Research Article Open Access

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Strongly not relatives Kähler manifolds

DOI 10.1515/coma-2017-0001
Received September 24, 2016; accepted November 6, 2016

Abstract: In this paper we study Kähler manifolds that are strongly not relative to any projective Kähler manifold, i.e. those Kähler manifolds that do not share a Kähler submanifold with any projective Kähler manifold even when their metric is rescaled by the multiplication by a positive constant. We prove two results which highlight some relations between this property and the existence of a full Kähler immersion into the infinite dimensional complex projective space. As application we get that the 1-parameter families of Bergman–Hartogs and Fock–Bargmann–Hartogs domains are strongly not relative to projective Kähler manifolds.

Keywords: Kähler manifolds; complex submanifolds; diastasis function

MSC: 53C55; 32H02

1 Introduction and statement of the main results

According to [6], two Kähler manifolds are called *relatives* when they share a common Kähler submanifold, i.e. if a complex submanifold of one of them with the induced metric is biholomorphically isometric to a complex submanifold of the other one with the induced metric. In his seminal paper [3], Calabi determined a criterion which characterizes Kähler manifolds admitting a Kähler immersion into finite or infinite dimensional complex space forms. The main tool he introduced is the *diastasis function* associated to a real analytic Kähler manifold, namely a particular Kähler potential characterized by being invariant under pull–back through a holomorphic map. Thanks to this property, the diastasis plays a key role in studying when two Kähler manifolds are relatives. In [16] Umehara proved that two finite dimensional complex space forms with holomorphic sectional curvatures of different signs can not be relatives. Although, as firstly pointed out by Bochner in [2], when the ambient space is allowed to be infinite dimensional, the situation is different: any Kähler submanifold of the infinite dimensional flat space $\ell^2(\mathbb{C})$ admits a Kähler immersion into the infinite dimensional complex projective space. Umehara's work has been generalized in the recent paper by X. Cheng and A. J. Di Scala [4], where the authors state necessary and sufficient conditions for complex space forms of finite dimension and different curvatures to not be relative to each others. In [6] A. J. Di Scala and A. Loi prove that a Hermitian symmetric space of noncompact type endowed with its Bergman metric is not relative to a projective Kähler manifold, i.e. a Kähler manifold which admit a local holomorphic and isometric (from now on *Kähler*) immersion into the *finite* dimensional complex projective space (see also [10] for the case of Hermitian symmetric spaces of noncompact type and Euclidean spaces), and their result has been generalized in [15] to homogeneous bounded domains of $\mathbb{C}^n$. Throughout the paper, we say that a Kähler manifold is *projectively induced* when it admits a Kähler immersion into $\mathbb{C}\mathrm{P}^{N\leq\infty}$. When we also specify that it is *infinite projectively induced*, we mean that the Kähler immersion is full into $\mathbb{C}\mathrm{P}^{\infty}$.

In this paper we are interested in studying when a Kähler manifold (M, g) is *strongly not relative* to any projective Kähler manifold, that is, when $(M, c\,g)$ is not relative to any projective Kähler manifold for any value of the constant $c > 0$ multiplying the metric.

Our first result can be viewed as a generalization of the results in [6, 15] and can be stated as follows:

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Theorem 1. *Let (M,g) be a Kähler manifold such that $(M,\beta g)$ is infinite projectively induced for any $\beta>\beta_0\geq 0$. If (M,g) and $\mathbb{C}\mathrm{P}^n$ are not relatives for any $n<\infty$, then (M,g) is strongly not relative to any projective Kähler manifold.*

Observe that in general there are not reasons for a Kähler manifold which is not relative to another Kähler manifold to remain so when its metric is rescaled. For example, consider that the complex projective space $(\mathbb{C}\mathrm{P}^2, c\,g_{FS})$ where g_{FS} is the Fubini–Study metric, for $c=\frac{2}{3}$ is not relative to $(\mathbb{C}\mathrm{P}^2, g_{FS})$, while for positive integer values of c it is (see [4] for a proof).

In order to state our second result, consider a d-dimensional Kähler manifold (M,g) which admits global coordinates $\{z_1,\dots,z_d\}$ and denote by M_j the 1-dimensional submanifold of M defined by:

$$M_j=\{z\in M|\, z_1=\dots=z_{j-1}=z_{j+1}=\dots=z_d=0\}.$$

When exists, a Kähler immersion $f\colon M\to\mathbb{C}\mathrm{P}^\infty$ is said to be *transversally full* when for any $j=1,\dots,d$, the immersion restricted to M_j is full into $\mathbb{C}\mathrm{P}^\infty$.

Theorem 2. *Let (M,g) be a Kähler manifold infinite projectively induced through a transversally full map. If for any $\alpha\geq\alpha_0>0$, $(M,\alpha\, g)$ is infinite projectively induced then (M,g) is strongly not relative to any projective Kähler manifold.*

As a consequence of Theorem 1 and Theorem 2 we get that the 1-parameter families of Bergman–Hartogs and Fock–Bargmann–Hartogs domains, which we describe in Section 4, are strongly not relative to any projective Kähler manifold (see corollaries 8 and 11 below).

The paper consts of three more sections. In the first one we briefly recall the definition of diastasis function and its properties we need and in the second one we prove Theorem 1 and Theorem 2. Finally, in the third and last section we apply our results to Bergman–Hartogs and Fock–Bargmann–Hartogs domains.

The author is very grateful to Prof. Andrea Loi for all the interesting discussions and comments that helped her to improve the contents and the exposition.

2 Calabi's diastasis function

Consider a real analytic Kähler manifold (M,g) and let $\varphi: U\to\mathbb{R}$ be a Kähler potential for g defined on a coordinate neighborhood U around a point $p\in M$. Consider the analytic extension $\tilde\varphi: W\to\mathbb{R}$, $\tilde\varphi(z,\bar z)=\varphi(z)$, of φ on a neighborhood W of the diagonal in $U\times\bar U$. The *diastasis function* $\mathrm{D}(z,w)$ is defined by the formula:

$$\mathrm{D}(z,w):=\tilde\varphi(z,\bar z)+\tilde\varphi(w,\bar w)-\tilde\varphi(z,\bar w)-\tilde\varphi(w,\bar z). \tag{2.1}$$

Observe that since:

$$\frac{\partial^2}{\partial z\partial\bar z}\mathrm{D}(z,w)=\frac{\partial^2}{\partial z\partial\bar z}\tilde\varphi\,(z,\bar z)=\frac{\partial^2}{\partial z\partial\bar z}\varphi\,(z)\,,$$

once one of its two entries is fixed, the diastasis is a Kähler potential for g. We denote by $\mathrm{D}_0(z)$ the diastasis centered at the origin. The following theorem due to Calabi [3], expresses the diastasis' property which is fundamental for our purpose.

Theorem 3 (E. Calabi)**.** *Let (M,g) and (S,G) be Kähler manifolds and assume G to be real analytic. Denote by ω and Ω the Kähler forms associated to g and G respectively. If there exists a holomorphic map $f\colon(M,g)\to(S,G)$ such that $f^*\Omega=\omega$, then the metric g is real analytic. Further, denoted by $\mathrm{D}^M_p: U\to\mathbb{R}$ and $\mathrm{D}^S_{f(p)}: V\to\mathbb{R}$ the diastasis functions of (M,g) and (S,G) around p and $f(p)$ respectively, we have $\mathrm{D}^S_{f(p)}\circ f=\mathrm{D}^M_p$ on $f^{-1}(V)\cap U$.*

Consider the complex projective space $\mathbb{C}\mathrm{P}^N_b$ of complex dimension $N\leq\infty$, with the Fubini-Study metric g_b of holomorphic bisectional curvature $4b$ for $b>0$. When $b=1$ we denote by g_{FS} and ω_{FS} the Fubini-Study metric and the Fubini-Study form respectively. Let $[Z_0, \dots, Z_N]$ be homogeneous coordinates, $p = [1, 0, \dots, 0]$ and $U_0 = \{Z_0 \neq 0\}$. Define affine coordinates $z_1, \dots, z_N$ on U_0 by $z_j = Z_j/(\sqrt{b}Z_0)$. The diastasis on U_0 centered at the origin reads:

$$\mathrm{D}_0^b(z) = \frac{1}{b}\log\left(1 + b\sum_{j=1}^{N}|z_j|^2\right). \tag{2.2}$$

Due to Th. 3 and the expression of $\mathbb{C}\mathrm{P}_b^N$'s diastasis (2.2), if $f : S \to \mathbb{C}\mathrm{P}_b^N$ is a holomorphic map, $f(z) = [f_0(z), f_1(z), \dots, f_N(z)]$, then the induced diastasis D_0^S in a neighborhood of a point $p \in S$ is given by:

$$\mathrm{D}_0^S(z) = \frac{1}{b}\log\left(1 + b\sum_{j=1}^{N}|f_j(z)|^2\right).$$

Further, if the Kähler map f is assumed to be *full*, i.e. the image $f(S)$ is not contained into any lower dimensional totally geodesic submanifold of $\mathbb{C}\mathrm{P}_b^N$, then f is univocally determined up to rigid motion of $\mathbb{C}\mathrm{P}_b^N$ [3, pp. 18]:

Theorem 4 (Calabi's Rigidity). *If a neighborhood V of a point p admits a full Kähler immersion into* $(\mathbb{C}\mathrm{P}_b^N, g_b)$, *then N is univocally determined by the metric and the immersion is unique up to rigid motions of* $(\mathbb{C}\mathrm{P}_b^N, g_b)$.

Observe that by Th. 4 above, a Kähler manifold which is infinite projectively induced does not admit a Kähler immersion into any finite dimensional complex projective space.

3 Proof of Theorems 1 and 2

Proof of Theorem 1. Observe first that due to Th. 3 it is enough to prove that $(M, c\,g)$ is not relative to $\mathbb{C}\mathrm{P}^n$ for any finite n and any $c > 0$. For any $c > 0$, we can choose a positive integer α such that $c\alpha > \beta_0$. Denote by ω the Kähler form on M associated to g. Let $F : M \to \mathbb{C}\mathrm{P}^\infty$ be a full Kähler map such that $F^\star\omega_{FS} = c\alpha\,\omega$. Then $\tilde{F} = F/\sqrt{\alpha}$ is a Kähler map of $(M, c\,g)$ into $\mathbb{C}\mathrm{P}_\alpha^\infty$. Let S be a common Kähler submanifold of $(M, c\,g)$ and $\mathbb{C}\mathrm{P}^n$. Then by Th. 3 for any $p \in S$ there exist a neighborhood U and two holomorhic maps $f : U \to M$ and $h : U \to \mathbb{C}\mathrm{P}^n$, such that $f^\star(c\omega)|_U = (\tilde{F} \circ f)^\star\omega_{FS}|_U = h^\star\omega_{FS}|_U$.

Thus, by (2.2) one has:

$$\log\left(1 + \sum_{j=1}^{n}|h_j|^2\right) = \frac{1}{\alpha}\log\left(1 + \sum_{j=1}^{\infty}|(F \circ f)_j)|^2\right).$$

i.e.:

$$\alpha\log\left(1 + \sum_{j=1}^{n}|h_j|^2\right) = \log\left(1 + \sum_{j=1}^{\infty}|(F \circ f)_j)|^2\right). \tag{3.1}$$

Since $F \circ f$ is full and α is a positive integer, this last equality and Calabi rigidity Theorem 4 imply $n = \infty$. □

Proof of Theorem 2. Due to Th. 1 and Th. 3 we need only to prove that a if a Kähler manifold is infinite projectively induced through a transversally full immersion then it is not relative to $\mathbb{C}\mathrm{P}^n$ for any n. Assume that S is a 1-dimensional Kähler submanifold of both $\mathbb{C}\mathrm{P}^n$ and (M, g). Then around each point $p \in S$ there exist an open neighborhood U and two holomorphic maps $\psi : U \to \mathbb{C}\mathrm{P}^n$ and $\varphi : U \to M$, $\varphi(\xi) = (\varphi_1(\xi), \dots, \varphi_d(\xi))$ where ξ are coordinates on U, such that $\psi^\star\omega_{FS}|_U = \varphi^\star(c\omega)|_U$. Without loss of generality we can assume $\frac{\partial\varphi_1(\xi)}{\partial\xi}(0) \neq 0$. Let $f : M \to \mathbb{C}\mathrm{P}^\infty$ be a Kähler map from (M, g) into $\mathbb{C}\mathrm{P}^\infty$. Since by assumption f is transversally full, $f = [f_0, \dots, f_j, \dots]$ contains for any $m = 1, 2, 3, \dots,$ a subsequence $\{f_{j_1}, \dots, f_{j_m}\}$ of functions which restricted to M_1 are linearly independent. The map $f \circ \varphi : U \to \mathbb{C}\mathrm{P}^\infty$ is full, in fact $f|_{M_1} \circ \varphi$ is full since $\varphi_1(\xi)$ is not constant and for any $m = 1, 2, 3, \dots,$ $\{f_{j_1}(\varphi_1(\xi)), \dots, f_{j_m}(\varphi_1(\xi))\}$ is a subsequence of $\{f|_{M_1} \circ \varphi\}$ of linearly independent functions. Conclusion follows by Calabi's rigidity Theorem 4. □

4 Applications

Let $(\Omega, \beta g_B)$, $\beta > 0$, denote a bounded domain of $\mathbb{C}^d$ endowed with a positive multiple of its Bergman metric g_B. Recall that g_B is the Kähler metric on Ω whose associated Kähler form ω_B is given by $\omega_B = \frac{i}{2}\partial\bar\partial \log \mathrm{K}(z,z)$, where $\mathrm{K}(z,z)$ is the reproducing kernel for the Hilbert space:

$$\mathcal{H} = \left\{ \varphi \in \mathrm{hol}(\Omega),\ \int_\Omega |\varphi|^2 \frac{\omega_0^d}{d!} < \infty \right\},$$

where $\omega_0 = \frac{i}{2}\sum_{j=1}^d dz_j \wedge d\bar z_j$ is the standard Kähler form of $\mathbb{C}^d$. It follows by (2.1) that the diastasis function for g_B is given by:

$$\mathrm{D}_0^\Omega(z) = \log \frac{\mathrm{K}(z,z)\,\mathrm{K}(0,0)}{|\,\mathrm{K}(z,0)|^2}. \tag{4.1}$$

Observe that the Bergman metric g_B admits a natural Kähler immersion into the infinite dimensional complex projective space (cfr. [11]). More precisely, if $\mathrm{K}(z,z) = \sum_{j=0}^\infty |\varphi_j(z)|^2$, the map:

$$\varphi\colon \Omega \to \mathbb{C}\mathrm{P}^\infty, \quad \varphi = (\varphi_0, \dots, \varphi_j, \dots), \tag{4.2}$$

is a Kähler immersion of (Ω, g_B) into $\mathbb{C}\mathrm{P}^\infty$, for $\varphi^* g_{FS} = g_B$, as it follows by:

$$\omega_B = \frac{i}{2}\partial\bar\partial \log(\mathrm{K}(z,z)) = \frac{i}{2}\partial\bar\partial \log\left(\sum_{j=0}^\infty |\varphi_j(z)|^2\right) = \varphi^*\omega_{FS}.$$

Further, such immersion is full since $\{\varphi_j\}$ is a basis for the Hilbert space $\mathcal{H}$ and a bounded domain does not admit a Kähler immersion into a finite dimensional complex projective space even when the metric is rescaled. Although the existence of a Kähler immersion of $(\Omega, \beta g_B)$ into $\mathbb{C}\mathrm{P}^\infty$ is strictly related to the constant β which multiplies the metric (see [13] for the case when Ω is symmetric). In [5] it is proven that the only homogeneous bounded domain which is projectively induced for all positive values of the constant multiplying the metric is a product of complex hyperbolic spaces. Although, the property of being projectively induced for a large enough constant is not so unusual and the following holds [12]:

Theorem 5 (A. Loi, R. Mossa). *Let (Ω, g) be a homogeneous bounded domain. Then, there exists $\alpha_0 > 0$ such that $(\Omega, \alpha g)$ is projectively induced for any $\alpha \geq \alpha_0 > 0$.*

Notice that it is an open question if the same statement holds dropping the homogeneous assumption.

Regarding the property of being relative to some projective Kähler manifold, we recall the following result due to A. J. Di Scala and A. Loi in [6], which plays a key role in the proof of Corollary 8.

Theorem 6 (A. J. Di Scala, A. Loi). *A bounded domain of $\mathbb{C}^n$ endowed with its Bergman metric and a projective Kähler manifold are not relatives.*

Observe that due to theorems 5 and 6, Theorem 1 implies that a bounded domain of $\mathbb{C}^n$ endowed with its Bergman metric and a projective Kähler manifold are *strongly* not relatives. Althought, this result has been proven in a more general context by R. Mossa in [15], where he shows that a homogeneous bounded domain and a projective Kähler manifold are not relatives.

Let us now describe the family of Bergman–Hartogs domains. For all positive real numbers μ a *Bergman-Hartogs domain* is defined by:

$$M_\Omega(\mu) = \left\{(z,w) \in \Omega \times \mathbb{C},\ |w|^2 < \tilde{\mathrm{K}}(z,z)^{-\mu}\right\}, \tag{4.3}$$

where $\tilde{\mathrm{K}}(z,z) = \frac{\mathrm{K}(z,z)\,\mathrm{K}(0,0)}{|\,\mathrm{K}(z,0)|^2}$ with K the Bergman kernel of Ω. Consider on $M_\Omega(\mu)$ the metric $g(\mu)$ whose associated Kähler form $\omega(\mu)$ can be described by the (globally defined) Kähler potential centered at the origin

$$\Phi(z,w) = -\log(\tilde{K}(z,z)^{-\mu} - |w|^2). \tag{4.4}$$

The domain Ω is called the *base* of the Bergman–Hartogs domain $M_\Omega(\mu)$ (one also says that $M_\Omega(\mu)$ is based on Ω). Observe that these domains include and are a natural generalization of Cartan–Hartogs domains which have been studied under several points of view (see e.g. [7, 17] and references therein). To the author knowledge, Bergman-Hartogs domains has been already considered in [8, 9, 18].

In [13] the author of the present paper jointly with A. Loi proved that when the base domain is symmetric $(M_\Omega(\mu), c\,g(\mu))$ admits a Kähler immersion into the infinite dimensional complex projective space if and only if $(\Omega, (c+m)\mu g_B)$ does for every integer $m \geq 0$. As pointed out in [8], a totally similar proof holds also when the base is a homogeneous bounded domain. This fact together with Theorem 5 proves that a Bergman–Hartogs domain $(M_\Omega(\mu), c\,g(\mu))$ is projectively induced for all large enough values of the constant c multiplying the metric. Further, the immersion can be written explicitely as follows (cfr. [14, Lemma 8]):

Lemma 7. *Let α be a positive real number such that the Bergman–Hartogs domain $(M_\Omega(\mu), \alpha\, g(\mu))$ is projectively induced. Then, the Kähler map f from $(M_\Omega(\mu), \alpha\, g(\mu))$ into $\mathbb{C}P^\infty$, up to unitary transformation of $\mathbb{C}P^\infty$, is given by:*

$$f = \left[1, s, h_{\mu\,\alpha}, \dots, \sqrt{\frac{(m+\alpha-1)!}{(\alpha-1)!m!}}\, h_{\mu(\alpha+m)} w^m, \dots\right], \tag{4.5}$$

where $s = (s_1, \dots, s_m, \dots)$ with

$$s_m = \sqrt{\frac{(m+\alpha-1)!}{(\alpha-1)!m!}}\, w^m,$$

and $h_k = (h_k^1, \dots, h_k^j, \dots)$ denotes the sequence of holomorphic maps on Ω such that the immersion $\tilde{h}_k = (1, h_k^1, \dots, h_k^j, \dots)$, $\tilde{h}_k \colon \Omega \to \mathbb{C}P^\infty$, satisfies $\tilde{h}_k^* \omega_{FS} = k\omega_B$, i.e.

$$1 + \sum_{j=1}^{\infty} |h_k^j|^2 = \tilde{K}^{-k}.$$

Proof. The proof follows essentially that of [14, Lemma 8] once considered that $\Phi(z,w) = -\log(\tilde{K}(z,z)^{-\mu} - |w|^2)$ is the diastasis function for $(M_\Omega(\mu), g(\mu))$ as follows readily applying (2.1). ☐

Observe that such map is full, as can be easily seen for example by considering that for any $m = 1, 2, 3, \dots,$ the subsequence $\{s_1, \dots, s_m\}$ is composed by linearly independent functions.

As a consequence of theorems 1, 2, 6 and Lemma 7, we get the following:

Corollary 8. *For any $\mu > 0$, a Bergman–Hartogs domain $(M_\Omega(\mu), g(\mu))$ is strongly not relative to any projective manifold.*

Proof. Observe first that due to Th. 3 it is enough to prove that $(M_\Omega(\mu), \alpha g(\mu))$ is not relative to $\mathbb{C}P^n$ for any finite n. Further, by Th. 1 and Th. 6, a common submanifold S of both $(M_\Omega(\mu), \alpha g(\mu))$ and $\mathbb{C}P^n$ is not contained into $(\Omega, \alpha g(\mu)|_\Omega)$, since $\alpha g(\mu)|_\Omega = \frac{\alpha\mu}{\gamma} g_B$ is a multiple of the Bergman metric on Ω. Thus, due to arguments totally similar to those in the proof of Th. 2, it is enough to check that the Kähler immersion $f\colon M_\Omega(\mu) \to \mathbb{C}P^\infty$ is transversally full with respect to the w coordinate. Conclusion follows then by (4.5). ☐

Finally, we describe what we need about the 1-parameter family of Fock–Bargmann–Hartogs domains, referring the reader to [1] and reference therein for details and further results. For any value of $\mu > 0$, a Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is a strongly pseudoconvex, nonhomogeneous unbounded domains in $\mathbb{C}^{n+m}$ with smooth real-analytic boundary, given by:

$$D_{n,m}(\mu) := \{(z,w) \in \mathbb{C}^{n+m} : ||w||^2 < e^{-\mu||z||^2}\}.$$

One can define a Kähler metric $\omega(\mu;\nu)$, $\nu > -1$ on $D_{n,m}(\mu)$ through the globally defined Kähler potential:

$$\Phi(z,w) := \nu\mu||z||^2 - \log(e^{-\mu||z||^2} - ||w||^2).$$

In [1], E. Bi, Z. Feng and Z. Tu prove that when $n = 1$ and $\nu = -\frac{1}{m+1}$, the metric $\omega(\mu;\nu)$ is infinite projectively induced whenever it is rescaled by a big enough constant. More precisely they prove the following:

Theorem 9 (E. Bi, Z. Feng, Z. Tu). *The metric $\alpha g(\mu;\nu)$ on the Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is balanced if and only if $\alpha > m+n$, $n = 1$, $\nu = -\frac{1}{m+1}$.*

Recall that a balanced Kähler metric is a particular projectively induced metric such that the immersion map is defined by a orthonormal basis of a weighted Hilbert space (see e.g. [14]).

In order to apply Th. 2 to Fock–Bargmann–Hartogs domains we need the following lemma:

Lemma 10. *For any $\mu > 0$ and any $\alpha > m+1$, a Fock–Bargmann–Hartogs domain $\left(D_{1,m}(\mu), \alpha\omega(\mu; -\frac{1}{m+1})\right)$ admits a transversally full Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$.*

Proof. A Kähler immersion exists due to Th. 9. In order to see that it is transversally full, observe that when $w_1 = \cdots = w_m = 0$, $\alpha\omega(\mu; -\frac{1}{m+1})|_{M_1}$ is a multiple of the flat metric, and when only one w_j is different from zero $\alpha\omega(\mu; -\frac{1}{m+1})|_{M_j}$ is a multiple of the hyperbolic metric. □

Corollary 11. *For any $\mu > 0$, a Fock–Bargmann–Hartogs domain $\left(D_{1,m}(\mu), \omega(\mu; -\frac{1}{m+1})\right)$ is strongly not relative to any projective manifold.*

Proof. If follows directly from Th. 2 and Lemma 10. □

Acknowledgement: This work has been partially supported by the group G.N.S.A.G.A. of I.N.d.A.M.

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