COLLOCATION METHODS FOR PANTOGRAPH-TYPE
VOLterra FUNCTIONAL EQUATIONS WITH MULTIPLE
DELAYS

H. BRUNNER

Abstract — We analyze the optimal superconvergence properties of piecewise polynomial collocation solutions on uniform meshes for Volterra integral and integro-differential equations with multiple (vanishing) proportional delays $\theta_j(t) = q_j t$ ($0 < q_1 < \cdots < q_r < 1$). It is shown that for delay integro-differential equations the recently obtained optimal order is also attainable. For integral equations with multiple vanishing delays this is no longer true.

2000 Mathematics Subject Classification: 65R20, 34K06, 34K28.

Keywords: Volterra functional integral and integro-differential equations, multiple vanishing delays, solution representations, collocation solutions, uniform meshes, optimal order of superconvergence.

1. Introduction

Let $\theta_j(t) := q_j t$ ($j = 1, \ldots, r$; $r \geq 2$) be given delay functions with $0 < q_1 < \cdots < q_r < 1$ and assume that the underlying interval is $I := [0, T]$. We shall consider analytical and numerical aspects of functional differential and integral equations corresponding to linear delay Volterra integral operators $\mathcal{V}_{\theta_j} : C(I) \to C(I)$ defined by

$$\mathcal{V}_{\theta_j}u(t) := \int_0^{\theta_j(t)} K_j(t, s) u(s) \, ds, \quad t \in I \text{ (} j = 1, \ldots, r).$$

(1.1)

The kernels $K_j$ are assumed to satisfy $K_j \in C^d(D_{\theta_j})$ for some integer $d \geq 0$; here,

$$D_{\theta_j} := \{(t, s) : 0 \leq s \leq \theta_j(t), \ t \in I\}.$$

The focus of the paper is on the Volterra functional integro-differential equation (VFIDE)

$$u'(t) = \sum_{j=1}^r b_j(t) u(\theta_j(t)) + \sum_{j=1}^r (\mathcal{V}_{\theta_j}u)(t) + g(t), \quad t \in I,$$

(1.2)
with initial condition $u(0) = u_0$, and the Volterra functional integral equation (VFIE)

$$u(t) = g(t) + \sum_{j=1}^{r}(V_{\theta_j}u)(t), \quad t \in I,$$  \hspace{1cm} (1.3)

The VFIDE (1.2) contains as an important special case the multi-delay pantographic delay differential equation (DDE),

$$u'(t) = au(t) + \sum_{j=1}^{r} b_j u(q_j t)$$  \hspace{1cm} (1.4)

(see, e.g., [12, 13, 17, 19, 24]; compare also [10] for the higher-order version of (1.4) and applications).

As the results in Section 2 will reveal, multi-delay pantograph-type VFIEs and VFIDEs with smooth data and delays vanishing at $t = 0$ have solutions that are (globally) smooth on the given interval $[0, T]$.

The aim of this paper is to analyze the optimal order of (local) superconvergence of piecewise polynomial collocation solutions (and, in the case of the VFIE (1.2), the corresponding iterated collocations solution) when the underlying meshes are uniform. It will be shown that results analogous to the ones in [7] hold for the VFIDE (1.2). For the multi-delay VFIE (1.3) the optimal orders of local superconvergence at the mesh points ([6]) are no longer valid.

In the last ten years, theory and numerical analysis (analysis of asymptotic stability) of the multi-delay pantograph DDE (1.4) have received some attention. We mention in particular the 2004 paper [17] where the solution representation for (1.4) (using Dirichlet series) and the numerical solution by the $\theta$-method (on certain geometric meshes) are analysed in detail. The reader may also wish to consult the related papers [16, 19] and [24]. The only work studying the convergence of a numerical method for the VFIE (1.3) and the DDE (1.4) appears to be [2]; there, these functional equations are solved by the spectral method.

In this paper we present a complete analysis of the convergence properties of collocation solutions on uniform meshes for the VFIDE (1.2) and the VFIE (1.3). Section 2 is dedicated to a discussion of the representation and the regularity properties of solutions to (1.2) and (1.3). In Section 3 we describe the collocation equations for the VFIDE (1.2) and the VFIE (1.3) when the meshes used in the respective spaces of piecewise polynomials are uniform. The results of Section 2 are then used in Section 4 to establish the optimal global and local (super-) convergence estimates for the VFIDE (1.3) (and hence the multi-delay pantograph DDE (1.4)). The concluding Section 5 deals with an open problem related to the oldest multi-delay (first-kind) VFIE.

2. Representation of solutions

It was shown in [8] (see also the classical 1914 paper [1]) that for continuous $g$ and $K$ the single-delay VFIE

$$u(t) = g(t) + \int_{0}^{qt} K(t, s)u(s) \, ds, \quad t \in I, \quad 0 < q < 1.$$  \hspace{1cm} (2.1)
possesses a unique solution \( u \in C(I) \) given by

\[
    u(t) = g(t) + \sum_{\nu=1}^{\infty} \int_0^{q^\nu t} K^{(\nu)}(t, s)g(s) \, ds, \quad t \in I,
\]

where the iterated kernels \( K^{(\nu)} \) of \( K =: K^{(1)} \) are defined by

\[
    K^{(\nu+1)}(t, s) := \int_0^{q^\nu t} K(t, v)K^{(\nu)}(v, s) \, ds, \quad (t, s) \in D^{(\nu+1)}_\theta (\nu \geq 1).
\]

An expression analogous to (2.2) can be derived for more general delay functions \( \theta = \theta(t) \) satisfying \( \theta(0) = 0 \), \( \theta \) continuous and strictly increasing on \( I \), and \( \theta(t) \leq qt \), \( t \in I \), for some \( q \in (0, 1) \) (see [5]): the solution of the VFIE

\[
    u(t) = g(t) + \int_0^{\theta(t)} K(t, s)u(s) \, ds, \quad t \in I,
\]

has the form

\[
    u(t) = g(t) + \sum_{\nu=1}^{\infty} \int_0^{\theta^{(\nu)}(t)} K^{(\nu)}(t, s)g(s) \, ds, \quad t \in I,
\]

with iterated kernels \( K^{(1)}(t, s) := K(t, s) \) and

\[
    K^{(\nu+1)}(t, s) := \int_0^{\theta^{(\nu)}(t)} K(t, v)K^{(\nu)}(v, s) \, dv.
\]

Here, \( \theta^j \) denotes the \( j \)-fold composition of \( \theta \) with itself, and \( \theta^{-1} \) is the inverse function of \( \theta \).

Since the (single-delay) VFIDE

\[
    u'(t) = b(t)u(\theta(t)) + g(t) + \int_0^{\theta(t)} K(t, s)u(s) \, ds, \quad t \in I,
\]

with \( u(0) = u_0 \), is equivalent to the VFIE

\[
    u(t) = g_0(t) + \int_0^{\theta(t)} \tilde{K}(t, s)u(s) \, ds, \quad t \in I,
\]

where

\[
    g_0(t) := u_0 + \int_0^t g(s) \, ds
\]

and

\[
    \tilde{K}(t, s) := b(\theta^{-1}(s))\theta'(\theta^{-1}(s)) + \int_{\theta^{-1}(s)}^t K(v, s) \, dv.
\]
its solution possesses the representation

\[ u(t) = g_0(t) + \sum_{\nu=1}^{\infty} \theta^{(\nu)}(t) \int_0^t \tilde{H}^{(\nu)}(t, s) g_0(s) \, ds, \quad t \in I. \]  

(2.5)

The kernel functions \( \{ \tilde{H}^{(\nu)} \} \) are the iterated kernels of the kernel \( \tilde{K} \) of the above VFIE (cf. (2.4)).

We shall see in Sections 2.1 and 2.2 that analogous (though of course rather more complex) solution representations exist for the multi-delay VFIE (1.3) and the multi-delay VFIDE (1.2).

The multi-delay VFIE (1.3). The first aim of this paper is to generalise the solution representation (2.3) to the multi-delay VFIE (1.3). Since the VFIDE (1.2) is equivalent to a VFIE of the form (1.3), Theorem 2.1 will then yield the representation formula for the solution of (1.2).

**Theorem 2.1.** Assume:

(a) The given kernel functions in the VFIE (1.3) satisfy \( K_j \in C(D_{\theta_j}) \) \( (j = 1, \ldots, r) \).

(b) \( \theta_1, \ldots, \theta_r \) are continuous delay functions on \( I \) with the properties \( \theta_j(0) = 0 \), \( \theta_j \) is strictly increasing, and \( \theta_j(t) \leq q_j t \) \( (t \in I, \quad j = 1, \ldots, r) \)

with \( 0 < q_1 < \cdots < q_r < 1 \).

Then for each \( g \in C(I) \) the VFIE (1.3) possesses a unique solution \( u \in C(I) \). This solution has the representation

\[ u(t) = g(t) + \sum_{\nu=1}^{\infty} \sum_{j=1}^{r} \sum_{k_1, \ldots, k_{\nu-1}} \int_0^t H^{(\nu-1)}_{j,k_1,\ldots,k_{\nu-1}}(t, s) g(s) \, ds, \quad t \in I, \]  

(2.6)

where we have used the notational conventions \( \theta_0 := 1 \) and \( \sum_{k_{\nu}=1}^{r} \int_0^{(\theta_k \circ \theta_j)(t)} \int_0^{\theta_j(t)} f(s \circ \theta_j)(t) = \int_0^{\theta_j(t)} f(s \circ \theta_j)(t) \). The infinite series converges absolutely and uniformly on \( I \). The functions \( H^{(\nu)}_{j,k_1,\ldots,k_{\nu-1}} \) are the iterated kernels corresponding to the given kernels \( K_1, \ldots, K_r \) and arising in the Picard iteration for the VFIE (1.3).

**Proof.** We begin by stating a lemma that contains one of the key results needed for deriving the formulas on the representations of the solutions to the VFIE (1.2) and the VFIDE (1.3). It describes the result of the composition of Volterra delay integral operators with vanishing delays. \( \square \)

**Lemma 2.1.** Let the delay function \( \theta_j \) be as in Theorem 2.1, and assume that \( \theta \) is a delay function satisfying \( \theta(0) = 0 \), \( \theta \) is strictly increasing and continuous on \( I \), and \( \theta(t) \leq q t \) \((t \in I) \) for some \( q \in (0, 1) \). Then

\[ \int_0^t \int_0^s \phi(s, v) \, dv \, ds = \int_0^t \int_0^{\theta_j(t)} \phi(s, v) \, ds \, dv, \]

whenever the integrand \( \phi \) is continuous on the domain of integration.
The above identities follow from an elementary result (Dirichlet’s formula) of integral calculus.

We now apply Picard iteration to the VFIE (1.3):

\[ u_{\nu+1}(t) := g(t) + \sum_{j=1}^{r} (V_{q_j} u_{\nu})(t), \quad t \in I \ (\nu \geq 0), \]

with \( u_0(t) := g(t) \). Specifically, using the expression for \( u_1(t) \),

\[ u_1(t) = g(t) + \sum_{j=1}^{r} \theta_j(t) \int_{0}^{t} K_j(t, s) g(s) \, ds, \]

we obtain

\[ u_2(t) = u_1(t) + \sum_{j=1}^{r} \theta_j(t) \int_{0}^{t} K_j(t, s) \left( g(s) + \sum_{k_1=1}^{r} \theta_{k_1}(s) \int_{0}^{s} K_{k_1}(s, v) g(v) \, dv \right) ds. \]

It follows from Lemma 2.1 that

\[ \theta_j(t) \int_{0}^{t} K_j(t, s) K_{k_1}(s, v) g(v) \, dv =: \int_{0}^{t} H^{(2)}_{j, k_1}(t, s) g(s) \, ds. \]

Thus,

\[ u_2(t) = u_1(t) + \sum_{j=1}^{r} \sum_{k_1=1}^{r} (\theta_{k_1} \circ \theta_j)(t) \int_{0}^{t} H^{(2)}_{j, k_1}(t, s) g(s) \, ds. \]

An induction argument, together with repeated application of Lemma 2.1, lead to the expression

\[ u_\ell(t) = u_1(t) + \sum_{\nu=1}^{\ell-1} \sum_{j=1}^{r} \sum_{k_1, \ldots, k_{\nu-1}=1}^{r} (\theta_{k_{\nu-1}} \circ \cdots \circ \theta_{k_1} \circ \theta_j)(t) \int_{0}^{t} H^{(\nu)}_{j, k_1, \ldots, k_{\nu-1}}(t, s) g(s) \, ds \quad (\ell \geq 2), \]

with obvious meaning of the kernels \( H^{(\nu)}_{j, k_1, \ldots, k_{\nu-1}}(t, s) := K_j(t, s) \).

The uniform convergence of the infinite series in (2.6) can be established along the lines of the proof of Theorem 2.2 in [7]; it requires the derivation of uniform bounds for the kernels \( H^{(\nu)}_{j, k_1, \ldots, k_{\nu-1}}(t, s) \) in (2.4). Since we have \( \theta_j(t) \leq q_r t \ (j = 1, \ldots, r) \), with \( q_r < 1 \), and since – owing to the fact that the operators \( V_{q_j} \) are Volterra integral operators, it is not difficult (though somewhat ‘messy’) to verify that the uniform kernel estimates are of the form \(|H^{(\nu)}_{j, k_1, \ldots, k_{\nu-1}}(t, s)| \leq \text{const.} / \nu! \) (again, compare the proof of Theorem 2.2 in [7], especially the estimates (2.14), (2.15)). We omit the details.
The uniqueness of the solution can be established by a Gronwall-type argument. Let \( u \) and \( w \) be two continuous solutions of the VFIE (1.3). Setting \( z(t) := u(t) - w(t) \), we obtain

\[ |z(t)| \leq \sum_{j=1}^{r} \int_{0}^{\theta_j(t)} |K_j(t, s)||z(s)|\, ds \leq \bar{K}r \int_{0}^{t} |z(s)|\, ds, \quad t \in I, \]

with

\[ \bar{K} := \max_{(j)} \max \{|K_j(t, s)| : (t, s) \in D_{\theta_j}\}. \]

This implies, by continuity, that \( w(t) = u(t) - w(t) \equiv 0 \) on \( I \).

The following regularity result for (1.3) is an immediate consequence of the representation formula (2.6) and the regularity of the iterated kernels \( H_{j,k_1,\ldots,k_{\nu}}^{(v)} \).

**Corollary 2.1.** Assume that \( g \) and kernel functions \( K_j \) \((j = 1, \ldots, r)\) in the VFIE (1.3) satisfy \( g \in C^{d} \) and \( K_j \in C^{d}(D_{\theta_j}) \), respectively, for some \( d \geq 1 \). Then its (unique) solution \( u \) lies in \( C^{d}(I) \).

Next, we turn to the VFIDE (1.2). It can be rewritten as an equivalent VFIE,

\[ u(t) = g_0(t) + \sum_{j=1}^{r} \int_{0}^{\theta_j(t)} \tilde{K}_j(t, s)u(s)\, ds, \quad t \in I, \tag{2.7} \]

where \( g_0(t) := u_0 + \int_{0}^{t} g(s)\, ds \) and \( \tilde{K}_j(t, s) := q_j^{-1}b(q_j^{-1}s) + \int_{q_j^{-1}s}^{t} K_j(v, s)\, dv \). Thus, we can resort to Theorem 2.1 to derive the following theorem on the solution of the VFIDE (1.2)

**Theorem 2.2.** Assume that the given functions \( g, b_j \) and \( K_j \) \((j = 1, \ldots, r)\) in the VFIDE (1.2) are continuous on their respective domains \( I \) and \( D_{\theta_j} \). Then for every initial value \( u_0 \) the VFIDE possesses a unique solution \( u \in C^{1}(I) \). For \( r = 2 \), this solution has the representation

\[ u(t) = u_0 + \int_{0}^{t} g(s)\, ds + \sum_{\nu=1}^{\infty} \sum_{j=1}^{r} \sum_{k_1,\ldots,k_{\nu-1},k_{\nu}=1}^{r} \int_{0}^{t} \tilde{H}_{j,k_1,\ldots,k_{\nu-1}}^{(\nu)}(t, s)g_0(s)\, ds, \quad t \in I, \tag{2.8} \]

where the kernel functions \( \tilde{H}_{j,k_1,\ldots,k_{\nu-1}}^{(\nu)} \) are the analogues of \( H_{j,k_1,\ldots,k_{\nu-1}}^{(\nu-1)} \) in (2.4) but now correspond to the kernels \( \tilde{K}_j \) in (2.5), using the notational convention of Theorem 2.1. The infinite series converge uniformly on \( I \).

**Corollary 2.2.** If \( g, b_j \in C_{d}(I) \) and \( K_j \in C_{d}(D_{\theta_j}) \) \((j = 1, \ldots, r)\), then the solution of the initial-value problem for the VFIDE (1.2) has the regularity \( u \in C^{d+1}(I) \).

**Remark 2.1.** It can be shown that the regularity results of Corollaries 2.1 and 2.2 remain valid for VFIDEs and VFIEs of the more general forms

\[ u'(t) = \sum_{j=1}^{r} b_j(t)u(\theta_j(t)) + \sum_{j=1}^{r} (W_{\theta_j}u)(t) + g(t), \quad t \in I, \]

and

\[ u(t) = g(t) + \sum_{j=1}^{r} (W_{\theta_j}u)(t), \quad t \in I, \]
corresponding to the Volterra delay operators

\[(W_{\theta_j}u)(t) := \int_{\theta_j(t)}^{t} K_j(t, s)u(s) \, ds.\]

3. Collocation in piecewise polynomial spaces

Let \( I_h := \{ t_n : t_n = nh \ (n = 0, 1, \ldots, N; \ t_N = T \} \) be a uniform mesh for the interval \( I = [0, T] \), and set \( \varepsilon_n := (t_n, t_{n+1}) \ (0 \leq n \leq N - 1) \). For given \( I_h \) and given integer \( m \geq 1 \), the collocation approximations \( u_h \) to the VFIDE (1.2) and the VFIE (1.3) will be sought in the (real) piecewise polynomial spaces

\[ S_m^{(0)}(I_h) := \{ v \in C(I) : v|_{\varepsilon_n} \in \pi_m \ (0 \leq n \leq N - 1) \} \]  

(3.1)

(for (1.2) and (1.4)) and in

\[ S_m^{(-1)}(I_h) := \{ v : v|_{\varepsilon_n} \in \pi_{m-1} \ (0 \leq n \leq N - 1) \} \]  

(3.2)

(for (1.3)). Here, \( \pi_m \) stands for the space of (real) polynomials (on \( \varepsilon_n \)) of degree not exceeding \( m \). Since the dimensions of these two linear spaces are related by

\[ \dim S_m^{(0)}(I_h) = \dim S_m^{(-1)}(I_h) + 1, \]

and since the collocation solution for (1.2) will be required to assume the given initial value \( u_0 \), the set \( X_h \) of collocation points will be chosen to be the same for both VFEs, namely

\[ X_h := \{ t_n + c_i h : 0 < c_1 < \cdots < c_m \leq 1 \ (0 \leq n \leq N - 1) \}, \]  

(3.3)

where the \( \{c_i\} \) is a prescribed set of collocation parameters. Obviously, the cardinality of \( X_h \) is \(|X_h| = Nm|\).

In order to derive the computational form of the collocation equations for the VFIDE (1.2) and the VFIE (1.3), we shall have to distinguish between the following three phases that will be described in terms of the integers

\[ q_j^I := \lceil \frac{q_j}{1 - q_j} c_1 \rceil, \quad q_j^{II} := \lceil \frac{q_j}{1 - q_j} \rceil \quad (j = 1, \ldots, r). \]

and

\[ q^I := \min\{q_j^I : 1 \leq j \leq r\}, \quad q^{II} := \max\{q_j^{II} : 1 \leq j \leq r\}. \]
Phase I: \(0 \leq n < q^I\).
Here, 
\[
\theta_j(t_n + c_i h) \in (t_n, t_{n+1}) \quad \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, r,
\]
that is, \(q_{n,i}^{(j)} = n\) for all \(i\) and all \(j\) (complete overlap). This is always true when \(n = 0\), regardless of the values of \(q_j\).

Phase II: \(q^I \leq n < q^{II}\).
During this phase we encounter partial overlap: some of the images \(\theta_j(t_n + c_i h)\) still lie in \((t_n, t_{n+1})\), while for the others we have \(\theta_j(t_n + c_i h) \leq t_n\).

Phase III: \(q^{II} \leq n \leq N - 1\).
This is the pure delay phase:
\[
\theta_j(t_n + c_i h) \leq t_n \quad \text{for all } i = 1, \ldots, m; \ j = 1, \ldots, r,
\]
i.e., \(q_{n,i}^{(j)} < n\) for all \(i\) and all \(j\) (no overlap).

As \(n\) moves from \(n = q^I\) to \(n = q^{II}\) (partial overlap), the matrices governing the linear algebraic systems determining the collocation solutions for the VFIDE (1.2) and the VFIE (1.3) change significantly, in contrast to VFEs with non-vanishing delays. Some of the details will be given in the next section.

### 3.1. Collocation equations for the VFIDE (1.2).

The collocation approximation \(u_h \in S_m(0) (I_h)\) for the VFIDE (1.2) has the local representation (on \([t_n, t_{n+1}]\))
\[
u_h(t_n + vh) = u_h(t_n) + h \sum_{j=1}^{m} \beta_j(v)Y_{n,j}, \quad v \in [0, 1], \tag{3.5}
\]
where \(Y_{n,j} := u_h'(t_n + c_j h)\) and \(\beta_j(v) := \int_0^v L_j(s) \, ds\). Here,
\[
L_j(v) := \prod_{k \neq j}^{m} \frac{v - c_k}{c_j - c_k}
\]
denotes the \(j\)th (local) Lagrange canonical polynomial with respect to the collocation parameters \(\{c_i\}\).

The coefficient vectors \(Y_n := (Y_{n,1}, \ldots, Y_{n,m})^T \in \mathbb{R}^m\) in the local representation (3.5) of \(u_h\) will be determined by the collocation equations for (1.2),
\[
u_h'(t) = g(t) + \sum_{j=1}^{r} b_j(t)u_h(\theta_j(t)) + \sum_{j=1}^{r} (V_{\theta_j} u_h)(t), \quad t \in X_h, \tag{3.6}
\]
and \(u_h(0) = u_0\), with respect to the subintervals \(e_n\) \((0 \leq n \leq N - 1)\). To be more precise, let \(t = t_{n,i} := t_n + c_i h\) be a collocation point in the subinterval \((t_n, t_{n+1}]\). For \(0 \leq n < q^I\) (Phase I: \(d_{n,i}^{(j)} = n\) for \(1 \leq i \leq m\) and \(1 \leq j \leq r\)), it follows from the collocation equation (3.6) and the local representation (3.5) of \(u_h\) that the left-hand side of the linear algebraic system for the coefficient vector \(Y_n\) in (3.5) has the form
\[
A_n Y_n := \left( I_m - h \sum_{j=1}^{r} B_{n,j}^I - h^2 \sum_{j=1}^{r} X_{n,j}^I \right) Y_n \quad (0 \leq n < q^I), \tag{3.7}
\]
with $m$-by-$m$ matrices

$$\mathcal{B}_{n,j}^I := \text{diag} \{ b_j(t_{n,1}), \ldots, b_j(t_{n,m}) \} \begin{bmatrix} \beta_k(\gamma_{n,i}^{(j)}) \\ (i, k = 1, \ldots, m) \end{bmatrix}$$

and

$$\mathcal{C}_{n,j}^I := \begin{bmatrix} \gamma_{n,i}^{(j)} \\ \int_0^h K_j(t_{n,i} + sh) \beta_k(s) \, ds \\ (i, k = 1, \ldots, m) \end{bmatrix};$$

$I_m$ denotes the identity matrix. Since the given functions $b_j$ and $K_j$ are continuous, the matrices $\mathcal{A}^I_n$ in (3.7) are invertible for all sufficiently small $h > 0$, thus assuring the existence and uniqueness of the collocation solution $u_h \in S^0_m(I_h)$ for (1.2) for $0 \leq n < q^I$.

For $q^I \leq n < q^{II}$ (partial overlap phase), the left-hand side of the linear algebraic system for $Y_n$ has a form similar to (3.7),

$$\mathcal{A}^{II}_n Y_n := \left(I_m - h \sum_{j=1}^r \mathcal{B}^{II}_{n,j} - h^2 \sum_{j=1}^r \mathcal{C}^{II}_{n,j} \right) Y_n \quad (q^I \leq n < q^{II}),$$

but now the matrices $\mathcal{B}^{II}_{n,j}$ and $\mathcal{C}^{II}_{n,j}$ have no longer full ranks: their ranks decrease to one as $n$ approaches $q^{II}$. However, for all sufficiently small $h > 0$ each system defines a unique $Y_n$ and hence a unique collocation solution. Details are left to the reader.

In Phase 3 (no overlap), the matrix of the linear algebraic systems for $Y_n$ ($q^{II} \leq n \leq N - 1$) becomes $I_m$. Thus, for all uniform meshes $I_h$ with sufficiently small $h > 0$, the collocation equation (3.6), together with (3.5), defines a unique collocation solution $u_h \in S^0_m(I_h)$ for the VFIDE (1.2) with $u_h(0) = u_0$.

### 3.2. Collocation and iterated collocation for the VFIE (1.3)

On the subinterval $e_n = (t_n, t_{n+1}]$ the collocation solution $u_h \in S^{(1)}_{m-1}(I_h)$ for the VFIE (1.3) has the representation

$$u_h = \sum_{j=1}^m L_j(v) U_{n,j}, \quad v \in (0, 1]; \quad U_{n,j} := u_h(t_n + c_j h). \quad (3.8)$$

In analogy to (3.6), the collocation equation for (1.3),

$$u_h(t) = g(t) + \sum_{j=1}^r (V_{\theta_j} u)(t), \quad t \in X_h,$$  \quad (3.9)

will determine the coefficient vectors $U_n := (U_{n,1}, \ldots, U_{n,m})^T \in \mathbb{R}^m$.

Once we know $u_h$ we can also define the iterated collocation solution $u_h^I$: it is given by

$$u_h^I(t) := g(t) + \sum_{j=1}^r (V_{\theta_j} u_h)(t), \quad t \in I.$$

(We observe that in practice one will usually compute $u_h^I$ only at special values of $t \in I$, for example, at $t = T = t_N$, or at mesh points $t = t_n$; see also Theorem 4.3 (c) below.)

The structure of the linear algebraic systems for the coefficient vectors $U_n := (U_{n,1}, \ldots, U_{n,m})^T \in \mathbb{R}^m$ in the local representations (3.8) possess a structure similar to the ones for $Y_n$.}

Unauthenticated
for \( Y_n \) in Section 3.1, as \( n \) moves from \( n = 0 \) through \( n = q^I \) and \( n = q^{II} \) to the final value \( n = N - 1 \). For all sufficiently small \( h > 0 \), each of these systems has a unique solution \( U_n \), thus assuring the existence of a unique collocation solution \( u_h \in S_{m-1}(I_h) \) for the VFIE (1.3). We leave the details to the reader.

4. Global and local superconvergence results for uniform meshes

We first recall two recently obtained results ([6, 7]) on the attainable order of (global and local) superconvergence of collocation solutions (on uniform meshes) for the single-delay VFIDE

\[
 u'(t) = b(t)u(\theta(t)) + g(t) + \int_0^{\theta(t)} K(t, s)u(s) \, ds, \quad t \in I, \tag{4.1}
\]

with \( \theta(t) = qt \) \((0 < q < 1)\) and the single-delay VFIE (2.1).

**Theorem 4.1.** Suppose that \( u_h \in S_m^0(I_h) \) is the collocation solution to the initial-value problem for the VFIDE (4.1), where the underlying mesh \( I_h \) is uniform. Let the collocation parameters \( \{c_i\} \) satisfy the orthogonality condition

\[
 J_\nu := \int_0^1 s^{\nu} \prod_{i=1}^m (s - c_i) \, ds = 0, \quad \nu = 0, \ldots, \kappa - 1, \tag{4.2}
\]

for some \( \kappa \) with \( 1 \leq \kappa \leq m \) (e.g., the \( \{c_i\} \) are the \( m \) Gauss points in \((0, 1)\), which correspond to \( \kappa = m \)). Then for all sufficiently smooth data \( g, b \) and \( K \), and for any \( q \in (0, 1) \) the resulting collocation error satisfies

\[
 \|u - u_h\|_\infty \leq C_m(q)h^{m+1},
\]

and

\[
 \max_{1 \leq n \leq N} |u(t_n) - u_h(t_n)| \leq C_m^*(q)h^{p^*},
\]

with

\[
 p^* = \begin{cases} 
 2m & \text{if } m = 1, 2, \\
 m + 2 & \text{if } m \geq 3.
\end{cases}
\]

The constants \( C_m(q) \) and \( C_m^*(q) \) do not depend on \( h \). The above local superconvergence orders \( p^* \) are best possible for nontrivial solutions.

For the single-delay VFIE (2.1) the local superconvergence order \( m+2 \) cannot be attained for all \( q \in (0, 1) \) ([6]).

**Theorem 4.2.** Assume that the given functions \( g \) and \( K \) in the VFIE (2.1) are arbitrarily smooth, and let \( u_h \in S_{m-1}^{(-1)}(I_h) \) be the collocation solution corresponding to collocation parameters \( \{c_i\} \) in (3.3) satisfying (4.2) (e.g., the \( m \) Gauss points), and with uniform mesh \( I_h \). Then the resulting iterated collocation solution defined by (2.11) exhibits the global order described by

\[
 \|u - u_h^{(I)}\|_\infty \leq C_m(q)h^{m+1},
\]

while its optimal local order depends on the value of \( q \):

\[
 \max\{|u(t) - u_h^{(I)}(t)| : t \in I_h \setminus \{0\}| \leq C_m^*(q)h^{p^*}
\]

with

\[
 p^* = \begin{cases} 
 2m & \text{if } m = 1, 2, \\
 m + 2 & \text{if } m \geq 3.
\end{cases}
\]
with
\[
p^* = \begin{cases} 
2m & \text{if } m = 1, 2 \\
m + 2 & \text{if } q = 1/2 \text{ and } m \text{ even,} \\
m + 1 & \text{otherwise.}
\end{cases}
\]

The constants $C_m(q)$ and $C^*_m(q)$ do not depend on $h$.

**Remark 4.1.** The optimal global and local superconvergence orders in Theorems 4.1 and 4.2 remain true if the general orthogonality condition (4.2) is replaced by

\[
J_0 := \int_0^1 \prod_{i=1}^m (s - c_i) \, ds = 0;
\]

that is, for any $\{c_i\}$ for which the associated $m$-point interpolatory quadrature formula possesses the degree of precision $m$.

**4.1. Optimal error estimates for the multi-delay VFIDE (1.2).** It is clear that the optimal orders of (global and local) convergence of the collocation solution $u_h \in S^{(0)}_m(I_h)$ for the multi-delay VFIDE (1.2) cannot exceed the ones of Theorem 4.1 for a single delay. However, as the following theorem shows, the results of Theorem 4.1 remain true when $r \geq 2$.

**Theorem 4.3.** Let $u_h \in S^{(0)}_m(I_h)$ be the collocation solution to the VFIDE (1.2) determined by (3.5) and (3.6). Assume that the given functions $g$, $b_j$ and $K_j$ ($j = 1, \ldots, m$) satisfy $g, b_j \in C^d(I)$ and $K_j \in C^d(D_{\theta_j})$, respectively.

(a) If $d \geq m$, then for all sufficiently small $h > 0$ and arbitrary $\{c_i\}$ the estimates

\[
\|u^{(k)} - u_h^{(k)}\|_\infty \leq C_m(q)h^m \quad (k = 0, 1)
\]

($q := (q_1, \ldots, q_r)$) hold.

(b) If the collocation parameters satisfy (4.2), then

\[
\|u - u_h\|_\infty \leq C_m(q)h^{m+1},
\]

whenever $d \geq m + 1$.

(c) Let

\[
J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i) \, ds = 0 \quad (\nu = 0, \ldots, \kappa - 1)
\]

hold for some integer $\kappa$ with $1 \leq \kappa \leq m$. If $d \geq m + \kappa + 1$, then we have the optimal local superconvergence estimate

\[
\max\{|u(t) - u_h(t)| : t \in I_h\} \leq C^*_m(q)h^{m+2};
\]

this holds for any $q$ with $0 < q_1 < \cdots < q_r < 1$.

**Proof.** In order to familiarise the reader with the key ideas underlying the proof of the optimal global and local superconvergence estimates of the collocation solution $u_h \in S^{(0)}_m(I_h)$ (with uniform mesh $I_h$) for multi-delay VFIDEs, we briefly recall the proof of the optimal local superconvergence result in Theorem 4.1 for single-delay VFIDEs ($r = 1$ in (1.2); cf. [7]).
The starting point is the equation for the collocation error $e_h := u - u_h$ corresponding to (4.1),

$$e_h'(t) = b(t)e_h(\theta(t)) + \delta_h(t) + \int_0^{\theta(t)} K(t, s)e_h(s) \, ds, \quad t \in I,$$

with $e_h(0) = 0$. By (2.5) its solution has the representation

$$e_h(t) = e_h(0) + \int_0^t \delta_h(s) \, ds + \sum_{\nu=1}^{\infty} \theta^{\nu}(t) \left( \int_0^s \delta_h(v) \, dv \right) \, ds.$$  

For a given mesh point $t = t_n$ ($n = 1, \ldots, N$) we set, in analogy to (3.4),

$$q_{k,n} := [q^k n], \quad \gamma_{k,n} := q^k n - q_k n, \quad k^*_n(q) := \max\{k : q_k n \geq 1\}. \quad (4.4)$$

Thus,

$$e_h(t_n) = \int_0^{t_n} \delta_h(s) \, ds + S^I_n + S^{II}_n, \quad (4.5)$$

where

$$S^I_n := \sum_{k=1}^{k^*_n(q)} \sum_{\ell=0}^{q_k n - 1} \tilde{H}^{(k)}(t_n, t_\ell + sh) \left( h \sum_{\nu=0}^{\ell-1} \delta_h(t_\nu + vh) + h \int_0^s \delta_h(t_\ell + vh) \, dv \right) \, ds.$$  

and

$$S^{II}_n := h \sum_{k=1}^{\infty} \sum_{\ell=0}^{\gamma_{k,n}} \tilde{H}^{(k)}(t_n, t_{q_k n} + sh) \left( \int_0^{t_{q_k n} + sh} \delta_h(v) \, dv \right) \, ds =$$

$$h \sum_{k=1}^{\infty} \sum_{\ell=0}^{\gamma_{k,n}} \tilde{H}^{(k)}(t_n, t_{q_k n} + sh) \left( h \sum_{\nu=0}^{q_{k,n} - 1} \delta_h(t_\ell + vh) + h \int_0^s \delta_h(t_{q_k n} + vh) \, dv \right) \, ds. \quad (4.7)$$

In order to estimate $|S^I_n|$ and $|S^{II}_n|$, we first observe that in the upper bound for $|S^I_n|$ we have

$$h \sum_{\nu=0}^{\ell-1} \left| \int_0^1 \delta_h(t_\nu + vh) \, dv \right| + h \left| \int_0^s \delta_h(t_\ell + vh) \, dv \right| \leq TQ_m h^{m+k} + h \cdot D_0 h^{m-1} =: \tilde{D}_0 h^{m+1}.$$  

Hence, it is easy to verify that

$$|S^I_n| \leq h \tilde{D}_0 h^{m+1} \sum_{k=1}^{k^*_n(q)} \sum_{\ell=0}^{q_k n - 1} \int_0^1 \tilde{H}^{(k)}(t_n, t_\ell + sh) \, ds \leq$$

$$\tilde{D}_0 h^{m+2} \sum_{k=1}^{k^*_n(q)} \frac{\beta + \tilde{K}_1 T^k k^{k-1}}{(k-1)!} q^{(k-1)(3k-4)/2} =: C_0 h^{m+2}, \quad (4.8)$$
where $\beta$ and $\bar{K}$ denote, respectively, upper bounds for $b(t)$ and $K(t, s)$ in (4.1) (compare also [7] for additional technical details).

Similarly, we can derive the estimate

$$|S_n^{II}| \leq C_1 h^{m+2} \quad (1 \leq n \leq N).$$

These estimates now readily yield the desired optimal local superconvergence result on $I_h$,

$$|e_h(t_n)| \leq (C_0 + C_1) h^{m+2} =: C^* h^{m+2}, \quad n = 1, \ldots, N;$$

as the above analysis shows, the power $h^{m+2}$ cannot be replaced by $h^{m+3}$, except in trivial cases.

Returning to the multi-delay VFIDE (1.2), the collocation error $e_h := u - u_h$ induced by the collocation solution $u_h \in S_n^{(0)}(I_h)$ is the solution of the initial-value problem

$$e_h(t) = d_0(t) + \sum_{j=1}^{r} \int_0^t \tilde{K}_j(t, s)e_h(s) \, ds, \quad t \in I,$$

with $e_h(0) = 0$ and with the defect function $\delta_h$ vanishing on the set $X_h$ of collocation points. Hence, the error equation (4.4) can be transformed into the equivalent VFIE

$$e_h(t) = d_0(t) + \sum_{j=1}^{r} \int_0^t \tilde{K}_j(t, s)e_h(s) \, ds, \quad t \in I,$$

where $d_0(t) := e_h(0) + \int_0^t \delta_h(s) \, ds$ and $\tilde{K}_j(t, s) := q_j^{-1} b(q_j^{-1} s) + \int_{q_j^{-1} s}^t K_j(v, s) \, dv$.

Thus, recalling Theorem 2.2 and observing that $e_h(0) = 0$, the collocation error can be expressed in the form

$$e_h(t) = \int_0^t \delta_h(s) \, ds + \sum_{\nu=1}^{\infty} \sum_{j=1}^{r} \sum_{k_{\nu-1}, \ldots, k_{\nu-1} = 1} (\theta_{k_{\nu-1}} \circ \cdots \circ \theta_{k_{\nu}})(t) \int_0^t \tilde{H}_{j, k_{\nu-1}, \ldots, k_{\nu-1}}(t, s) e_h(s) \, ds, \quad t \in I,$$

where $(\theta_{k_{\nu-1}} \circ \cdots \circ \theta_{k_{1}} \circ \theta_{j})(t) = q_{k_{1}} \cdots q_{k_{\nu}} q_j t$. In analogy to the techniques for VFIDEs with a single delay ($r = 1$) described in the proof of Theorem 4.3, let $t = t_n = nh$ ($n = 1, \ldots, N$) in (4.11) and consider a typical integral with upper limit of integration

$$(\theta_{k_{\nu-1}} \circ \cdots \circ \theta_{k_{1}} \circ \theta_{j})(t_n) = q_{k_{\nu-1}} \cdots q_{k_{1}} q_j t_n =: t_{\tilde{q}_{\nu-1}, j} + \tilde{\gamma}_{n, j} h,$$

where we have set

$$\tilde{q}_{\nu-1, j} := [q_{k_{\nu-1}} \cdots q_{k_{1}} q_j n] \quad \text{and} \quad \tilde{\gamma}_{n, j} := q_{k_{\nu-1}} \cdots q_{k_{1}} q_j n - \tilde{q}_{\nu, j} \in [0, 1].$$

Hence, the integral terms in (4.11) can be written as the sums

$$\int_{0}^{t_{\tilde{q}_{\nu, j}}} \cdots ds + h \int_{0}^{\tilde{\gamma}_{n, j}} \cdots ds.$$

The proof of the optimal local superconvergence order in Theorem 4.3 is now completed by adapting the techniques for the case $r = 1$ that we described at the beginning of the proof.
Corollary 4.1. Let $u_h \in S_{m}^{(0)}(I_h)$ be the collocation solution, with respect to a uniform mesh $I_h$ and collocation parameters $\{c_i\}$ satisfying the orthogonality conditions (4.3) for the multi-delay pantograph DDE (1.4) ($r \geq 2$). Then the local superconvergence estimate
\[
\max\{|u(t) - u_h(t)| : t \in I_h\} \leq C_m^r(q)h^{m+2}
\]
is optimal.

Remark 4.2. The local superconvergence estimate in (c) is in particular true for the Gauss points $\{c_i\}$ (that is, the zeros of the shifted Legendre polynomial $P_m(2s-1)$). The order $p^* := m + 2$ is optimal for nontrivial data.

Remark 4.3. For $r = 1$ the above optimal order estimates were established in [7].

4.2. Optimal error estimates for the multi-delay VFIE (1.3). As we have already pointed out, the optimal local $O(h^{m+2})$-estimates of Theorem 4.2 ($r = 1$) will no longer be true for the VFIE (1.3) with $r \geq 2$ vanishing delays. The proof is obvious.

Theorem 4.4. Let the given functions $g$ and $K_j$ ($j = 1, \ldots, m$) in the VFIE (1.3) satisfy
\[
g \in C^d(I) \quad \text{and} \quad K_j \in C^d(D_{\theta_j}),
\]
respectively, for some integer $d$. Then for sufficiently small $h > 0$ the collocation solution $u_h \in S_{m-1}^{(1-1)}(I_h)$ and the associated iterated collocation solution $u^{\text{it}}_h$ for the VFIE (defined in (3.9), (2.**) exhibit the following (optimal) convergence orders:

(a) For arbitrary $\{c_i\}$ and $d \geq m$,
\[
\|u - u_h\|_\infty \leq C_m(q)h^m \quad \text{and} \quad \|u - u^{\text{it}}_h\|_\infty \leq C_m(q)h^m.
\]

(b) If $J_0 = 0$ (cf (4.2) in Theorem 4.1) and $d \geq m + 1$,
\[
\|u - u^{\text{it}}_h\|_\infty \leq C_m(q)h^{m+1}.
\]

(c) Assume that (4.3) holds. Then for any $d \geq m + \kappa 1$, the local estimate
\[
\max\{|u(t) - u^{\text{it}}_h(t)| : t \in I_h \setminus \{0\}\} \leq C_m(q)h^{m+1}
\]
is optimal: the local order $p^* = m + 2$ cannot be attained when $r \geq 2$, in contrast to the case $r = 1$ (single vanishing delay (cf. Theorem 4.2) and [6]).

Remark 4.4. For the more general (ill-posed) VFIE
\[
u(t) = g(t) + \sum_{j=1}^r b_j(t)u(\theta_j(t)) + \sum_{j=1}^r (V_{\theta_j}u)(t), \quad t \in I,
\]
the (global and local) convergence analyses for collocation and iterated collocation solutions remains to be established, also for the special case when $V_{\theta_j} = 0$, and even when $r = 1$. 

Unauthenticated
5. Concluding remark

The first multi-delay VFIE occurring in the literature is an equation of the first kind,

\[ \int_{q_1t}^{q_2t} K(t,s)u(s) \, ds = g(t), \quad t \in I := [0, T], \quad 0 < q_1 < q_2 < 1. \] (5.1)

It was studied in detail by Lalesco ([14, 15]); see also Volterra [23] (pp. 95-97). If \( K \) and \( g \) are smooth and satisfy \( |K(t,s)| \geq \kappa_0 > 0 \) and \( g(0) = 0 \), then (5.1) is equivalent to the VFIE

\[ q_2 K(t,q_2t)u(q_2t) - q_1 K(t,q_1t)u(q_1t) + \int_{q_1t}^{q_2t} \frac{\partial K(t,s)}{\partial t} u(s) \, ds = g'(t), \quad t \in I. \] (5.2)

Observe that this VFIE is closely related to (4.12) with \( r = 1 \).

Although the theory of (5.1) and (5.2) is well understood (compare also [9]), the numerical analysis of these VFIEs is completely open.

**Acknowledgements.** The research was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC Discovery Grant No. 9406). The author also gratefully acknowledges the hospitality extended to him by Professor Houde Han and the Department of Mathematical Sciences during a recent visit to Tsinghua University (Beijing) where a substantial part of this work was carried out.

**References**

8. Ll. G. Chambers, *Some properties of the functional equation \( \phi(x) = f(x) + \int_0^x g(x,y,\phi(y)) \, dy \)*, Internat. J. Math. Math. Sci., **14** (1990), pp. 27–44.


