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Some applications of the theory of harmonic integrals

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Abstract: In this survey, we present recent techniques on the theory of harmonic integrals to study the cohomology groups of the adjoint bundle with the multiplier ideal sheaf of singular metrics. As an application, we give an analytic version of the injectivity theorem.

Keywords: Injectivity theorems; Singular metrics; Multiplier ideal sheaves; The theory of harmonic integrals; L^2-methods

MSC: 14F18; 32L10; 32L20

1 Introduction

In this survey, we mainly study higher cohomology groups, which naturally appear as obstructions when we approach certain fundamental problems in complex geometry. For example, when we consider the problem of extending (holomorphic) sections of a (holomorphic) line bundle F from a subvariety $S \subset X$ to the ambient complex manifold X, the long exact sequence induced by the following sequence

$$0 \to F \otimes \mathcal{I}_S \to F \to F|_S \to 0$$

tells us that every section of $F|_S$ on S can be extended to a section of F on X if the first cohomology group $H^1(X, F \otimes \mathcal{I}_S)$ vanishes. The same conclusion holds under the weaker assumption that the induced map $H^1(X, F \otimes \mathcal{I}_S) \to H^1(X, F)$ is injective. Therefore it is important to find good conditions that imply the vanishing or the injectivity of higher cohomology groups. The Kodaira vanishing theorem is one of the most celebrated vanishing theorems:

Theorem 1.1 (The Kodaira vanishing theorem). *Let F be a positive line bundle on a smooth projective variety X. Then*

$$H^q(X, K_X \otimes F) = 0 \quad \textit{for any } q > 0.$$

Here K_X denotes the canonical bundle of X.

The following theorem is the so-called injectivity theorem, which can be seen as a generalization of the above theorem to semi-positive line bundles.

Theorem 1.2 ([10] (resp. [5])). *Let F be a semi-ample* (*resp. semi-positive*) *line bundle on a smooth projective variety* (*resp. a compact Kähler manifold*) *X. Then for a* (*non-zero*) *section s of a positive multiple F^m of the line bundle F, the multiplication map induced by the tensor product with s*

$$\Phi_s : H^q(X, K_X \otimes F) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1})$$

is injective for any q.

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In the proof of Kodaira's vanishing theorem and Enoki's injectivity theorem, the theory of harmonic integrals plays a crucial role, which enables us to study the vanishing theorem and the injectivity theorem from the viewpoint of complex differential geometry.

The purpose of this survey is to present recent techniques on the theory of harmonic integrals to study the cohomology group $H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$, where $\mathcal{I}(h)$ is the multiplier ideal sheaf of a singular (hermitian) metric h on F. As an application, we generalize Enoki's injectivity theorem to line bundles equipped with singular metrics (Theorem 1.3). A line bundle is said to be *pseudo-effective* if it admits a singular metric with semi-positive curvature. Therefore Theorem 1.3 can be seen as an injectivity theorem for pseudo-effective line bundles.

Theorem 1.3 ([11, Theorem 1.3]). *Let F be a line bundle on a compact Kähler manifold X and h be a singular metric with semi-positive curvature on F. Then for a (non-zero) section s of a positive multiple F^m satisfying $\sup_X |s|_{h^m} < \infty$, the multiplication map*

$$\Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$$

is (well-defined and) injective for any q. Here $\mathcal{I}(h)$ denotes the multiplier ideal sheaf associated to the singular metric h.

The multiplication map is well-defined thanks to the assumption of $\sup_X |s|_{h^m} < \infty$. When h is a metric with minimal singularities on F, this assumption is automatically satisfied for any section s of F^m (see [1] for metrics with minimal singularities). Metrics with minimal singularities or singular metrics obtained from a suitable limit of a family of metrics play an important role when we study algebraic geometry by using transcendental methods, however they do not always have algebraic singularities. We can apply the above theorem to such singular metrics since we do not assume that h has algebraic singularities, which is one of our advantages.

When we consider the problem of extending sections from subvarieties to the ambient space, we need to refine the above formulation (see [9, Theorem 1.3]). Our injectivity theorem can be seen as an improvement of [5], [7], [10], [12], [16]. For the proof, we take an analytic approach for the cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{I}(h)$, which includes techniques of [5], [7], [12], [13], [15]. The proof is based on a technical combination of the L^2-method for the $\overline{\partial}$-equation and the theory of harmonic integrals. To handle transcendental (non-algebraic) singularities, after regularizing a given singular metric, we investigate the asymptotic behavior of the harmonic forms with respect to a family of the regularized metrics. See [12] and [9] for more details.

This survey is organized as follows: In Section 2, we give techniques on the theory of harmonic integrals to study $H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$ by using harmonic differential forms. In Section 3, after we give a proof of the special case of Theorem 1.3, we discuss Theorem 1.3 and its generalization.

2 The Theory of Harmonic Integrals

In this section, recent techniques on the theory of harmonic integrals are given, whose purpose is to study $H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$ by using harmonic differential forms. By Proposition 2.5, we know that, if a singular metric h is a smooth on a Zariski open set Y, cohomology classes can be represented by harmonic forms on Y (not X). For this reason, we need to formulate the theory of harmonic integrals on non-compact manifolds.

2.1 Harmonic differential forms in L^2-spaces

In this subsection, we recall basic facts on the theory of harmonic integrals. Let Y be a (not necessarily compact) complex manifold with a hermitian form $\widetilde{\omega}$ and F be a line bundle on Y with a smooth (hermitian) metric h. For F-valued (p, q)-forms u and v, the point-wise inner product $\langle u, v\rangle_{h,\widetilde{\omega}}$ can be defined, and the (global) inner product $\langle\!\langle u, v\rangle\!\rangle_{h,\widetilde{\omega}}$ can also be defined by

$$\langle\!\langle u, v\rangle\!\rangle_{h,\widetilde{\omega}} := \int_Y \langle u, v\rangle_{h,\widetilde{\omega}} \frac{\widetilde{\omega}^n}{n!}.$$

Denote by $L^{p,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$ the L^2-space of F-valued (p, q)-forms on Y, namely

$$L^{p,q}_{(2)}(Y, F)_{h,\widetilde{\omega}} := \{u \mid u \text{ is an } F\text{-valued } (p, q)\text{-form on } Y \text{ with } \|u\|_{h,\widetilde{\omega}} < \infty.\}.$$

The Chern connection D_h of F is defined by the holomorphic structure and the hermitian metric h of F, and further D_h can be written as $D_h = D'_h + D''_h$ with the $(1, 0)$-connection D'_h and the $(0, 1)$-connection D''_h. Remark that $D''_h = \overline{\partial}$ by the definition. We consider the maximal Hilbert extension of the connections D'_h and D''_h, which we denote by the same notation. For example, D'_h is densely defined closed operator on $L^{p,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$, whose domain is the following subspace

$$\operatorname{Dom} D'_h := \{u \in L^{p,q}_{(2)}(Y, F)_{h,\widetilde{\omega}} \mid D'_h u \in L^{p+1,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}.\}.$$

Here $D'_h u \in L^{p+1,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$ means that there exists $v \in L^{p+1,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$ such that $D'_h u = v$ in the sense of distributions. The Hilbert adjoint operators D'^{*}_h and D''^{*}_h in the sense of von Neumann can be defined as follows: For every u in $\operatorname{Dom} D'^{*}_h$ defined by

$$\operatorname{Dom} D'^{*}_h := \{u \in L^{p,q}_{(2)}(Y, F)_{h,\widetilde{\omega}} \mid \operatorname{Dom} D'_h \ni v \mapsto \langle\!\langle u, D'_h v\rangle\!\rangle_{h,\widetilde{\omega}} \text{ is bounded.}\},$$

there uniquely exists $w \in L^{p-1,q}_{(2)}(Y, F)$ such that $\langle\!\langle w, v\rangle\!\rangle = \langle\!\langle u, D'_h v\rangle\!\rangle$ for any $v \in \operatorname{Dom} D'_h$. Then we put $D'^{*}_h u := w$. Now we have the following orthogonal decomposition:

$$L^{p,q}_{(2)}(Y, F)_{h,\widetilde{\omega}} = \overline{\operatorname{Im} \overline{\partial}} \oplus \mathcal{H}^{p,q}_{h,\widetilde{\omega}}(Y, F) \oplus \overline{\operatorname{Im} D''^{*}_h}.$$

Here $\mathcal{H}^{p,q}_{h,\widetilde{\omega}}(Y, F)$ is the space of harmonic forms defined by

$$\mathcal{H}^{p,q}_{h,\widetilde{\omega}}(Y, F) := \{u \mid u \text{ is an } F\text{-valued } (p, q)\text{-form such that } \overline{\partial} u = D''^{*}_h u = 0.\}.$$

In some cases, we can prove that the subspaces $\operatorname{Im} \overline{\partial}$ and $\operatorname{Im} D''^{*}_h$ are closed, and then we have

$$\operatorname{Ker}\overline{\partial}/\operatorname{Im} \overline{\partial} \cong \mathcal{H}^{p,q}_{h,\widetilde{\omega}}(Y, F).$$

In particular, equivalence classes in the left hand side can be represented by the associated harmonic forms. For example, when Y is a compact complex manifold, we have

$$H^q(Y, \Omega^p_Y \otimes F) \cong \operatorname{Ker}\overline{\partial}/\operatorname{Im} \overline{\partial} \cong \mathcal{H}^{p,q}_{h,\widetilde{\omega}}(Y, F).$$

Therefore given cohomology classes can be represented by the harmonic forms. Under suitable assumptions, such an isomorphism can be proved for $H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$. See Proposition 2.5.

2.2 Bochner-Nakano-Kodaira's identity

The Kodaira vanishing theorem and Enoki's injectivity theorem are derived from Bochner-Nakano-Kodaira's identity. In this subsection, we formulate some results obtained from Bochner-Nakano-Kodaira's identity, which can be applied in the proof of our injectivity theorem.

If $\widetilde{\omega}$ is a *complete* form on Y, the Hilbert adjoints D'^{*}_h and D''^{*}_h coincide with the maximal Hilbert extensions of the formal adjoints (see [2, (3,2) Theorem in Chapter 8]). The following proposition follows from Bochner-Nakano-Kodaira's identity and the density lemma (see [2, ChapterVIII], [3, Lemma 4.3]).

Proposition 2.1. *Assume that $\widetilde{\omega}$ is a complete Kähler form on Y and $\sqrt{-1}\Theta_h(F) \geq -C\widetilde{\omega}$ for some positive constant $C > 0$. Then for every $u \in L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$ satisfying $u \in \mathrm{Dom}\, D''^{\star}_h \cap \mathrm{Dom}\, \overline{\partial}$, the following equality holds:*

$$\|D''^{\star}_h u\|^2_{h,\widetilde{\omega}} + \|\overline{\partial} u\|^2_{h,\widetilde{\omega}} = \|D'^{\star}_h u\|^2_{h,\widetilde{\omega}} + \langle\!\langle \sqrt{-1}\Theta_h(F)\Lambda_{\widetilde{\omega}} u, u\rangle\!\rangle_{h,\widetilde{\omega}}.$$

Here n denotes the dimension of Y and $\Lambda_{\widetilde{\omega}}$ denotes the adjoint operator of the wedge product $\widetilde{\omega} \wedge \cdot$.

Proof. This proposition can be obtained from Bochner-Nakano-Kodaira's identity and the density lemma. Since $\widetilde{\omega}$ is a Kähler form, we have Bochner-Nakano-Kodaira's identity:

$$\Delta'' = \Delta' + [\sqrt{-1}\Theta_h(F), \Lambda_{\widetilde{\omega}}].$$

Here Δ' (resp. Δ'') is the Laplacian operator defined by $\Delta' := D'_h D'^{\star}_h + D'^{\star}_h D'_h$ (resp. $\Delta'' := D''_h D''^{\star}_h + D''^{\star}_h D''_h$) and $[\cdot\, , \cdot]$ is the graded Lie bracket. Therefore the equality in the proposition holds if u is smooth and compactly supported.

Since $\widetilde{\omega}$ is complete, we can take a family of cut-off functions $\{\psi_\ell\}_{\ell=1}^{\infty}$ with $|d\psi_\ell|_{\widetilde{\omega}} \leq 1$. For a given u satisfying $u \in \mathrm{Dom}\, D''^{\star}_h \cap \mathrm{Dom}\, \overline{\partial}$, by putting $u_\ell := u\psi_\ell$ and by considering convolution with regularizing kernels ρ_ε, we can obtain $u_{\ell,\varepsilon} := u_\ell \star \rho_\varepsilon$ satisfying the following properties:

- $u_{\ell,\varepsilon}$ is smooth and compactly supported.
- $u_{\ell,\varepsilon}$ converges to u in $L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$.
- $D''^{\star}_h u_{\ell,\varepsilon}$ (resp. $\overline{\partial} u_{\ell,\varepsilon}$) converges to $D''^{\star}_h u$ (resp. $\overline{\partial} u$) in $L^{n,\bullet}_{(2)}(Y, F)_{h,\widetilde{\omega}}$.

The third property comes from $|d\psi_\ell|_{\widetilde{\omega}} \leq 1$ (completeness of $\widetilde{\omega}$). As ε goes to zero, we have the equality in the proposition for u_ℓ. Further by the assumption of $\sqrt{-1}\Theta(F) \geq -C\widetilde{\omega}$, the second term of the right hand side

$$\langle\!\langle \sqrt{-1}\Theta_h(F)\Lambda_{\widetilde{\omega}} u_\ell, u_\ell\rangle\!\rangle_{h,\widetilde{\omega}} = \int_Y \langle \sqrt{-1}\Theta_h(F)\Lambda_{\widetilde{\omega}} u_\ell, u_\ell\rangle_{h,\widetilde{\omega}} \frac{\widetilde{\omega}^n}{n!}.$$

is bounded below. Therefore we obtain the conclusion by Lebesgue's monotone convergence theorem. □

The following proposition is also used in the proof of Theorem 1.3.

Proposition 2.2. *Let g be a smooth metric on a line bundle G and s be a section with $\sup_Y |s|_g < \infty$. Assume that $\widetilde{\omega}$ is a complete Kähler form on Y and $\sqrt{-1}\Theta_h(F)$, $\sqrt{-1}\Theta_g(G) \geq -C\widetilde{\omega}$ for some positive constant $C > 0$. Then*

$$su \in \mathrm{Dom}\, D''^{\star}_{hg} \cap \mathrm{Dom}\, \overline{\partial} \subset L^{n,q}_{(2)}(Y, F \otimes G)_{hg,\widetilde{\omega}}$$

if u belongs to $\mathrm{Dom}\, D''^{\star}_h \cap \mathrm{Dom}\, \overline{\partial} \subset L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$.

Proof. If u is smooth, we can easily see that $\overline{\partial} su = s\overline{\partial} u$ and $D'^{\star}_{hg} su = sD'^{\star}_h u$. Here we used $D'^{\star}_{hg} = -\star\overline{\partial}\star$, where $\star$ is the Hodge star operator with respect to $\widetilde{\omega}$. Even if u is not smooth, for every $u \in \mathrm{Dom}\, D''^{\star}_h \cap \mathrm{Dom}\, \overline{\partial}$, we can prove that $\overline{\partial} su = s\overline{\partial} u$ and $D'^{\star}_{hg} su = sD'^{\star}_h u$, by taking a smooth and compactly supported u_k such that $u_k \to u$, $D''^{\star}_h u_k \to D''^{\star}_h u$ and $\overline{\partial} u_k \to \overline{\partial} u$ in $L^{n,\bullet}_{(2)}(Y, F)_{h,\widetilde{\omega}}$. Indeed, by the assumption of $\sup_Y |s|_g < \infty$, we have

$$\|su_k - su\|_{hg,\widetilde{\omega}} = \sup_Y |s|_g \|u_k - u\|_{h,\widetilde{\omega}} \to 0,$$
$$\|s\overline{\partial} u_k - s\overline{\partial} u\|_{hg,\widetilde{\omega}} = \sup_Y |s|_g \|\overline{\partial} u_k - \overline{\partial} u\|_{h,\widetilde{\omega}} \to 0.$$

For a smooth and compactly supported w, we obtain

$$\begin{aligned}\langle\!\langle \overline{\partial} su, w\rangle\!\rangle_{hg,\widetilde{\omega}} &= \lim_{k\to\infty} \langle\!\langle su_k, D''^{\star}_{hg} w\rangle\!\rangle_{hg,\widetilde{\omega}}\\ &= \lim_{k\to\infty} \langle\!\langle s\overline{\partial} u_k, w\rangle\!\rangle_{hg,\widetilde{\omega}}\\ &= \langle\!\langle s\overline{\partial} u, w\rangle\!\rangle_{hg,\widetilde{\omega}},\end{aligned}$$

which implies $\overline{\partial} su = s\overline{\partial} u$. By the same argument we have $D_{hg}^{'\star} su = sD_h^{'\star} u$, and thus $su \in \mathrm{Dom}\,\overline{\partial} \cap \mathrm{Dom}\, D_{hg}^{'\star}$. It remains to show $su \in \mathrm{Dom}\, D_{hg}^{''\star}$. For every $w \in \mathrm{Dom}\,\overline{\partial}$, we take a smooth and compactly supported w_ℓ such that $w_\ell \to w$ and $\overline{\partial} w_\ell \to \overline{\partial} w$ in $L^{n,\bullet}_{(2)}(Y, F \otimes G)_{hg,\widetilde{\omega}}$. Then an easy computation yields

$$\begin{aligned}\langle\!\langle su, \overline{\partial} w\rangle\!\rangle_{hg,\widetilde{\omega}} &= \lim_{k\to\infty}\lim_{l\to\infty} \langle\!\langle su_k, \overline{\partial} w_\ell\rangle\!\rangle_{hg,\widetilde{\omega}} \\ &= \lim_{k\to\infty}\lim_{l\to\infty} \langle\!\langle D_{hg}^{''\star} su_k, w_\ell\rangle\!\rangle_{hg,\widetilde{\omega}} \\ &\le \lim_{k\to\infty} \|D_{hg}^{''\star} su_k\|_{h^{m+1},\widetilde{\omega}} \|w\|_{hg,\widetilde{\omega}}.\end{aligned}$$

Therefore it is sufficient to check that $\|D_{hg}^{''\star} su_k\|_{hg,\widetilde{\omega}}$ is uniformly bounded. Putting $g_k := \langle \sqrt{-1}\Theta_h(F)\Lambda_{\widetilde{\omega}} u_k, u_k\rangle_{h,\widetilde{\omega}}$ and applying Proposition 2.1 to su_k, we have

$$\begin{aligned}\|D_{hg}^{''\star} su_k\|^2_{hg,\widetilde{\omega}} &\le \sup_Y |s|_g^2 \|D_h^{'\star} u_k\|^2_{h,\widetilde{\omega}} + \int_Y |s|_g^2\, g_k \frac{\widetilde{\omega}^n}{n!} \\ &\le \sup_Y |s|_g^2 \left\{ \|D_h^{'\star} u_k\|^2_{h,\widetilde{\omega}} + \int_{\{g_k \ge 0\}} g_k \frac{\widetilde{\omega}^n}{n!} \right\}.\end{aligned}$$

On the other hand, by applying Proposition 2.1 to u_k, we obtain

$$\begin{aligned}-C\|u_k\|^2_{h,\widetilde{\omega}} &\le \int_{\{g_k \ge 0\}} g_k \frac{\widetilde{\omega}^n}{n!} + \int_{\{g_k \le 0\}} g_k \frac{\widetilde{\omega}^n}{n!} \\ &\le \|\overline{\partial} u_k\|^2_{h,\widetilde{\omega}} + \|D_h^{''\star} u_k\|^2_{h,\widetilde{\omega}}.\end{aligned}$$

Here we used the assumption of the curvature. Therefore $\|D_{hg}^{''\star} su_k\|_{hg,\widetilde{\omega}}$ is uniformly bounded.

□

2.3 Singular metrics and their multiplier ideal sheaves

First we recall the definition of singular metrics, curvature currents, and multiplier ideal sheaves.

Definition 2.3 (Singular metrics, curvature currents)**.**

(1) A (hermitian) metric h on F is called a *singular metric*, if for a local trivialization $\theta : F|_U \cong U \times \mathbb{C}$ and a local section ξ of F, we have

$$|\xi|_h = |\theta(\xi)| e^{-\varphi}$$

for some L^1_{loc}-function φ on U. Here φ is called the local *weight* of h with respect to the trivialization.

(2) The *curvature current* $\sqrt{-1}\Theta_h(F)$ associated to h is defined by

$$\sqrt{-1}\Theta_h(F) = dd^c \varphi,$$

where φ is a local weight of h.

(3) The curvature current $\sqrt{-1}\Theta_h(F)$ is said to be *semi-positive* if $\sqrt{-1}\Theta_h(F) \ge 0$ in the sense of currents.

For simplicity we abbreviate the singular metric (resp. the curvature current) to the metric (resp. the curvature). The Levi form $dd^c\varphi$ is taken in the sense of distributions, and thus the curvature is a $(1,1)$-current but not always a smooth $(1,1)$-form. The Levi form $dd^c\varphi$ is semi-positive if and only if the function φ is a plurisubharmonic function (psh function for short).

We consider only metrics h such that $\sqrt{-1}\Theta_h(F) \ge \gamma$ for some smooth $(1,1)$-form γ. Under this condition, the weight function φ becomes a quasi-psh function. In particular φ is upper semi-continuous and hence is bounded above. Then we define multiplier ideal sheaves, which are coherent ideal sheaves.

Definition 2.4. Let h be a singular metric on F such that $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$-form γ. Then the ideal sheaf $\mathcal{I}(h)$ defined to be

$$\mathcal{I}(h)(U) := \mathcal{I}(\varphi)(U) := \{f \in \mathcal{O}_Y(U) \mid |f|e^{-\varphi} \in L^2_{\mathrm{loc}}(U)\}$$

for every open set U, is called the *multiplier ideal sheaf* associated to h.

From now on, let X be a compact Kähler manifold with a Kähler form ω and F be a line bundle on X with a (singular) metric h on F. In this survey, we mainly study the cohomology group $H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$ by applying the theory of harmonic integrals. For this purpose, we introduce the space of the harmonic forms that is isomorphic to the cohomology group $H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$. If h is a smooth metric on a (non-empty) Zariski open $Y \subset X$, then we can represent a cohomology class by the associated harmonic form by taking a suitable complete Kähler form $\widetilde{\omega}$ on Y satisfying the following properties:

(a) $\widetilde{\omega}$ is larger or equal to than ω on Y.
(b) There is a bounded function Φ with $\widetilde{\omega} = dd^c\Phi$ on a neighborhood of every point.

Proposition 2.5. *Under the same situation as above, we have the following orthogonal decomposition:*

$$L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}} = \mathrm{Im}\,\overline{\partial} \oplus \mathcal{H}^{n,q}_{h,\widetilde{\omega}}(Y,F) \oplus \mathrm{Im}\,D''^{*}_h.$$

Moreover we have the following isomorphism:

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \cong \mathrm{Ker}\overline{\partial}/\mathrm{Im}\,\overline{\partial} \cong \mathcal{H}^{n,q}_{h,\widetilde{\omega}}(Y,F).$$

Proof. The proof is same as that of [7, Claim 1]. In general we have

$$L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}} = \overline{\mathrm{Im}\,\overline{\partial}} \oplus \mathcal{H}^{n,q}_{h,\widetilde{\omega}}(Y,F) \oplus \overline{\mathrm{Im}\,D''^{*}_h}.$$

Therefore it is sufficient to show that $\mathrm{Im}\,\overline{\partial}$ and $\mathrm{Im}\,D''^{*}_h$ are closed subspaces, and

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \cong \mathrm{Ker}\overline{\partial}/\mathrm{Im}\,\overline{\partial}.$$

First we prove the above isomorphism, by chasing the De Rham-Weil isomorphism. Fix an open cover $\mathcal{U} := \{U_j\}_{j=1}^N$ of X with an open ball U_j. Then we have the isomorphism

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \cong \check{H}^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)),$$

where the right hand side is the Čech cohomology group calculated by $\mathcal{U}$. For simplicity, we put $U_{j_0 j_1 \dots j_q} := U_{j_0} \cap \cdots \cap U_{j_q}$. We consider a q-cocycle $u = \{u_{j_0 j_1 \dots j_q}\}$ satisfying

$$u_{j_0 j_1 \dots j_q} \in H^0(U_{j_0 j_1 \dots j_q}, K_X \otimes F \otimes \mathcal{I}(h)) \quad \text{and} \quad \delta u = 0,$$

where δ is the coboundary operator of the Čech complex. Then, by using a partition $\{\rho_j\}_{j=1}^N$ of unity associated to $\mathcal{U}$, we define $u^1 := \{u_{j_0 j_1 \dots j_{q-1}}\}$ by $u_{j_0 j_1 \dots j_{q-1}} := \sum_{j=1}^N \rho_j u_{j j_0 \dots j_{q-1}}$. By the construction, we have $\delta u^1 = u$ and $\delta\overline{\partial}u^1 = 0$. From the same argument, we can obtain u^2 with $\delta u^2 = \overline{\partial}u^1$. By repeating this process, we obtain u^q. Then $\overline{\partial}u^q$ determines the F-valued (n,q)-form on X with $\|\overline{\partial}u_q\|_{h,\omega} < \infty$ thanks to $\delta\overline{\partial}u^q = 0$. Further it is easy to see that $\overline{\partial}u^q$ belongs to $L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}}$ (that is, $\|\overline{\partial}u_q\|_{h,\widetilde{\omega}} < \infty$). Indeed, for every F-valued (n,q)-form u, we have the inequality $|u|^2_{h,\widetilde{\omega}}\,\widetilde{\omega}^n \leq |u|^2_{h,\omega}\omega^n$ by property (a) of $\widetilde{\omega}$. From the above argument, we have obtained the (well-defined) map

$$\check{H}^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)) \to \mathrm{Ker}\overline{\partial}/\mathrm{Im}\,\overline{\partial}.$$

Now we see that this map is actually isomorphic by using the L^2-method for the $\overline{\partial}$-equation (for example, see [3, Théorème 4.1]). For every $w \in L^{n,q}_{(2)}(Y,F) \cap \mathrm{Ker}\overline{\partial}$, we define $w^0 := \{w_{j_0}\}$ by $w_{j_0} := w|_{U_{j_0} \setminus Z}$. By the L^2-method for the $\overline{\partial}$-equation, we obtain $w^1 = \{w^1_{j_0}\}$ such that

$$\overline{\partial}w^1 = w^0 \quad \text{on } U_{j_0} \setminus Z,$$

$$\|w^1\|^2_{h,\widetilde{\omega}} := \sum_{j_0=1}^{N} \int_{U_{j_0} \setminus Z} |w^1_{j_0}|^2_{h,\widetilde{\omega}}\,\widetilde{\omega}^n \leq C_1 \|w\|^2_{h,\widetilde{\omega}}.$$

Here C_1 is a positive constant independent of w. By the construction, we have $\overline{\partial}\delta w^1 = 0$. Therefore by the same method, we can obtain $w^2 = \{w^2_{j_0 j_1}\}$ such that

$$\overline{\partial} w^2 = \delta w^1 \quad \text{on } U_{j_0 j_1} \setminus Z,$$
$$\|w^2\|^2_{h,\widetilde{\omega}} := \sum_{j_0,\, j_1=1}^{N} \int_{U_{j_0 j_1}\setminus Z} |w^2_{j_0 j_1}|^2_{h,\widetilde{\omega}}\, \widetilde{\omega}^n \leq C_2 \|w\|^2_{h,\widetilde{\omega}}.$$

By repeating this process, we obtain w^q such that $\overline{\partial} w^q = \delta w^{q-1}$ and $\|\delta w^q\|_{h,\widetilde{\omega}} \leq C\|w\|_{h,\widetilde{\omega}}$ for some positive constant $C > 0$. Then δw^q determines the q-cocycle since holomorphic $(n, 0)$-forms on $U \setminus Z$ with bounded L^2-norm can be extended (by the Riemann extension theorem). Here we used the special characteristics of the canonical bundle. Hence we have obtained

$$\beta : \mathrm{Ker}\overline{\partial}/\mathrm{Im}\,\overline{\partial} \to \check{H}^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)).$$

It is easy to see that these maps give an isomorphism.

It remains to show that $\mathrm{Im}\,\overline{\partial}$ is a closed subspace in $L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$. Take a sequence $\{\overline{\partial} v_k\}_{k=1}^{\infty}$ in $\mathrm{Im}\,\overline{\partial}$ that converges to $w \in L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$. Since β is continuous by the construction, we have $\|\beta(w - \overline{\partial} v_k)\| \to 0$ as $k \to \infty$. Then $\beta(w)$ is zero in $\check{H}^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ since $\check{H}^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ is a finite dimensional vector space. Therefore we know $w \in \mathrm{Im}\,\overline{\partial}$ by the above isomorphism, which implies that $\mathrm{Im}\,\overline{\partial}$ is closed. It follows from this fact that $\mathrm{Im}\, D''^{*}_{h}$ is also closed. □

For a given singular metric h, thanks to [4, Theorem 2.3.], we can approximate h by singular metrics h_ε that are smooth on a Zariski open set, without changing the multiplier ideal sheaf. Moreover if the set $\{x \in X \mid \nu(h, x) > 0\}$ is contained in a subvariety, then we can assume that the singular metrics h_ε are smooth on a Zariski open set independent of ε. Here $\nu(h, x)$ denotes the Lelong number of the weight φ of h at x. We reformulate [4, Theorem 2.3.] with our notation and give an additional property.

Theorem 2.6. *([4, Theorem 2.3.]) Let X be a compact Kähler manifold and F be a line bundle with a singular metric h such that $\sqrt{-1}\Theta_h \geq \gamma$ for some smooth $(1, 1)$-form γ. Then there exist metrics $\{h_\varepsilon\}_{1\gg\varepsilon>0}$ on F with the following properties:*

(a) h_ε is smooth on $X \setminus Z_\varepsilon$, where Z_ε is a subvariety on X.
(b) $h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h$ holds for any $0 < \varepsilon_1 < \varepsilon_2$.
(c) $\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$.
(d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq \gamma - \varepsilon\omega$.

Moreover, if the set $\{x \in X \mid \nu(h, x) > 0\}$ is contained in a subvariety Z, then we can add the property that Z_ε is contained in Z for any $\varepsilon > 0$.

Proof. Fix a smooth metric g on F. Then there exists an L^1-function φ on X with $h = ge^{-\varphi}$. By applying [4, Theorem 2.3.] to φ, we obtain quasi-psh functions φ_ν with equisingularities. For a given $\varepsilon > 0$, by taking large $\nu = \nu(\varepsilon)$, we define h_ε by $h_\varepsilon := ge^{-\varphi_{\nu(\varepsilon)}}$. Then the metric h_ε satisfies properties (a), (b), (c), (d).

The latter conclusion follows from the proof in [4]. We see this fact shortly, by using the notation in [4]. In their proof, they locally approximate φ by $\varphi_{\varepsilon,\nu,j}$ with logarithmic pole. By inequality (2.5) in [4], the Lelong number of $\varphi_{\varepsilon,\nu,j}$ is less than or equal to that of φ. Hence $\varphi_{\varepsilon,\nu,j}$ is smooth on $X \setminus Z$ since $\varphi_{\varepsilon,\nu,j}$ has a logarithmic pole. Since φ_ν is obtained from Richberg's regularization of the supremum of these functions (see around (2.5) and (2.7)), we obtain the latter conclusion. □

3 Applications

In this section, we give an idea of the proof of Theorem 1.3 (see also [7]). The following theorem is an injectivity theorem with multiplier ideal sheaves under the regularity assumption for singular metrics, whose proof provides a rough strategy to prove Theorem 1.3.

Theorem 3.1 ([12, Theorem 1.5]). *Let (L, h_L) and (M, h_M) be singular hermitian line bundles on a compact Kähler manifold X. Assume the following conditions:*

(1) *There exists a subvariety Z on X such that h_L and h_M are smooth on $X \setminus Z$.*
(2) *$\sqrt{-1}\Theta_{h_L}(L) \geq \gamma$ and $\sqrt{-1}\Theta_{h_M}(M) \geq \gamma$ on X for some smooth $(1,1)$-form γ on X.*
(3) *$\sqrt{-1}\Theta_{h_L}(L) \geq 0$ on $X \setminus Z$.*
(4) *$\sqrt{-1}\Theta_{h_L}(L) \geq \varepsilon\sqrt{-1}\Theta_{h_M}(M)$ on $X \setminus Z$ for some positive number $\varepsilon > 0$.*

Then for a (non-zero) section s of M with $\sup_X |s|_{h_M} < \infty$, then the multiplication map induced by the tensor product with s

$$\Phi_s : H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)) \xrightarrow{\otimes s} H^q(X, K_X \otimes L \otimes M \otimes \mathcal{I}(h_L h_M))$$

is (well-defined and) injective for any i.

Proof. Fix a Kähler form ω on X and a complete Kähler form $\widetilde{\omega}$ on $Y := X \setminus Z$ satisfying the following properties:

(a) $\widetilde{\omega}$ is larger or equal to than ω on Y,
(b) There is a bounded function Φ with $\widetilde{\omega} = dd^c\Phi$ on a neighborhood of every point.

We prove that the multiplication map Ψ_s from $\mathcal{H}^{n,q}(Y, L)_{h_L,\widetilde{\omega}}$ to $\mathcal{H}^{n,q}(Y, L \otimes M)_{h_L h_M,\widetilde{\omega}}$ is well-defined, by using the theory of harmonic integrals in Section 2. In other words, we prove that su belongs to $\mathcal{H}^{n,q}(Y, L \otimes M)_{h_L h_M,\widetilde{\omega}}$ for every $u \in \mathcal{H}^{n,q}(Y, L)_{h_L,\widetilde{\omega}}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^q(X, K_X \otimes L \otimes \mathcal{I}(h_L)) & \xrightarrow{\cong} & \mathcal{H}^{n,q}(Y, L)_{h_L,\widetilde{\omega}} \\ {\scriptstyle\Phi_s}\downarrow & & {\scriptstyle\Psi_s}\downarrow \\ H^q(X, K_X \otimes L \otimes M \otimes \mathcal{I}(h_L h_M)) & \xrightarrow{\cong} & \mathcal{H}^{n,q}(Y, L \otimes M)_{h_L h_M,\widetilde{\omega}} \end{array}$$

We can easily see that the multiplication Φ_s is injective if $\Psi_s$is injective.

For a given $u \in \mathcal{H}^{n,q}(Y, L)_{h_L,\widetilde{\omega}}$. we first see that su is L^2-bounded with respect to $h_L h_M$. Since $\sup_X |s|_{h_M}$ is finite by the assumption, an easy computation yields

$$\begin{aligned} \|su\|^2_{h_L h_M,\widetilde{\omega}} &= \int_Y |s|^2_{h_M} |u|^2_{h_L,\widetilde{\omega}} \frac{\widetilde{\omega}^n}{n!} \\ &\leq \sup_X |s|^2_{h_M} \int_Y |u|^2_{h_L,\widetilde{\omega}} \frac{\widetilde{\omega}^n}{n!} \\ &= \sup_X |s|^2_{h_M} \|u\|^2_{h_L,\widetilde{\omega}}. \end{aligned}$$

Therefore su is L^2-bounded. By Proposition 2.2, we can apply Proposition 2.1 to su, and thus we obtain

$$\|D''^{*}_{h_L h_M} su\|^2_{h_L h_M,\widetilde{\omega}} = \|D'^{*}_{h_L h_M} su\|^2_{h_L h_M,\widetilde{\omega}} + \langle\!\langle \sqrt{-1}\Theta_{h_L h_M}(L \otimes M)\Lambda_{\widetilde{\omega}} su, su \rangle\!\rangle_{h_L h_M,\widetilde{\omega}}.$$

Here we used $\overline{\partial} su = s\overline{\partial} u = 0$. In order to show that su is harmonic, it is sufficient to prove that the right hand side is zero.

On the other hand, by applying Proposition 2.1 to u, we have

$$0 = \|D'^{*}_{h_L} u\|^2_{h_L,\widetilde{\omega}} + \langle\!\langle \sqrt{-1}\Theta_{h_L}(L)\Lambda_{\widetilde{\omega}} u, u \rangle\!\rangle_{h_L,\widetilde{\omega}}.$$

Remark that the left hand side is equal to zero since u is harmonic. Since the curvature $\sqrt{-1}\Theta_{h_L}(L)$ is semi-positive on Y by assumption (3), the integrand $\langle\sqrt{-1}\Theta_{h_L}(L)\Lambda_{\widetilde{\omega}}u, u\rangle_{h_L,\widetilde{\omega}}$ is a semi-positive function. Therefore from the above equality we obtain

$$|D'^{\star}_{h_L}u|_{h_L,\widetilde{\omega}} = 0 \quad \text{and} \quad \langle\sqrt{-1}\Theta_{h_L}(L)\Lambda_{\widetilde{\omega}}u, u\rangle_{h_L,\widetilde{\omega}} = 0 \quad \text{on } Y.$$

Since s is a holomorphic section, we can easily see that $D'^{\star}_{h_L h_M}su = sD'^{\star}_{h_L}u = 0$. Moreover by assumption (4) we have

$$\begin{aligned}\langle\sqrt{-1}\Theta_{h_L h_M}(L\otimes M)\Lambda_{\widetilde{\omega}}su, su\rangle &= |s|^2_{h_M}\langle\sqrt{-1}\Theta_{h_L h_M}(L\otimes M)\Lambda_{\widetilde{\omega}}u, u\rangle\\ &= |s|^2_{h_M}\langle\sqrt{-1}\Theta_{h_M}(M)\Lambda_{\widetilde{\omega}}u, u\rangle\\ &\leq \frac{1}{\varepsilon}|s|^2_{h_M}\langle\sqrt{-1}\Theta_{h_L}(L)\Lambda_{\widetilde{\omega}}u, u\rangle = 0.\end{aligned}$$

This implies $\langle\!\langle\sqrt{-1}\Theta_{h_L h_M}(L\otimes M)\Lambda_{\widetilde{\omega}}su, su\rangle\!\rangle_{h_L h_M,\widetilde{\omega}} = 0$. We have proved that su is harmonic for every harmonic form $u \in \mathcal{H}^{n,q}(Y, L)_{h_L,\widetilde{\omega}}$. It is obvious that the multiplication map $\Psi_s : \mathcal{H}^{n,q}(Y, L)_{h_L,\widetilde{\omega}} \to \mathcal{H}^{n,q}(Y, L\otimes M)_{h_L h_M,\widetilde{\omega}}$ is injective. This completes the proof. □

The following theorem is a generalization of Theorem 3.1 and Theorem 1.3. At the end of this survey, we give some remarks and the sketch of the proof of Theorem 3.2.

Theorem 3.2 ([9, Theorem 1.3]). *Let (F, h_F) and (L, h_L) be singular hermitian line bundles with semi-positive curvature on a compact Kähler manifold X. Assume that there exists an effective $\mathbb{R}$-divisor Δ with*

$$h_F = h_L^a \cdot h_\Delta,$$

where a is a positive real number and h_Δ is the singular metric defined by Δ.

Then for a (non-zero) section s of L satisfying $\sup_X |s|_{h_L} < \infty$, the multiplication map

$$\Phi_s : H^q(X, K_X\otimes F\otimes\mathcal{I}(h_F)) \xrightarrow{\otimes s} H^q(X, K_X\otimes F\otimes L\otimes\mathcal{I}(h_F h_L))$$

is (well-defined and) injective for any q.

Remark 3.3. (1) The case of $\Delta = 0$ corresponds to Theorem 1.3. When we apply our injectivity theorem to the problem of extending holomorphic sections from subvarieties to the ambient space, it is important to consider the case of $\Delta \neq 0$. See [9] for more details.

(2) If h_L and h_F are smooth on a Zariski open set, the same conclusion holds under the weaker assumption of $\sqrt{-1}\Theta_{h_F}(F) \geq a\sqrt{-1}\Theta_{h_L}(L)$. See Theorem 3.1.

(3) When (L, h_L)=(F^m, h^m), the above theorem agrees with Theorem 1.3.

We shortly give the sketch of the proof (see [11] for the precise proof). In our situation, we must consider singular metrics with transcendental (non-algebraic) singularities. It is quite difficult to directly handle transcendental singularities, and thus in the first step, we approximate a given metric h_F by metrics $\{h_\varepsilon\}_{\varepsilon>0}$ that are smooth on a Zariski open set, by using Theorem 2.6. Then we represent a given cohomology class in $H^q(X, K_X\otimes F\otimes\mathcal{I}(h_F))$ by the associated harmonic form u_ε with respect to h_ε on the Zariski open set (see Theorem 2.5). We want to show that su_ε is also harmonic by using the same method as the proof of Enoki or Theorem 3.1. However, the same argument fails since the curvature of h_ε is not semi-positive. For this reason, in the second step, we investigate the asymptotic behavior of the harmonic forms u_ε with respect to a family of the regularized metrics $\{h_\varepsilon\}_{\varepsilon>0}$. Then we show that the L^2-norm $\|D''^{\star}_{h_\varepsilon h_{L,\varepsilon}}su_\varepsilon\|$ converges to zero as ε goes to zero, where $h_{L,\varepsilon}$ is a suitable approximation of h_L. Moreover, in the third Step, we construct solutions γ_ε of the $\overline{\partial}$-equation $\overline{\partial}\gamma_\varepsilon = su_\varepsilon$ such that the L^2-norm $\|\gamma_\varepsilon\|$ is uniformly bounded, by applying the Čech complex with the topology induced by the local L^2-norms. In the final step, we prove

$$\|su_\varepsilon\|^2 = \langle\!\langle su_\varepsilon, \overline{\partial}\gamma_\varepsilon\rangle\!\rangle \leq \|D''^{\star}_{h_\varepsilon h_{L,\varepsilon}}su_\varepsilon\|\|\gamma_\varepsilon\| \to 0 \quad \text{as } \varepsilon \to 0.$$

From these observations, we can conclude that u_ε converges to zero in a suitable sense. This completes the proof.

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