \in S_{(k)h} \cap S.
\end{cases}
\]

Assume \( \overline{G}_{kh} = \overline{G}_{(k)h} \cup \{ \overline{G}_{k-1,h} \setminus \overline{G}_{(k)} \} \),

\[
z_k(x, t) = \begin{cases}
z_{(k)}(x, t), & (x, t) \in \overline{G}_{(k)h}, \\
z_{k-1}(x, t), & (x, t) \in \overline{G}_{k-1,h} \setminus \overline{G}_{(k)}.
\end{cases}
\]

If for some value \( k = K_0 \) it occurs that the discrete solution \( z_k(x, t), (x, t) \in \overline{G}_{kh} \) well approximates on \( \overline{G}_{kh} \) the solution of the differential problem, then for \( k \geq K_0 + 1 \) we believe that the sets \( \overline{G}_{(k)} \) are empty and further the functions \( z_{(k)}(x, t) \) are not computed.

For example, for \( k \geq K_0 \) we have \( z_k(x, t) = z_{K0}(x, t), \overline{G}_{kh} = \overline{G}_{K0h} \).

The computations are stopped also in the case where for some value of \( k = K_0 \) the condition \( d_k - d_{K_0} \geq d_k - d_{K_0-1} \) holds, which means that the solution cannot be further improved. For \( k \geq K_0 + 1 \) the function \( z_{(k)}(x, t) \) is not computed; for \( k \geq K_0 \) we assume \( z_k(x, t) = z_{K0}(x, t), \overline{G}_{kh} = \overline{G}_{K0h} \).

For \( k = K \), where \( K \) is a given fixed number, \( K \geq 1 \), we suppose that

\[
\overline{G}_h^K = \overline{G}_{Kh} \equiv \overline{G}_h, \quad z^K(x, t) = z_K(x, t) \equiv z(x, t).
\]

We call the function \( z_{(5,3f)}(x, t), (x, t) \in \overline{G}_{h(5,3f)}, \) the solution of scheme (3.4), (5.3), and the functions \( z_k(x, t), (x, t) \in \overline{G}_{kh}, k = 1, ..., K, \) the components of the solution to the difference scheme.

The above algorithm (we call it \( A_{(5,3)} \)) allows us to construct meshes condensing in transition layers. The value \( N_K + 1 \), i.e., the number of nodes in the mesh \( \omega^K = \omega_K \) used to construct the function \( z^K(x, t) \), does not exceed the value \( N(K) = N(K + 1) \).
In schemes (3.4), (5.3), in solving the intermediate problems (5.3e), it does not require the interpolation in order to determinate the values of the functions \( z_k(x, t) \) on the boundary \( S_{(k)h} \).

3. The meshes \( \mathcal{G}_{kh} \), \( k = 1, \ldots, K \) generated by the algorithm \( A_{(5.3)} \) are defined by the rule of choosing the values \( d_k \), \( d_k \) and also by the values of \( K \) and \( N, N_0 \).

Thus, the algorithm \( A_{(5.3)} \) defines the class of difference schemes, i.e., the class of schemes (3.4), (5.3). In this class of schemes, the boundary of the subdomain in which the refinement of the mesh is derived pass through nodes of more roughly refined mesh. Note that the smallest step size in the mesh \( \mathcal{G}_K \) is not less than the value of \( dN^{-K} \). In the meshes generated by the algorithm \( A_{(5.3)} \), the values of \( d_k \) are defined on the basis of the intermediate results obtained in the computing process; such meshes \( \mathcal{G}_{kh} \) are a posteriori condensing meshes.

For schemes from the class (3.4), (5.3), the maximum principle is valid. Note that in this class there are no schemes whose solutions converge \( \varepsilon \)-uniformly to the solution of (2.2), (2.1), (2.5).

6. Adaptive scheme based on the estimate of the solution gradient

To construct a posteriori condensing meshes, we use indicators (auxiliary functionals from the solutions of intermediate problems) which help us to define the boundaries of the mesh domain subjected to a refinement. We show the construction of an indicator based on the estimate for the gradient of the solution.

1. We define the width of the boundary layer for problem (2.2), (2.1). Let the following estimate hold for the component \( U(x, t) \) from the representation of (9.1):

\[
\left| \frac{\partial}{\partial x} U(x, t) \right| \leq M_1, \quad (x, t) \in \mathcal{G}. \tag{6.1a}
\]

Suppose that the values of the parameter \( \varepsilon \) are sufficiently small, \( \varepsilon \leq \varepsilon_0 \). We say that \( \sigma_0^L = \sigma_0^L(\varepsilon, M_0) \) and \( \sigma_0^R = \sigma_0^R(\varepsilon, M_0) \), where \( M_0 \) is an arbitrary large number, \( M_0 > M_1 \), are the left and right boundaries of the transition layer in the neighbourhood of the interface boundary \( \gamma \), if \( \sigma_0^L \) and \( \sigma_0^R \) are respectively the maximum and minimum values of \( \sigma^L \) and \( \sigma^R \) for which we have the estimate

\[
\left| \frac{\partial}{\partial x} u(x, t) \right| \leq M_0, \quad (x, t) \in \mathcal{G}, \quad x \notin (\sigma^L, \sigma^R); \tag{6.1b}
\]

we call the quantity \( \sigma_0 = \sigma_0^R - \sigma_0^L \) the width of the layer.

We consider such a boundary value problem as a model example:

\[
\varepsilon u''(x) + b(x)u'(x) = 1, \quad x \in \Omega = (-1, 1), \quad u(-1) = 1, \quad u(1) = 0, \tag{6.2}
\]

where \( b(x) = 1, \quad x \in \overline{\Omega}^2, \quad b(x) = -1, \quad x \in \overline{\Omega}^1 \). The solution of problem (6.2) can be decomposed into its regular \( U(x) \) and singular \( V(x) \) components: \( u(x) = U(x) + V(x), \quad x \in \overline{\Omega}^1, \quad r = 1, 2, \) the transition layer appears in the neighbourhood of the point \( x = 0 \).

In the case of problem (6.2), the width of the layer \( \sigma_0(6.1) \) has the asymptotic behavior

\[
\sigma_0 \approx \varepsilon \ln \varepsilon^{-1} \quad \text{for} \quad \varepsilon = o(1).
\]
The following estimate also holds:

\[ \sigma_0 \leq M \varepsilon \ln(\varepsilon^{-1}M_0^{-1}), \quad \varepsilon \in (0, \varepsilon_0], \quad \varepsilon_0 = \varepsilon_0(M_0), \quad \varepsilon_0 \leq mM_0^{-1}, \]

where \( M_0, \ m \) are any constants satisfying the conditions \( M_0 > 1, \ m < 1, \ M > 1. \)

2. We define the width of the transition layer for the difference scheme (3.4), (3.1). We denote by \( z_v(x, t), (x, t) \in \overline{G}_h \) the solution of the difference problem

\[
\begin{align*}
\Lambda_{(3.4)} z(x, t) &= L_{(2.2)} v(x, t), \quad (x, t) \in G_h^r, \\
z(x, t) &= v(x, t), \quad (x, t) \in S_h^r, \quad r = 1, 2,
\end{align*}
\]

where \( v(x, t) \) is any sufficiently smooth function, \( v \in C^{2,1}(G^r) \cap C(\overline{G}) \). The solution of problem (3.4), (3.1) can be represented as the sum of the functions

\[
z(x, t) = z_U(x, t) + z_V(x, t), \quad (x, t) \in \overline{G}_h^r, \quad r = 1, 2,
\]

where \( z_U(x, t), (x, t) \in \overline{G}_h^r \) is a discrete function approximating the component \( U(x, t) \) from the representation of (9.1). Let the component \( z_U(x, t) \) satisfy the following estimate:

\[
|\delta_z z_U(x, t)| \leq M_1, \quad (x, t) \in \overline{G}_h^r, \quad r = 1, 2; \quad x \neq 0, d. \tag{6.3a}
\]

We say that \( \sigma_0^L = \sigma_0^L(M_0) = \sigma_0^L(M_0; D, \varepsilon, \overline{G}_h), \sigma_0^R = \sigma_0^R(M_0) = \sigma_0^R(M_0; D, \varepsilon, \overline{G}_h) \) are the left and right boundaries of the discrete transition layer in the neighbourhood of the interface boundary \( \gamma \) if \( \sigma_0^L \) and \( \sigma_0^R \) are respectively the maximum and minimum values of \( \sigma^L \) and \( \sigma^R \), for which we have the estimate

\[
|\delta_z z(x, t)|, \ |\delta_{\tau z}(x, t)| \leq M_0, \quad (x, t) \in \overline{G}_h, \quad x \neq (\sigma^L, \sigma^R). \tag{6.3b}
\]

where \( \varepsilon \in (0, \varepsilon_0], \ M_0 \) and \( \varepsilon_0 \) are sufficiently large and small constants, \( M_0 > M_1, \varepsilon_0 = \varepsilon_0(M_0) \), we call the quantity \( \sigma_0 = \sigma_0^R - \sigma_0^L \) the width of the layer. Thus, the functions \( \sigma_0(M_0), \sigma_0^L(M_0), \sigma_0^R(M_0) \) are constructed.

In the case of the difference scheme

\[
\varepsilon \delta_{\tau z} z(x) + b^+(x) \delta z(x) + b^-(x) \delta_{\tau z} z(x) = 1, \quad x \in \Omega_h, \quad z(-1) = 1, \quad z(1) = 0,
\]

which approximates problem (6.2), for the width of the boundary layer on uniform (with the step size \( h \)) meshes we have the asymptotic

\[
\sigma_0 \approx \begin{cases} 
\varepsilon \ln \varepsilon^{-1}, & h \leq M \varepsilon, \\
\varepsilon \ln(1 + \varepsilon^{-1}h) \ln h^{-1}, & h > m \varepsilon; \quad \varepsilon, h = o(1).
\end{cases}
\]

The following estimate is valid:

\[
\sigma_0 \leq M \left[ \varepsilon \ln \varepsilon^{-1} + h \ln h^{-1} \right], \quad \varepsilon \in (0, \varepsilon_0], \quad h \leq h_0,
\]

where \( \varepsilon_0, \ h_0 \) are sufficiently small values, \( \varepsilon_0 = \varepsilon_0(M_0), \ h_0 = h_0(M_0) \).

3. In order that the formal mesh construction (3.4), (5.3) is constructive, it is required to give the values of \( K \) and \( d_k, d^k, k = 1, 2, \ldots, K. \)

Let \( K \geq 1. \) We define the values of \( d_k(5.3), d^k(5.3). \) Assume

\[
d_1 = \sigma_1^L, \quad d^1 = \sigma_1^R, \tag{6.4a}
\]
where \( \sigma_1^{(L,R)} = \sigma_0^{(L,R)}(M_0; D, \varepsilon, \Gamma_h) \), \( D = D_{(2)}, \Gamma_h = \Gamma_{h(3,3)} \), \( M_0 \) is a sufficiently large number. Let the values of \( d_{k-1}, d^{k-1} \) be found. Further we find the values of \( \sigma_k^{(L,R)} \):

\[
\sigma_k^{(L,R)} = \sigma_0^{(L,R)}(kM_0; D_{(k)}, \varepsilon, \Gamma_{(k)h}), \quad k \geq 2,
\]

(6.4b)

where \( \sigma_0^{(L,R)}(M; D, \varepsilon, \Gamma_h) = \sigma_0^{(L,R)}(M; D, \varepsilon, \Gamma_{(k)h}) \), \( M = kM_0 \), \( D_{(k)} = D_{(k)(5,3)} \), \( \Gamma_{(k)h} = \Gamma_{(k)(h)(3,3)} \).

If the relation \( \sigma_k \leq m_0 \sigma_{k-1} \) where \( \sigma_k = \sigma_k^L - \sigma_k^R \), is valid, then we assume

\[
d_k = \sigma_k^L, \quad d^k = \sigma_k^R;
\]

(6.4c)

here \( m_0 \) is a sufficiently small number. If for some value of \( k = k_0 \) it occurs that \( \sigma_{k_0} > m_0 \sigma_{k_0-1} \), then we assume \( d_k = d_{k_0}, d^k = d^{k_0} \) for \( k \geq k_0 \).

The difference scheme (3.4), (5.3), (6.4) is the scheme on the adaptive meshes constructed on the basis of the estimate for the gradient of the discrete solutions obtained in the process of intermediate computations. The mesh refinement is realized only in the neighbourhood of the transition layer; the diameter of such a neighbourhood (the width of the transition layer) narrows with increasing value of \( k \).

7. Analysis of scheme (3.4), (5.3), (6.4)

1. We now give some estimates for the solution of the difference scheme (3.4), (3.3). We denote by \( W(x) \) and \( z_W(x) \) the solutions of the problems

\[
W(x) \equiv \left\{ \varepsilon a_{(1)} \frac{d^2}{dx^2} + b_{(1)} \text{sign } x \frac{d}{dx} \right\} u(x) = 0, \quad x \in D, \quad x \neq 0,
\]

\[
W(0) = 1, \quad W(x) = 0, \quad x \in \Gamma;
\]

\[
\Lambda z(x) \equiv \left\{ \varepsilon a_{(1)} \delta_x + b_{(1)} \text{sign } x \right\} z(x) = 0, \quad x \in D_h, \quad x \neq 0, \quad z(x) = 0, \quad x \in \Gamma_h,
\]

where \( D_h = \Omega_{(3,3)} \), \( a_{(1)} = \max_{\Gamma} a(x, t) \), \( b_{(1)} = \min_{\Gamma} |b(x, t)| \).

1.1. The function \( z_W(x) \) satisfies the estimate

\[
z_W(x) \leq q^{-r_1 h^{-1}}, \quad x \in \Omega_h, \quad (7.1a)
\]

where \( q = 1 + a_{(1)}^{-1} b_{(1)}^{-1} \varepsilon^{-1} h \), \( r_1 = r(x, \beta(t = 0)) = |x|, r(x, x^*) \) is the distance between the points \( x \) and \( x^* \). Thus, we have the estimate

\[
z_W(x) \leq \left\{ \begin{array}{ll}
M \exp(-m \varepsilon^{-1} r_1), & h \leq M \varepsilon \\
M (\varepsilon h^{-1})^{mh^{-1} r_1}, & h > M \varepsilon
\end{array} \right\}, \quad x \in \Omega_h.
\]

(7.1b)

The function \( z_W(x) \) is the majorant for the component \( z_V(x, t) \), corresponding to \( V(x, t) \) from the representation of (9.1)

\[
|z_V(x, t)| \leq M z_W(x), \quad (x, t) \in \Gamma_h, \quad r = 1, 2.
\]

(7.2)

1.2. The solution of the difference scheme (3.4), (3.3) satisfies the estimate

\[
|u(x, t) - z(x, t)| \leq M \left( \varepsilon + N^{-1} \right) N^{-1} + N_0^{-1}, \quad (x, t) \in \Omega_h.
\]

(7.3a)

\[
|u(x, t) - z(x, t)| \leq M \left( z_W(x) + N^{-1} + N_0^{-1} \right), \quad (x, t) \in \Omega_h.
\]

(7.3b)
It follows from estimates (7.2), (7.3) that under condition (3.8) the scheme converges on $\overline{G}_h$, and also the scheme converges $\varepsilon$-uniformly outside the $\sigma_0$-neighbourhood of the set $\gamma$:

$$|u(x,t) - z(x,t)| \leq M \left[ N^{-1/2} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h,$$

(7.4a)

for $r(x,\gamma) \geq \sigma_0$, where $\sigma_0 = \sigma_{0(6,3)}(M_0; D, \varepsilon, \overline{G}_h)$

$$\sigma_0 \leq M \left[ \varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N \right].$$

(7.4b)

The neighbourhood, beyond which estimate (7.4a) holds, becomes narrow for $\varepsilon \to 0$, $N \to \infty$.

**Theorem 7.1.** Let the solution of the boundary value problem (2.2), (2.1) satisfy condition (2.5) and the estimates of Theorem 9.1. Then the solution of the difference scheme (3.4), (3.3) converges on $\overline{G}$ to the solution of the boundary value problem under condition (3.8) and also $\varepsilon$-uniformly (with the rate $O(N^{-1/2} + N_0^{-1})$) outside the $\sigma_0$-neighbourhood of the set $\gamma$. The discrete solution satisfies estimates (7.2)–(7.4).

2. Let us consider the difference scheme (3.4), (5.3), (6.4).

For the component $z_1(x,t) = z(x,t)$ of the solution to this scheme estimate (7.3) is valid. Taking into account estimate (7.1), for the function $z_W(x)$ we find the following estimate for the value of $\sigma_1$, i.e., the width of the transition layer:

$$\sigma_1 \leq M \left[ \varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N \right], \quad \varepsilon \in (0, \varepsilon_0], \quad h \leq h_0.$$

The value $h_2$, i.e., the step size of the mesh $\varpi(2)$, satisfies the estimate

$$h_2 \leq M N^{-1} \left[ \varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N \right].$$

Taking into consideration estimate (7.3b), we estimate $u(x,t) - z_1(x,t)$ on the boundary on the set $\overline{G}_h^2$ and also $u(x,t) - z_2(x,t)$ on the whole set $\overline{G}_h^0$. For $u(x,t) - z_2(x,t)$ we have the estimate

$$|u(x,t) - z_2(x,t)| \leq M \left[ N^{-1/2} + \varepsilon^{-1} N^{-2} \ln N + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h^2,$$

and outside the $\sigma_2$-neighbourhood of the set $\gamma$ we have

$$|u(x,t) - z_2(x,t)| \leq M \left[ N^{-1/2} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h^2, \quad r(x,\gamma) \geq \sigma_2.$$

The value $\sigma_2$ satisfies the estimate

$$\sigma_2 \leq M N^{-1} \ln N.$$

Likewise, we find the estimates

$$|u(x,t) - z_k(x,t)| \leq M \left[ N^{-1/2} + \varepsilon^{-1} N^{-k} \ln^{k-1} N + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h^k;$$

$$|u(x,t) - z_k(x,t)| \leq M \left[ N^{-1/2} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h^k, \quad r(x,\gamma) \geq \sigma_k;$$

$$\sigma_k \leq \begin{cases} M \varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N, & k = 1 \\ M N^{-k+1} \ln^{k-1} N, & k \geq 2 \end{cases}, \quad k = 1, 2, \ldots, K;$$

(7.5)

$$|u(x,t) - z(x,t)| \leq M \left[ N^{-1/2} + \varepsilon^{-1} N^{-K} \ln^{K-1} N + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h^K;$$

$$|u(x,t) - z(x,t)| \leq M \left[ N^{-1/2} + N_0^{-1} \right], \quad (x,t) \in \overline{G}_h^K, \quad r(x,\gamma) \geq \sigma_K;$$

$$\sigma_K \leq \begin{cases} M \varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N, & K = 1 \\ M N^{-K+1} \ln^{K-1} N, & K \geq 2 \end{cases};$$

(7.6)
where \( z(x,t) = z_{(5,3e)}(x,t) \), \( \mathcal{G}_h = \mathcal{G}_{h(5,3e)} \).

The functions \( z(x,t) \) and \( z_k(x,t) \) for \( N, N_0 \to \infty \) converge (to the solution of the boundary value problem (2.2), (2.1)) \( \varepsilon \)-uniformly outside the \( \sigma_K \)- and \( \sigma_k \)-neighbourhoods of the set \( \gamma \), and also on the sets \( \mathcal{G}_{kh} \) and \( \mathcal{G}_h \) for sufficiently small (but not too small) values of the parameter \( \varepsilon \), namely, under the condition

\[
\varepsilon \geq \varepsilon_0(N), \quad \varepsilon_0^{-1}(N) = o(N^k \ln^{-K+1} N); \quad \varepsilon \geq \varepsilon_k(N), \quad \varepsilon_k^{-1}(N) = o(N^k \ln^{-k+1} N), \quad k = 1,2,\ldots,K.
\]

Thus, the difference scheme (3.4), (5.3), (6.4), i.e., the scheme on the adaptive meshes, converges almost \( \varepsilon \)-uniformly. To ensure that the convergence defect for the function \( z(x,t) \) does not exceed the values of \( \nu_{(2,7)} \), it is required to choose the value of \( K \) satisfying the condition

\[
K > K(\nu), \quad K(\nu) = \nu^{-1}.
\]

**Theorem 7.2.** Let the hypothesis of Theorem 7.1 be fulfilled. Then the functions \( z(x,t), (x,t) \in \mathcal{G}_h \) and \( z_k(x,t), (x,t) \in \mathcal{G}_{kh}, k = 1,\ldots,K \), i.e., the solution of the difference scheme (3.4), (5.3), (6.4) and its components converge on \( \mathcal{G} \) to the solution of the boundary value problem (2.2), (2.1) under condition (7.7) and also \( \varepsilon \)-uniformly (with the rate \( O(N^{-1/2}+N_0^{-1}) \)) outside the \( \sigma_K \)- and \( \sigma_k \)-neighbourhoods of the set \( \gamma \); the solution of scheme (3.4), (5.3), (6.4), (7.8) converges to the solution of the boundary value problem almost \( \varepsilon \)-uniformly with the defect \( \nu \). For discrete solutions, estimates (7.5), (7.6) are valid.

**Remark 7.1.** For the interpolant \( \bar{z}(x,t), (x,t) \in \mathcal{G} \) (linear on triangular elements) constructed from the function \( z(x,t), (x,t) \in \mathcal{G}_h \), we have an estimate similar to estimate (7.6):

\[
|u(x,t) - \bar{z}(x,t)| \leq M \left[ N^{-1/2} + \varepsilon^{-1}N^{-K} \ln^{K-1} N + N_0^{-1} \right], \quad (x,t) \in \mathcal{G}.
\]

\[
|u(x,t) - \bar{z}(x,t)| \leq M \left[ N^{-1/2} + N_0^{-1} \right], \quad (x,t) \in \mathcal{G}, \quad r(\gamma) \geq \sigma_K;
\]

\[
\sigma_K \leq \left\{ \begin{array}{ll}
M[\varepsilon \ln^{-1} N + N^{-1} \ln N], & K = 1 \\
MN^{-K+1} \ln^{K-1} N, & K \geq 2
\end{array} \right\}.
\]

8. **Special scheme for problem (2.2), (2.1)**

In the case of problem (2.2), (2.1) with a moving interface boundary \( \gamma \) we pass to the system of new coordinates \( \xi, t \). Further, for problem (9.2), (9.3) we construct a classical scheme without focusing on the conjugation on condition in the approximation.

1. On the set \( \mathcal{G} \) we construct meshes. First, we introduce the basic mesh

\[
\mathcal{G}_h^B = \bar{\omega}_1 \times \bar{\omega}_0,
\]

where \( \bar{\omega}_1 \) is a mesh on the axis \( \xi \); \( \bar{\omega}_0 = \bar{\omega}_{0(3,1)} \); the mesh \( \bar{\omega}_1 \) is a mesh with any distribution of the nodes satisfying only the condition

\[
h_\xi \leq MN^{-1},
\]

where \( h_\xi = \max h_\xi^i; \quad h_\xi^i = \xi_i+1 - \xi_i, \quad \xi_i, \xi_{i+1} \in \bar{\omega}_1, \quad N+1 \) is the maximal number of nodes on an interval of unit length on the axis \( \xi \). As a basic grid, we use the following mesh uniform with respect to \( \xi, t \):

\[
\mathcal{G}_h^B, \quad \text{where } \bar{\omega}_1, \bar{\omega}_0 \text{ are uniform meshes}.
\]
The interior nodes are defined by the relation \( \tilde{G}_h = \tilde{G} \cap \tilde{G}_h^B \); the boundary nodes are generated by the intersection of the lines \( t = t^j \in \bar{\omega}_0 \) with a lateral boundary \( \tilde{S}_L \) and the lines \( \xi = \xi^j \in \bar{\omega} \) with the lower part of the boundary \( \tilde{S}_0; \tilde{S}_h = \tilde{S}_{0h} \cup \tilde{S}_h^L \). On the set \( \tilde{G} \) we introduce the mesh

\[
\tilde{G}_h = \tilde{G}_h \cup \tilde{S}_h; \quad \tilde{G}_h = \tilde{G}_h(\tilde{G}_h^B).
\] (8.3)

Problem (9.2), (9.3) is approximated by the implicit difference scheme

\[
\tilde{A}Z(\xi, t) = \left\{ \varepsilon \tilde{a}(\xi, t) \delta_\xi + B^+(\xi, t) \delta_\xi + B^-(\xi, t) \delta_{\bar{\xi}} - \tilde{c}(\xi, t) - \tilde{p}(\xi, t) \delta_{\bar{\xi}} \right\} \tilde{Z}(\xi, t) = \tilde{f}(\xi, t),
\] (8.4)

\( (\xi, t) \in \tilde{G}_h, \)

\( \tilde{Z}(\xi, t) = \tilde{\varphi}(\xi, t), \quad (\xi, t) \in \tilde{S}_h. \)

The difference scheme (8.4), (8.3), (8.1) is monotone.

Using the algorithm \( A(5,3) \) for scheme (8.4), we construct the meshes

\[
\tilde{G}_{kh}, \quad k = 1, 2, \ldots, K, \quad \tilde{G}_h,
\] (8.5a)

where \( \tilde{G}_{1h} = \tilde{G}_{h(8.3)}(\tilde{G}_h(8.2), \tilde{G}_h = \tilde{G}_{Kh} \), and then we find the functions

\[
Z_k(\xi, t), \quad (\xi, t) \in \tilde{G}_{kh}, \quad Z(x, t), \quad (\xi, t) \in \tilde{G}_h,
\] (8.5b)

where \( Z(\xi, t) = Z_K(\xi, t) \). The meshes are defined by the law of choice of the values

\[
d_k, d^k, \quad k = 1, 2, \ldots, K,
\] (8.5c)

and also by the values of \( K \) and \( N, N_0 \).

In the class of difference schemes (8.4), (8.5), a more precise discrete solution is produced on simple domains, i.e., domains with stationary boundaries; the boundary of the domain in which the mesh refinement is realized pass through the nodes of the refine mesh.

For the schemes from class (8.4), (8.5) the maximum principle is valid.

The values

\[
\sigma^L_k, \quad \sigma^R_k, \quad \sigma_k, \quad k = 1, 2, \ldots, K,
\] (8.6a)

defining the left and right boundaries of the transition layer and its width are constructed similarly to the values \( \sigma^L_{k(6,4)}, \sigma^R_{k(6,4)}, \sigma_k(6,4) \). The parameters \( d_k, d^k \) are defined similarly to the parameters \( d_{k(6,4)}, d^k_{(6,4)} \) as

\[
d_k = \sigma^L_k, \quad d^k = \sigma^R_k, \quad k = 1, 2, \ldots, K.
\] (8.6b)

The difference scheme (8.4), (8.5), (8.6) is the scheme on the a posteriori adaptive meshes constructed on the basis of the gradient of the intermediate discrete solutions.

2. For the function \( Z_1(\xi, t) = Z(8.4,8.3;8.2) \), i.e., the solution of difference scheme (8.4), (8.3), (8.2), we have the estimates

\[
|\tilde{u}(\xi, t) - Z_1(\xi, t)| \leq M \left[ (\varepsilon + N^{-1})^{-1} N^{-1} + N_0^{-1} \right],
\]

\[
|\tilde{u}(\xi, t) - Z_1(\xi, t)| \leq M \left[ (Z_W(\xi) + N^{-1} + N_0^{-1}) \right], \quad (\xi, t) \in \tilde{G}_h.
\]
Here $Z_W(\xi)$ is the solution of the problem

$$
\Lambda Z(\xi) \equiv \left\{ \varepsilon a_{(1)} \delta \xi + (B_{(1)} \text{sign} \xi)^{+} \delta \xi + (B_{(1)} \text{sign} \xi)^{-} \delta \xi \right\} Z(\xi) = 0, \quad \xi \in \tilde{D}_h^B, \ \xi \neq 0,
$$

$$
Z(0) = 1, \quad Z(\xi) \to 0 \text{ for } |\xi| \to \infty,
$$

$$
a_{(1)} = \max_{G} a(x, t), \quad B_{(1)} = \min_{G} |B_{(9.2)}(\xi, t)|, \quad \tilde{D}_h^B \text{ is a uniform mesh on the axis } \xi \text{ with a step size } h_\xi = N^{-1}.
$$

The function $Z_W(\xi)$ satisfies the estimate

$$
Z_W(\xi) \leq \begin{cases} 
M \exp(-m\varepsilon^{-1}r_2), & h_\xi \leq M\varepsilon \\
M(\varepsilon h_\xi^{-1})^{m\varepsilon^{-1}r_2}, & h_\xi > M\varepsilon
\end{cases}, \quad \xi \in \tilde{D}_h^B,
$$

where $r_2 = r(\xi, \xi(\beta(0), 0)) = |\xi|$.

With regard for the a priori estimates for the solution of problem (9.2), (9.3), for the solutions of the difference scheme (8.4), (8.5), (8.6) we establish the estimates

$$
|\bar{u}(\xi, t) - Z_k(\xi, t)| \leq M \left[ N^{-1/2} + \varepsilon^{-1}N^{-k} \ln^{k-1}N + N_0^{-1} \right], \quad (\xi, t) \in \overline{G}_kh;
$$

$$
|\bar{u}(\xi, t) - Z_k(\xi, t)| \leq M \left[ N^{-1/2} + N_0^{-1} \right], \quad (\xi, t) \in \overline{G}_kh, \quad r(\xi, \gamma) \geq \sigma_k;
$$

$$
\sigma_k \leq \begin{cases} 
M \left[ \varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N \right], & k = 1 \\
M N^{-k+1} \ln^{k-1}N, & k \geq 2
\end{cases}, \quad k = 1, \ldots, K;
$$

$$
|\bar{u}(\xi, t) - Z(\xi, t)| \leq M \left[ N^{-1/2} + \varepsilon^{-1}N^{-K} \ln^{K-1}N + N_0^{-1} \right], \quad (\xi, t) \in \overline{G}_h;
$$

$$
|\bar{u}(\xi, t) - Z(\xi, t)| \leq M \left[ N^{-1/2} + N_0^{-1} \right], \quad (\xi, t) \in \overline{G}_h, \quad r(\xi, \gamma) \geq \sigma_K;
$$

$$
\sigma_K \leq \begin{cases} 
M \left[ \varepsilon \ln \varepsilon^{-1} + N^{-1} \ln N \right], & K = 1 \\
M N^{-K+1} \ln^{K-1}N, & K \geq 2
\end{cases},
$$

where $Z(\xi, t) = Z_{(8.5)}(\xi, t), \quad \overline{G}_h = \overline{G}_{h(8.5)}$.

The function $Z(x, t)$ for $N, N_0 \to \infty$ converges (to the solution of problem (9.2), (9.3)) $\varepsilon$-uniformly outside the $\sigma_K$-neighbourhood of the set $\gamma$, and also on $\overline{G}$ for the values of the parameter satisfying the condition $(N^{-K} \ln^{K-1}N << \varepsilon)$

$$
\varepsilon \geq \varepsilon_0(N), \quad \varepsilon_0^{-1}(N) = o(N^{-K} \ln^{K-1}N).
$$

Thus, the difference scheme (8.4), (8.5), (8.6) converges almost $\varepsilon$-uniformly for large $K$; to ensure that the convergence defect of the function $Z(\xi, t)$ does not exceed the value of $\nu(2,7)$, it is required to choose the value of $K$ satisfying the condition

$$
K > K(\nu), \quad K(\nu) = \nu^{-1}.
$$

Theorem 8.1. Let the hypothesis of Theorem 7.1 be fulfilled. Then the solution of the difference scheme (8.4), (8.5), (8.6) converges to the solution of problem (9.2), (9.3) under condition (8.9): the solution of this scheme under condition (8.10) converges almost $\varepsilon$-uniformly. For discrete solutions, estimates (8.7), (8.8) are valid.

Remark 8.1. For the interpolant $\overline{Z}(\xi, t), \ (\xi, t) \in \overline{G}$ (linear on triangular elements), the following estimate holds:

$$
|\bar{u}(\xi, t) - \overline{Z}(\xi, t)| \leq M \left[ N^{-1/2} + \varepsilon^{-1}N^{-K} \ln^{K-1}N + N_0^{-1} \right], \quad (\xi, t) \in \overline{G}.
$$
In the variables $x, t$ we have

$$\left| u(x, t) - Z_x(x, t) \right| \leq M \left[ N^{-1/2} + \varepsilon^{-1} N^{-K} \ln^{K-1} N + N_0^{-1} \right], \quad (x, t) \in \{ \overline{G}_h \}_x;$$

$$\left| u(x, t) - (\overline{Z})_x(x, t) \right| \leq M \left[ N^{-1/2} + \varepsilon^{-1} N^{-K} \ln^{K-1} N + N_0^{-1} \right], \quad (x, t) \in \overline{G},$$

where $Z_x(x, t) = Z(\xi(x, t), t), \{ \overline{G}_h \}_x$ is the mesh on $\overline{G}$ corresponding to the mesh $\overline{G}_h$ on $\overline{S}$.

**Remark 8.2.** For problem (9.2), (9.3), when we have an almost $\varepsilon$-uniform convergent scheme on the rectangular (in the variables $\xi, t$) *a posteriori* adaptive meshes, it is possible to rewrite the given constructions in the variables $x, t$. In this case, we pass to a difference scheme approximating problem (2.2), (2.1) on the meshes generated by a family of (sufficiently smooth) curves adapted to the interface boundary $\gamma$. An almost $\varepsilon$-uniform convergence of the thus constructed schemes for the approximation of problem (2.2), (2.1) is ensured by the *a posteriori* condensation of the mesh in the neighbourhood of the set $\gamma$.

**9. A priori estimates**

In this section, we consider the *a priori* estimates for the solution of problem (2.2), (2.1) used in our constructions (see also [2, 5, 6, 9]).

On the set $\overline{G}^r$ the solution can be decomposed into its regular and singular components

$$u(x, t) = U(x, t) + V(x, t), \quad (x, t) \in \overline{G}^r, \quad r = 1, 2,$$

which are defined below.

It is convenient to transform problem (2.2), (2.1) to the variables $\xi = \xi(x, t) = x - \beta(t), t$ as follows:

$$\tilde{L} \tilde{u}(\xi, t) = \tilde{f}(\xi, t), \quad (x, t) \in \tilde{G}^{(s)},$$

$$[\tilde{u}(\xi, t)] = 0, \quad \tilde{l} \tilde{u}(\xi, t) = 0, \quad (\xi, t) \in \tilde{\gamma}, \quad \tilde{u}(\xi, t) = \tilde{\varphi}(\xi, t), \quad (\xi, t) \in \tilde{S}.$$  

Here $\tilde{G}_0 = \{ G_0 \}_t = G_0$ is the image of the set $G_0 \subseteq \overline{G}$ in the variables $\xi, t, \xi = \xi(x, t)$;

$$\tilde{\gamma} = \{(\xi, t) : \xi = 0, t \in (0, T]\}, \quad \tilde{v}(\xi, t) = v(\xi, t, t) = v(\xi + \beta(t), t);$$

$$\tilde{L}_{(9.2)} \equiv \varepsilon \tilde{a}(\xi, t) \frac{\partial^2}{\partial \xi^2} + B(\xi, t) \frac{\partial}{\partial \xi} - \tilde{c}(\xi, t) - \tilde{p}(\xi, t) \frac{\partial}{\partial t}, \quad (\xi, t) \in \overline{G}^r,$$

$$\tilde{l}_{(9.2)} \tilde{u}(\xi, t) \equiv \varepsilon \left[ \tilde{a}(\xi, t) \frac{\partial}{\partial \xi} \tilde{u}(\xi, t) \right], \quad (\xi, t) \in \tilde{\gamma}; \quad B(\xi, t) = B^r(\xi, t), \quad (\xi, t) \in \overline{G}^r,$$

$$B^1(\xi, t) = -\tilde{b}^1(\xi, t) + \beta'(t) \tilde{p}^1(\xi, t), \quad B^2(\xi, t) = \tilde{b}^2(\xi, t) + \beta'(t) \tilde{p}^2(\xi, t);$$

$$\tilde{a}(\xi, t) = \tilde{a}^r(\xi, t), \ldots \quad \tilde{f}(\xi, t) = \tilde{f}^r(\xi, t), \quad (\xi, t) \in \overline{G}^r, \quad r = 1, 2.$$  

The domain

$$\overline{G} = \overline{G}^1 \cup \overline{G}^2, \quad \overline{G}^1 \cap \overline{G}^2 \neq \emptyset$$

is a domain, in general, with curvilinear lateral boundaries; the interface boundary $\tilde{\gamma}$ is immovable, moreover, the distance between the lateral boundary $\tilde{S}^L$ and the set $\gamma$ is not less than $\min[d - \beta_0, d - \beta_0]$. 
The solution of problem \( (9.2) \) can be differentiated with respect to \( t \) on \( \tilde{G}^r \) and with respect to \( \xi \) on \( \tilde{G}^r \) (see, e.g., [12]), and it is \( \varepsilon \)-uniformly bounded on \( \tilde{G} \) together with its derivatives with respect to \( t \) (under a suitable smoothness condition for the data of problem \( (2.2), (2.1) \)). We write the function \( \tilde{u}(\xi, t) \) on the set \( \tilde{G}^r \) as the sum of the functions

\[
\tilde{u}(\xi, t) = \tilde{U}(\xi, t) + \tilde{V}(\xi, t), \quad (\xi, t) \in \tilde{G}^r, \quad r = 1, 2,
\]

where \( \tilde{U}(\xi, t) \) and \( \tilde{V}(\xi, t) \) are the regular and singular (interior layer) components of the solution. The function \( \tilde{U}(\xi, t) \) is the restriction on \( \tilde{G}^r \) of the function \( \tilde{U}^{0r}(\xi, t) \), \( (\xi, t) \in \tilde{G}^{0r} \), which is the (bounded) solution of the problem

\[
\tilde{L}^{0r} \tilde{U}^{0r}(\xi, t) = \tilde{f}^{0r}(\xi, t), \quad (\xi, t) \in \tilde{G}^{0r},
\]

\[
\tilde{U}^{0r}(\xi, t) = \tilde{\varphi}^{0r}, \quad (\xi, t) \in \tilde{S}^{0r}, \quad r = 1, 2;
\]

the set \( \tilde{G}^{0r} \) is obtained by an extension of the set \( \tilde{G}^r \) beyond the interface boundary \( \tilde{\gamma} \), \( \tilde{G}^{01} = \tilde{G}^1 \cup \{ [0, \infty) \times (0, T] \} \), \( \tilde{G}^{02} = \tilde{G}^2 \cup \{ (-\infty, 0] \times (0, T] \} \), the operator \( \tilde{L}^{0r} \) and the functions \( \tilde{f}^{0r}(\xi, t) \), \( \tilde{\varphi}^{0r}(\xi, t) \) are continuations of the operator \( \tilde{L}^r \) and of the functions \( \tilde{f}(\xi, t) \), \( \tilde{\varphi}(\xi, t) \) from the sets \( \tilde{G}^r \) and \( \tilde{S} \cap \tilde{S}^{0r} \) onto the sets \( \tilde{G}^{0r} \) and \( \tilde{S}^{0r} \) which preserve the smoothness and boundedness properties, i.e.,

\[
a_0 \leq \tilde{a}^{0r}(\xi, t) \leq a^0, \quad B_0 \leq |B^{0r}(\xi, t)| \leq B^0, \ldots,
\]

\[
|\tilde{f}^{0r}(\xi, t)| \leq M, \quad (\xi, t) \in \tilde{G}^{0r}, \quad |\tilde{\varphi}^{0r}(\xi, t)| \leq M, \quad (\xi, t) \in \tilde{S}^{0r}.
\]

The function \( \tilde{V}(\xi, t) \) is the solution of the problem

\[
\tilde{L} \tilde{V}(\xi, t) = 0, \quad (\xi, t) \in \tilde{G}^r,
\]

\[
\tilde{V}(\xi, t) = \tilde{u}(\xi, t) - \tilde{U}(\xi, t), \quad (\xi, t) \in \tilde{S}^r, \quad r = 1, 2.
\]

For the functions \( \tilde{U}(\xi, t) \), \( \tilde{V}(\xi, t) \), we obtain the estimates

\[
\left| \frac{\partial^{k_1 + k_0}}{\partial \xi^{k_1} \partial t^{k_0}} \tilde{U}(\xi, t) \right| \leq M, \quad \left| \frac{\partial^{k_1 + k_0}}{\partial \xi^{k_1} \partial t^{k_0}} \tilde{V}(\xi, t) \right| \leq M \varepsilon^{-k_1} \exp(-m_1 \varepsilon^{-1}|\xi|),
\]

\[
(\xi, t) \in \tilde{G}^r, \quad k_1 + 2k_0 \leq 4, \quad r = 1, 2,
\]

where \( m_1 \in (0, m_0) \), \( m_0 = \min \{ a(x, t)^{-1} \left| b(x, t) - p(x, t)(d/dt)\alpha(t) \right| \} \). Returning to the variables \( x, t \), we find

\[
\left| \frac{\partial^{k_1 + k_0}}{\partial x^{k_1} \partial t^{k_0}} U(x, t) \right| \leq M, \quad (x, t) \in \tilde{G}^r,
\]

\[
\left| \frac{\partial^{k_1 + k_0}}{\partial x^{k_1} \partial t^{k_0}} V(x, t) \right| \leq M \varepsilon^{-k_1 - k_0} \exp(-m_1 \varepsilon^{-1} |x - \beta(t)|),
\]

\[
(x, t) \in \tilde{G}^r, \quad k_1 + 2k_0 \leq 4, \quad r = 1, 2, \quad m_1 = m_1(\varepsilon, 1).
\]

**Theorem 9.1.** Let \( a, b, c, p, f \in C^{4+\alpha}(G^r) \), \( \varphi \in C^{4+\alpha}(S^r) \cap C^{4+\alpha}(S_0) \), \( \beta \in C^{3+\alpha/2}([0, T]) \), and also \( u \in C^{4+\alpha,2+\alpha/2}(G^r) \), \( \alpha > 0 \), \( r = 1, 2 \), and let condition \( (2.4) \) hold. Then the components of the solution to problem \( (2.2), (2.1) \) from the representation of \( (9.1) \) satisfy estimates \( (9.4), (9.5) \).
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References


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