Complex Manifolds

Research Article

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A family of complex nilmanifolds with infinitely many real homotopy types

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Abstract: We find a one-parameter family of non-isomorphic nilpotent Lie algebras $\mathfrak{g}_a$, with $a \in [0, \infty)$, of real dimension eight with (strongly non-nilpotent) complex structures. By restricting a to take rational values, we arrive at the existence of infinitely many real homotopy types of 8-dimensional nilmanifolds admitting a complex structure. Moreover, balanced Hermitian metrics and generalized Gauduchon metrics on such nilmanifolds are constructed.

Keywords: Nilmanifold, Nilpotent Lie algebra, Complex structure, Hermitian metric, Homotopy theory, Minimal model

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1 Introduction

Let $\mathfrak{g}$ be an even-dimensional real nilpotent Lie algebra. A complex structure on $\mathfrak{g}$ is an endomorphism $J: \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $J^2 = -\mathrm{Id}$ and the integrability condition given by the vanishing of the Nijenhuis tensor, i.e.

$$N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0, \tag{1}$$

for all $X, Y \in \mathfrak{g}$. The classification of nilpotent Lie algebras endowed with such structures has interesting geometrical applications; for instance, it allows to construct complex nilmanifolds and study their geometric properties. Let us recall that a nilmanifold is a compact quotient $\Gamma\backslash G$ of a connected, simply connected, nilpotent Lie group G by a lattice Γ of maximal rank in G. If the Lie algebra $\mathfrak{g}$ of G has a complex structure J, then a compact complex manifold $X = (\Gamma\backslash G, J)$ is defined in a natural way.

The problem of determining which nilpotent Lie algebras admit a complex structure is completely solved only up to dimension 6. In real dimension 4 there are two nilpotent Lie algebras with a complex structure, namely, the abelian Lie algebra and the Lie algebra underlying the Kodaira-Thurston manifold. The classification of 6-dimensional nilpotent Lie algebras having a complex structure was achieved by Salamon in [25], and there are precisely 18 isomorphism classes. In higher dimensions only partial results are known (see [18] and the references therein). In particular, note that there is no classification of real nilpotent Lie algebras of dimension 8.

In [5, 6] Dotti and Fino study the 8-dimensional nilpotent Lie algebras $\mathfrak{g}$ that admit a *hypercomplex* structure, i.e. a pair of anticommuting complex structures $\{J_i\}_{i=1,2}$. They prove that if $\mathfrak{g}$ admits such a

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structure, then $\mathfrak{g}$ must be 2-step nilpotent and have first Betti number $b_1(\mathfrak{g}) \geq 4$. They also find an explicit description of the 8-dimensional nilpotent Lie algebras admitting a hypercomplex structure in terms of families which depend on several real parameters. However, it seems to be unclear whether these families contain an infinite number of pairwise non-isomorphic Lie algebras or not. Motivated by this fact, we address the following more general problem:

Question 1.1. *Do there exist infinite non-isomorphic nilpotent Lie algebras in dimension 8 admitting complex structures?*

In this paper we provide an affirmative answer to this question. In addition, we show that there are infinitely many real homotopy types of 8-dimensional nilmanifolds admitting a complex structure. Moreover, the nilmanifolds that we construct can be endowed with both generalized Gauduchon and balanced Hermitian metrics. It should be noted that our results are based on the theory of strongly non-nilpotent complex structures developed in [18].

The paper is structured as follows. In Section 2 we review the main results on complex structures found in [18]. It turns out that the essentially new complex structures on nilpotent Lie algebras that arise in each even real dimension are those of strongly non-nilpotent type. A complex structure J on a nilpotent Lie algebra $\mathfrak{g}$ is said to be *strongly non-nilpotent* (SnN for short) if the center of $\mathfrak{g}$ does not contain any non-trivial J-invariant ideal. In [18, Theorem 4.1] a structure result is obtained for the 8-dimensional nilpotent Lie algebras admitting an SnN complex structure in terms of the dimensions of their ascending central series. Furthermore, a complete description of the SnN complex geometry is given (see [18, Propositions 4.12, 4.13, and 4.14]).

We make use of [18, Proposition 4.12] to answer Question 1.1. Such result describes the generic complex equations of any SnN complex structure on 8-dimensional nilpotent Lie algebras $\mathfrak{g}$ with ascending central series $\{\mathfrak{g}_k\}_k$ of dimensions $(1, 3, 8)$, $(1, 3, 5, 8)$, $(1, 3, 6, 8)$, or $(1, 3, 5, 6, 8)$ (see Proposition 2.5). In Section 3 we present a specific choice of complex parameters that allows us to construct a family $\mathfrak{g}_a$, $a \in [0, \infty)$, of pairwise non-isomorphic 8-dimensional nilpotent Lie algebras endowed with complex structures (see Theorem 3.2). Note that the ascending central series of each $\mathfrak{g}_a$ is of type $(1, 3, 5, 8)$, thus $\mathfrak{g}_a$ is 4-step nilpotent and has first Betti number $b_1(\mathfrak{g}_a) = 3$.

In Section 4 we consider the family $\mathfrak{g}_a$ with rational values of the parameter a in order to show that in eight dimensions there are infinitely many *real* homotopy types of nilmanifolds admitting a complex structure (see Theorem 4.2). Notice that eight is the lowest dimension where this can occur, since for any even dimension less than or equal to 6, only a finite number of real homotopy types of nilmanifolds exists. Indeed, in six dimensions, Bazzoni and Muñoz prove in [1, Theorem 2] that there are infinitely many *rational* homotopy types of nilmanifolds, but only 34 different real homotopy types. We also compute the de Rham cohomology of the given nilmanifolds N_a, which allows us to show that N_a does not admit any symplectic structure for any a.

The last section is devoted to study the existence of special Hermitian metrics on the nilmanifolds N_a endowed with the strongly non-nilpotent complex structure $J_{a,1}$ found in Section 3. We prove in Theorem 5.2 that the complex nilmanifolds $X_a = (N_a, J_{a,1})$ have Hermitian metrics satisfying the k-th Gauduchon condition for every k [10]. Moreover, X_a also admits balanced [20] (hence, strongly Gauduchon [23, 24]) Hermitian metrics (see Theorem 5.3). Therefore, there are infinitely many real homotopy types of 8-dimensional nilmanifolds with both generalized Gauduchon and balanced metrics.

2 Complex structures on nilpotent Lie algebras

In this section we collect some known results about real nilpotent Lie algebras (NLA for short) endowed with complex structures. In particular, we pay special attention to real dimension 8 when the complex structure is of strongly non-nilpotent type, recalling the main ideas in [18].

A well-known invariant of a Lie algebra $\mathfrak{g}$ is its *ascending central series* $\{\mathfrak{g}_k\}_k$, whose terms are given by

$$\begin{cases} \mathfrak{g}_0 = \{0\}, \text{ and} \\ \mathfrak{g}_k = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{g}_{k-1}\}, \text{ for } k \geq 1. \end{cases} \tag{2}$$

Notice that $\mathfrak{g}_1 = Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is *nilpotent* if there is an integer $s \geq 1$ such that $\mathfrak{g}_k = \mathfrak{g}$, for every $k \geq s$. In such case, the smallest integer s satisfying the condition is called the *nilpotency step* of $\mathfrak{g}$, and the Lie algebra is said to be *s-step nilpotent*.

Let J be a complex structure on an NLA $\mathfrak{g}$, that is, an endomorphism $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ fulfilling $J^2 = -\mathrm{Id}$ and the integrability condition (1). Observe that the terms $\mathfrak{g}_k$ in the series (2) may not be invariant under J. For this reason, a new series $\{\mathfrak{a}_k(J)\}_k$ adapted to the complex structure J is introduced in [3]:

$$\begin{cases} \mathfrak{a}_0(J) = \{0\}, \text{ and} \\ \mathfrak{a}_k(J) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{a}_{k-1}(J) \text{ and } [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{k-1}(J)\}, \text{ for } k \geq 1. \end{cases}$$

This series $\{\mathfrak{a}_k(J)\}_k$ is called the *ascending J-compatible series of* $\mathfrak{g}$. Observe that every $\mathfrak{a}_k(J) \subseteq \mathfrak{g}_k$ is an even-dimensional J-invariant ideal of $\mathfrak{g}$, and $\mathfrak{a}_1(J)$ is indeed the largest subspace of the center $\mathfrak{g}_1$ which is J-invariant.

Depending on the behaviour of the series $\{\mathfrak{a}_k(J)\}_k$, complex structures on NLAs can be classified into different types:

Definition 2.1 ([3, 18]). *A complex structure J on a nilpotent Lie algebra $\mathfrak{g}$ is said to be*

(i) strongly non-nilpotent, *or* SnN *for short, if* $\mathfrak{a}_1(J) = \{0\}$*;*

(ii) quasi-nilpotent, *if* $\mathfrak{a}_1(J) \neq \{0\}$*; moreover, J is called*

(ii.1) nilpotent, *if there exists an integer $t > 0$ such that* $\mathfrak{a}_t(J) = \mathfrak{g}$,

(ii.2) weakly non-nilpotent, *if there is an integer $t > 0$ satisfying* $\mathfrak{a}_t(J) = \mathfrak{a}_l(J)$*, for every $l \geq t$, and* $\mathfrak{a}_t(J) \neq \mathfrak{g}$.

One can see that quasi-nilpotent complex structures on NLAs of a given dimension can be constructed from other complex structures defined on (strictly) lower dimensional NLAs (see [18, Section 2] for details). Therefore, the essentially new complex structures that arise in each even real dimension are those of strongly non-nilpotent type. That is to say, SnN complex structures constitute the remaining piece to completely understand complex geometry on nilpotent Lie algebras.

In real dimension 4 it is well known that SnN complex structures do not exist, whereas in dimension 6 one has the following result:

Theorem 2.2 ([27, 28]). *Let $\mathfrak{g}$ be an NLA of real dimension 6. If $\mathfrak{g}$ admits an SnN complex structure, then the terms of its ascending central series $\{\mathfrak{g}_k\}_k$ have dimensions* $(\dim \mathfrak{g}_k)_k = (1, 3, 6)$ *or* $(1, 3, 4, 6)$.

In fact, all the pairs $(\mathfrak{g}, J)$ with $\dim \mathfrak{g} = 6$ and $\mathfrak{a}_1(J) = \{0\}$ have been classified by means of their complex structure equations. It should be noted that only two NLAs of this dimension admit SnN complex structures, namely, $\mathfrak{h}_{19}^-$ and $\mathfrak{h}_{26}^+$ in the notation of [27, 28].

Concerning higher dimensions, [18] provides several general restrictions on the terms of the ascending central series of NLAs admitting SnN complex structures. Among them, we highlight the following one:

Theorem 2.3 ([18, Theorem 3.11]). *Let $(\mathfrak{g}, J)$ be a $2n$-dimensional nilpotent Lie algebra, with $n \geq 4$, endowed with a strongly non-nilpotent complex structure J. Then,* $1 \leq \dim \mathfrak{g}_1 \leq n - 3$.

Thanks to this result and using the *doubly adapted basis method* developed in [18], a structural result in the spirit of Theorem 2.2 is proved for dimension 8:

Theorem 2.4 ([18, Theorem 4.1]). *Let $\mathfrak{g}$ be an NLA of real dimension 8. If $\mathfrak{g}$ admits an SnN complex structure, then the terms of its ascending central series $\{\mathfrak{g}_k\}_k$ have dimensions* $(\dim \mathfrak{g}_k)_k = (1, 3, 8)$, $(1, 3, 5, 8)$, $(1, 3, 6, 8)$, $(1, 3, 5, 6, 8)$, $(1, 4, 8)$, $(1, 4, 6, 8)$, $(1, 5, 8)$, *or* $(1, 5, 6, 8)$.

Moreover, the complex structure equations of all the previous pairs $(\mathfrak{g}, J)$ with $\dim \mathfrak{g} = 8$ and $\mathfrak{a}_1(J) = \{0\}$ are parametrized (see [18, Propositions 4.12, 4.13, and 4.14]). For the aim of this paper, we focus on the proposition:

Proposition 2.5 ([18, Proposition 4.12])**.** *Let J be a strongly non-nilpotent complex structure on an 8-dimensional nilpotent Lie algebra $\mathfrak{g}$ such that $\dim \mathfrak{g}_1 = 1$ and $\dim \mathfrak{g}_2 = 3$. Then, there is a complex basis $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ of bidegree $(1,0)$ with respect to J satisfying the structure equations*[1]

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = A\,\omega^{1\bar{1}} - B(\omega^{14} - \omega^{1\bar{4}}), \\ d\omega^3 = (C-D)\,\omega^{12} - E\,(\omega^{14} - \omega^{1\bar{4}}) + F\,\omega^{1\bar{1}} + (G+D)\,\omega^{1\bar{2}} - H\,(\omega^{24} - \omega^{2\bar{4}}) + (C-G)\,\omega^{2\bar{1}} + K\,\omega^{2\bar{2}}, \\ d\omega^4 = L\,\omega^{1\bar{1}} + M\,\omega^{1\bar{2}} + N\,\omega^{1\bar{3}} - \bar{M}\,\omega^{2\bar{1}} + i\,s\,\omega^{2\bar{2}} + P\,\omega^{2\bar{3}} - \bar{N}\,\omega^{3\bar{1}} - \bar{P}\,\omega^{3\bar{2}}, \end{cases}$$

for some coefficients $s \in \mathbb{R}$ and $A, B, C, D, E, F, G, H, K, L, M, N, P \in \mathbb{C}$.

Notice that the coefficients above must fulfill the (non-linear) equations that guarantee the Jacobi identity of the Lie algebra, i.e. $d(d\omega^k) = 0$ for $1 \le k \le 4$.

3 A family of non-isomorphic 8-dimensional nilpotent Lie algebras with complex structures

In this section we find an infinite family of (non-isomorphic) 8-dimensional nilpotent Lie algebras admitting complex structures, providing an affirmative answer to Question 1.1.

In the complex equations given in Proposition 2.5, we choose the following particular values of the parameters:

$$A = E = K = L = P = 0, \quad B = N = s = 1, \quad C = \frac{i}{2}, \quad D = \frac{3\,i}{2}, \quad F = a, \quad G = -\frac{i}{2}, \quad H = -i, \quad M = ib,$$

where $a, b \in \mathbb{R}$. That is to say, we consider the complex structure equations

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\omega^{14} + \omega^{1\bar{4}}, \\ d\omega^3 = a\,\omega^{1\bar{1}} - i\,(\omega^{12} - \omega^{1\bar{2}} - \omega^{2\bar{1}}) + i\,(\omega^{24} - \omega^{2\bar{4}}), \\ d\omega^4 = i\,b\,(\omega^{1\bar{2}} + \omega^{2\bar{1}}) + \omega^{1\bar{3}} + i\,\omega^{2\bar{2}} - \omega^{3\bar{1}}. \end{cases} \tag{3}$$

It is easy to see that the Jacobi identity holds, so for each pair $(a, b) \in \mathbb{R}^2$, these structure equations define a nilpotent Lie algebra of real dimension 8 endowed with a complex structure. We will study the underlying real nilpotent Lie algebras and show that there is an infinite number of them.

Let $\{e^i\}_{i=1}^8$ be the real basis determined by

$$e^1 = -2\,\mathfrak{Re}\,\omega^1, \quad e^2 = 2\,\mathfrak{Im}\,\omega^1, \quad e^3 = 2\,\mathfrak{Im}\,\omega^4, \quad e^4 = 2\,\mathfrak{Im}\,\omega^2,$$
$$e^5 = 2\,\mathfrak{Re}\,\omega^2, \quad e^6 = -4\,\mathfrak{Re}\,\omega^3, \quad e^7 = 4\,\mathfrak{Im}\,\omega^3, \quad e^8 = 4\,\mathfrak{Re}\,\omega^4.$$

Equivalently,

$$\omega^1 = \frac{1}{2}\,(-e^1 + i\,e^2), \quad \omega^2 = \frac{1}{2}\,(e^5 + i\,e^4), \quad \omega^3 = \frac{1}{4}\,(-e^6 + i\,e^7), \quad \omega^4 = \frac{1}{4}\,(e^8 + 2i\,e^3).$$

[1] NOTATION: $\omega^{ij} = \omega^i \wedge \omega^j$ and $\omega^{i\bar{k}} = \omega^i \wedge \omega^{\bar{k}}$, where $\omega^{\bar{k}}$ is the conjugate of ω^k.

Then, for any $(a,b) \in \mathbb{R}^2$, one has a real nilpotent Lie algebra $\mathfrak{g}_{a,b}$ defined by the following structure equations coming from (3):

$$\mathfrak{g}_{a,b}: \begin{cases} de^1 = de^2 = de^3 = 0, \\ de^4 = e^{13}, \\ de^5 = e^{23}, \\ de^6 = 3\,e^{14} + e^{25} - 2\,e^{35}, \\ de^7 = 2\,a\,e^{12} + e^{15} + e^{24} + 2\,e^{34}, \\ de^8 = -2b\,e^{14} + e^{16} - 2b\,e^{25} + e^{27} - 2\,e^{45}, \end{cases} \tag{4}$$

where $e^{ij} = e^i \wedge e^j$. The integrable almost complex structure on $\mathfrak{g}_{a,b}$ is expressed in this real basis as

$$J_{a,b}(e^1) = e^2, \quad J_{a,b}(e^3) = \frac{1}{2}e^8, \quad J_{a,b}(e^4) = e^5, \quad J_{a,b}(e^6) = e^7. \tag{5}$$

In the next lines, we will prove the following result:

Proposition 3.1. *If the nilpotent Lie algebras $\mathfrak{g}_{a,b}$ and $\mathfrak{g}_{a',b'}$ are isomorphic, then there is a non-zero real number ρ such that*

$$a' = \pm\,\rho\,a, \quad \text{and} \quad b' = \rho\,b.$$

As a consequence, we are able to provide a positive answer to Question 1.1:

Theorem 3.2. *The real Lie algebras $\mathfrak{g}_{a,1}$, $a \in [0,\infty)$, define a family of pairwise non-isomorphic nilpotent Lie algebras admitting complex structures.*

Proof. Using Proposition 3.1 for $b' = b = 1$ and for non-negative values of a and a', we see that if $\mathfrak{g}_{a,1}$ is isomorphic to $\mathfrak{g}_{a',1}$ then $\rho = 1$ and $a' = a$. Hence, different values of a in $[0,\infty)$ produce non-isomorphic Lie algebras $\mathfrak{g}_{a,1}$. Since $J_{a,1}$ in (5) is an integrable almost complex structure on $\mathfrak{g}_{a,1}$, any nilpotent Lie algebra in this family has complex structures. □

Remark 3.3. *Any Lie algebra $\mathfrak{g}_{a,b}$ is isomorphic to one and only one of the following: $\mathfrak{g}_{0,0}$, $\mathfrak{g}_{1,0}$, or $\mathfrak{g}_{a,1}$, with $a \in [0,\infty)$. Indeed:*

- *If $a \neq 0$ and $b = 0$, then $\mathfrak{g}_{a,0}$ is isomorphic to $\mathfrak{g}_{1,0}$. In fact, it is enough to apply to the equations (4) the isomorphism given by* $\operatorname{diag}(a^{-1}, a^{-1}, a^{-1}, a^{-2}, a^{-2}, a^{-3}, a^{-3}, a^{-4}) \in \mathrm{GL}(8,\mathbb{R})$.
- *If $b \neq 0$, then the Lie algebra $\mathfrak{g}_{a,b}$ is isomorphic to $\mathfrak{g}_{|\frac{a}{b}|,1}$. To see this, we first apply to the equations (4) the transformation defined by* $\operatorname{diag}(b^{-1}, b^{-1}, b^{-1}, b^{-2}, b^{-2}, b^{-3}, b^{-3}, b^{-4}) \in \mathrm{GL}(8,\mathbb{R})$ *in order to get an isomorphism from $\mathfrak{g}_{a,b}$ to $\mathfrak{g}_{\frac{a}{b},1}$ Now,* $\operatorname{diag}(1,-1,-1,-1,1,-1,1,-1)$ *gives an isomorphism between the Lie algebras $\mathfrak{g}_{\frac{a}{b},1}$ and $\mathfrak{g}_{-\frac{a}{b},1}$.*
- *Finally, applying Proposition 3.1 to the Lie algebras $\mathfrak{g}_{0,0}$, $\mathfrak{g}_{1,0}$, and $\mathfrak{g}_{a,1}$, $a \in [0,\infty)$, one concludes that any two of them are never isomorphic.*

Proof of Proposition 3.1. The rest of this section is devoted to the proof of Proposition 3.1. First, we normalize several coefficients in the structure equations (4) in order to get simpler equations for the Lie algebras. With this aim, let us introduce the new basis $\{v^i\}_{i=1}^8$ given by

$$v^1 = e^1, \quad v^2 = \frac{1}{\sqrt{3}}e^2, \quad v^3 = \frac{2}{\sqrt{3}}e^3, \quad v^4 = \frac{2}{\sqrt{3}}e^4,$$

$$v^5 = \frac{2}{3}e^5, \quad v^6 = \frac{2}{3\sqrt{3}}e^6, \quad v^7 = \frac{2}{3}e^7, \quad v^8 = \frac{2}{3\sqrt{3}}e^8 + \frac{4b}{9\sqrt{3}}e^6.$$

In terms of this basis, the structure equations (4) become

$$\begin{cases} dv^1 = dv^2 = dv^3 = 0, \\ dv^4 = v^{13}, \\ dv^5 = v^{23}, \\ dv^6 = v^{14} + v^{25} - v^{35}, \\ dv^7 = \alpha\, v^{12} + v^{15} + v^{24} + v^{34}, \\ dv^8 = v^{16} - 2\beta\, v^{25} + v^{27} - \beta\, v^{35} - v^{45}, \end{cases} \tag{6}$$

where $\alpha = \frac{4a}{\sqrt{3}}$ and $\beta = \frac{2b}{3}$. From now on, we will denote the real Lie algebra by $\mathfrak{m}_{\alpha,\beta}$ (instead of $\mathfrak{g}_{a,b}$) to indicate that we are considering the structure equations (6) above.

We first observe that the algebras $\mathfrak{m}_{\alpha,\beta}$ share the same values for the usual invariants of nilpotent Lie algebras. More precisely, it can be seen that the dimensions of the terms in the ascending central series are $(1, 3, 5, 8)$, and those of the descending central series are $(8, 5, 3, 1)$. Furthermore, the Lie algebras cannot be distinguished by their cohomology groups (see Section 4 for more details). Hence, we are led to directly study the existence of an isomorphism between the real Lie algebras $\mathfrak{m}_{\alpha,\beta}$ in this family.

Let $\mathfrak{m}_{\alpha,\beta}$ and $\mathfrak{m}_{\alpha',\beta'}$ be two nilpotent Lie algebras defined by the structure equations (6) for (α, β) and (α', β'), respectively. Let $f\colon \mathfrak{m}_{\alpha,\beta} \longrightarrow \mathfrak{m}_{\alpha',\beta'}$ be a homomorphism of Lie algebras. Hence, the dual map $f^*\colon \mathfrak{m}^*_{\alpha',\beta'} \longrightarrow \mathfrak{m}^*_{\alpha,\beta}$ extends to a map $F\colon \bigwedge^* \mathfrak{m}^*_{\alpha',\beta'} \longrightarrow \bigwedge^* \mathfrak{m}^*_{\alpha,\beta}$ that commutes with the differentials, i.e. $F \circ d = d \circ F$.

Suppose that there exists a Lie algebra isomorphism between $\mathfrak{m}_{\alpha,\beta}$ and $\mathfrak{m}_{\alpha',\beta'}$, and let $\{v^k\}_{k=1}^8$ (resp. $\{v'^k\}_{k=1}^8$) be a basis for $\mathfrak{m}^*_{\alpha,\beta}$ (resp. $\mathfrak{m}^*_{\alpha',\beta'}$) satisfying equations (6) with (α, β) (resp. (α', β')). In terms of these bases, any Lie algebra isomorphism is defined by

$$F(v'^i) = \sum_{j=1}^{8} \lambda^i_j v^j, \quad i = 1, \dots, 8,$$

satisfying the conditions

$$F(dv^i) = d(F(v'^i)), \text{ for each } 1 \le i \le 8, \tag{7}$$

where the matrix $\Lambda = (\lambda^i_j)_{i,j=1,\dots,8}$ belongs to $\mathrm{GL}(8, \mathbb{R})$.

In what follows, we will prove that $\lambda^1_1 \neq 0$, and

$$\alpha' = \pm\, \lambda^1_1\, \alpha, \qquad \beta' = \lambda^1_1\, \beta.$$

Notice that this result implies Proposition 3.1, since a, b, a', b' are related to $\alpha, \beta, \alpha', \beta'$ by $\alpha = \frac{4a}{\sqrt{3}}$, $\beta = \frac{2b}{3}$, $\alpha' = \frac{4a'}{\sqrt{3}}$ and $\beta' = \frac{2b'}{3}$.

Lemma 3.4. *The elements $F(v'^i) \in \mathfrak{m}^*_{\alpha,\beta}$ satisfy the following conditions:*

$$\begin{aligned} &F(v'^i) \wedge v^{123} = 0, \quad \textit{for } i = 1, 2, 3; \\ &F(v'^i) \wedge v^{12345} = 0, \quad \textit{for } i = 4, 5; \\ &F(v'^i) \wedge v^{1234567} = 0, \quad \textit{for } i = 6, 7. \end{aligned}$$

In particular, the matrix $\Lambda = (\lambda^i_j)_{i,j} \in \mathrm{GL}(8, \mathbb{R})$ that determines F is a block triangular matrix, and thus

$$\det(\lambda^i_j)_{i,j=1,2,3} \cdot \det(\lambda^i_j)_{i,j=4,5} \cdot \det(\lambda^i_j)_{i,j=6,7} \cdot \lambda^8_8 = \det \Lambda \neq 0.$$

Proof. First, we observe that from the equations (6), the equalities (7) for $i = 1, 2, 3$ simply read as $d(F(v'^i)) = 0$. Performing this calculation, one immediately gets

$$\lambda^i_j = 0, \text{ for } 1 \le i \le 3 \text{ and } 4 \le j \le 8, \tag{8}$$

and thus we have $F(v'^i) \in \langle v^1, v^2, v^3\rangle$, or equivalently $F(v'^i) \wedge v^{123} = 0$, for $i = 1, 2, 3$.

Bearing in mind the values (8), one can see that (7) for $i = 4, 5$ can be written as

$$\begin{aligned} 0 = d(F(v'^i)) - F(dv'^i) = {} & (\lambda_2^{i-3}\lambda_1^3 - \lambda_1^{i-3}\lambda_2^3 + \alpha\lambda_7^i)\, v^{12} + (\lambda_3^{i-3}\lambda_1^3 - \lambda_1^{i-3}\lambda_3^3 + \lambda_4^i)\, v^{13} \\ & + \lambda_6^i v^{14} + \lambda_7^i v^{15} + \lambda_8^i v^{16} + (\lambda_3^{i-3}\lambda_2^3 - \lambda_2^{i-3}\lambda_3^3 + \lambda_5^i)\, v^{23} + \lambda_7^i v^{24} \\ & + (\lambda_6^i - 2\beta\lambda_8^i)\, v^{25} + \lambda_8^i v^{27} + \lambda_7^i v^{34} - (\lambda_6^i + \beta\lambda_8^i)\, v^{35} - \lambda_8^i v^{45}. \end{aligned} \tag{9}$$

As a consequence, the vanishing of the coefficients in v^{13}, v^{14}, v^{15}, v^{16}, and v^{23}, implies that $\lambda_6^i = \lambda_7^i = \lambda_8^i = 0$, and

$$\lambda_4^i = \lambda_1^{i-3}\lambda_3^3 - \lambda_3^{i-3}\lambda_1^3, \qquad \lambda_5^i = \lambda_2^{i-3}\lambda_3^3 - \lambda_3^{i-3}\lambda_2^3, \qquad \text{for } i = 4, 5. \tag{10}$$

Therefore, $F(v'^i) \in \langle v^1, \dots, v^5\rangle$, for $i = 4, 5$, as desired. Note that (10) will be used again in the proof of the next lemma.

Let us now consider $i = 6, 7$ in the equalities (7). It is easy to check that

$$d(F(v'^i)) - F(dv'^i) = \lambda_8^i v^{16} + \xi_i,$$

where $\xi_i \in \langle v^{16}\rangle^\perp \subset \bigwedge^2\langle v^1, \dots, v^8\rangle$. Here, and in what follows, the orthogonal is taken with respect to the inner product in $\bigwedge^2\langle v^1, \dots, v^8\rangle$ defined by declaring the standard basis $\{v^{ij}\}_{1\le i<j\le 6}$ to be orthonormal. Therefore, the equalities (7) for $i = 6, 7$ in particular imply

$$\lambda_8^6 = \lambda_8^7 = 0.$$

Hence, $F(v'^i) \in \langle v^1, \dots, v^7\rangle$, for $i = 6, 7$, and the proof of the lemma is complete. □

We now make use of Lemma 3.4 to prove the following sharper result.

Lemma 3.5. *The elements $F(v'^i) \in \mathfrak{m}^*_{\alpha,\beta}$ additionally satisfy $F(v'^i) \wedge v^i = 0$, for $i = 1, 2, 3$, and*

$$F(v'^4) \wedge v^{1234} = 0, \quad F(v'^5) \wedge v^{1235} = 0, \quad F(v'^6) \wedge v^{123456} = 0, \quad \text{and} \quad F(v'^7) \wedge v^{123457} = 0.$$

Therefore, $\Lambda = (\lambda_j^i)_{i,j} \in \mathrm{GL}(8, \mathbb{R})$ is a lower triangular matrix, and

$$\prod_{i=1}^{8} \lambda_i^i = \det \Lambda \neq 0.$$

Proof. Let us first recall that all the conditions in Lemma 3.4 and those included along its proof must be satisfied in order to ensure that F is an isomorphism of Lie algebras.

A direct calculation shows that

$$d(F(v'^8)) - F(dv'^8) = -(\lambda_3^1\lambda_6^6 + \lambda_3^2\lambda_6^7)\, v^{36} - (\lambda_3^1\lambda_7^6 + \lambda_3^2\lambda_7^7)\, v^{37} + \xi,$$

where $\xi \in \langle v^{36}, v^{37}\rangle^\perp \subset \bigwedge^2\langle v^1, \dots, v^8\rangle$. Now, the condition (7) for $i = 8$ implies the vanishing of the coefficients in v^{36} and v^{37} above, so the following homogeneous system must be satisfied:

$$\begin{pmatrix} \lambda_6^6 & \lambda_6^7 \\ \lambda_7^6 & \lambda_7^7 \end{pmatrix} \begin{pmatrix} \lambda_3^1 \\ \lambda_3^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since by Lemma 3.4 one has $\det(\lambda_j^i)_{i,j=6,7} \neq 0$, we conclude that $\lambda_3^1 = \lambda_3^2 = 0$. Therefore,

$$F(v'^i) \in \langle v^1, v^2\rangle, \text{ for } i = 1, 2. \tag{11}$$

By the equations (9) for $i = 4, 5$, one observes that the vanishing of the coefficient in the term v^{12} gives the following homogeneous system

$$\begin{pmatrix} \lambda_2^1 & -\lambda_1^1 \\ \lambda_2^2 & -\lambda_1^2 \end{pmatrix} \begin{pmatrix} \lambda_1^3 \\ \lambda_2^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By Lemma 3.4 and (11) we get $\det(\lambda_j^i)_{i,j=1,2}\cdot\lambda_3^3 = \det(\lambda_j^i)_{i,j=1,2,3} \neq 0$, hence $\lambda_1^3 = \lambda_2^3 = 0$. Consequently,

$$F(v'^3) \in \langle v^3\rangle, \tag{12}$$

as stated in the lemma. Now, (10) reduces to

$$\lambda_4^4 = \lambda_1^1\,\lambda_3^3, \qquad \lambda_5^4 = \lambda_2^1\,\lambda_3^3, \qquad \lambda_4^5 = \lambda_1^2\,\lambda_3^3, \qquad \lambda_5^5 = \lambda_2^2\,\lambda_3^3. \tag{13}$$

A direct computation bearing in mind (11) and (12) gives

$$\begin{aligned} d(F(v'^6)) - F(dv'^6) &= \left(\lambda_6^6 - \lambda_2^1\,\lambda_5^4 - \lambda_2^2\,\lambda_5^5\right)v^{25} + \left(\lambda_7^6 + \lambda_3^3\,\lambda_5^4\right)v^{34} - \left(\lambda_6^6 - \lambda_3^3\,\lambda_5^5\right)v^{35} + \vartheta_1,\\ d(F(v'^7)) - F(dv'^7) &= \left(\lambda_6^7 - \lambda_2^1\,\lambda_5^5 - \lambda_2^2\,\lambda_5^4\right)v^{25} + \left(\lambda_7^7 - \lambda_3^3\,\lambda_4^4\right)v^{34} - \left(\lambda_6^7 + \lambda_3^3\,\lambda_5^4\right)v^{35} + \vartheta_2, \end{aligned}$$

where $\vartheta_1, \vartheta_2 \in \langle v^{25}, v^{34}, v^{35}\rangle^\perp \subset \bigwedge^2\langle v^1,\dots,v^8\rangle$. Therefore, the equalities (7) for $i = 6, 7$ in particular imply that the coefficients in v^{34} and v^{35} above are zero, and using (13) we obtain

$$\lambda_6^6 = \lambda_2^2\,(\lambda_3^3)^2, \qquad \lambda_7^6 = -\lambda_1^2\,(\lambda_3^3)^2, \qquad \lambda_6^7 = -\lambda_2^1\,(\lambda_3^3)^2, \qquad \lambda_7^7 = \lambda_1^1\,(\lambda_3^3)^2. \tag{14}$$

Furthermore, the coefficients in v^{25} must also vanish, so taking into account (13) together with the fact that $\lambda_3^3 \neq 0$, we arrive at the following equations:

$$(\lambda_2^1)^2 + (\lambda_2^2)^2 - \lambda_2^2\,\lambda_3^3 = 0, \qquad \lambda_2^1\,(\lambda_3^3 + 2\,\lambda_2^2) = 0.$$

We observe that $\lambda_2^1 = 0$. Indeed, if λ_2^1 is non-zero, then the second expression gives $\lambda_3^3 = -2\,\lambda_2^2$ and replacing it in the first one, we arrive at $\det(\lambda_j^i)_{i,j=1,2} = 0$, which is a contradiction. Hence, one must have $\lambda_2^1 = 0$, which in turn implies $\lambda_5^4 = \lambda_6^7 = 0$ and thus allows us to ensure that

$$F(v'^1) \in \langle v^1\rangle, \qquad F(v'^4) \in \langle v^1,\dots,v^4\rangle, \quad \text{and} \quad F(v'^7) \in \langle v^1,\dots,v^5,v^7\rangle,$$

as desired. In particular, $\lambda_1^1 \neq 0$.

Finally, using (13) we get

$$d(F(v'^7)) - F(dv'^7) = -\left(\lambda_1^1\,\lambda_4^5 + \lambda_1^2\,\lambda_4^4\right)v^{14} + \zeta = -2\,\lambda_1^1\,\lambda_1^2\,\lambda_3^3\,v^{14} + \zeta,$$

where $\zeta \in \langle v^{14}\rangle^\perp \subset \bigwedge^2\langle v^1,\dots,v^8\rangle$. Now, the equality (7) for $i = 7$ implies $\lambda_1^2 = 0$, and consequently, also $\lambda_4^5 = \lambda_7^6 = 0$. Therefore,

$$F(v'^2) \in \langle v^2\rangle, \qquad F(v'^5) \in \langle v^1, v^2, v^3, v^5\rangle, \quad \text{and} \quad F(v'^6) \in \langle v^1,\dots,v^6\rangle,$$

concluding the proof of the lemma. □

We next make use of the previous result to relate the values of the parameters (α,β) and (α',β').

Lemma 3.6. *If the nilpotent Lie algebras* $\mathfrak{m}_{\alpha,\beta}$ *and* $\mathfrak{m}_{\alpha',\beta'}$ *are isomorphic, then*

$$\alpha' = \pm\lambda_1^1\,\alpha, \qquad \beta' = \lambda_1^1\,\beta.$$

Proof. Let us first recall that all the conditions in Lemma 3.5 and in its proof must be satisfied in order to have an isomorphism F of the Lie algebras $\mathfrak{m}_{\alpha,\beta}$ and $\mathfrak{m}_{\alpha',\beta'}$. In particular, the expressions (13) and (14) are reduced to

$$\lambda_4^4 = \lambda_1^1\,\lambda_3^3, \qquad \lambda_5^4 = \lambda_2^2\,\lambda_3^3, \qquad \lambda_6^6 = \lambda_2^2\,(\lambda_3^3)^2, \qquad \lambda_7^7 = \lambda_1^1\,(\lambda_3^3)^2. \tag{15}$$

In order to prove the lemma, we need to study more deeply the conditions (7) for $i = 6, 7, 8$ (notice that (7) are trivially fulfilled for $1 \le i \le 5$).

Let us start with $i = 6$. A direct computation applying (15) gives us

$$d(F(v'^6)) - F(dv'^6) = \lambda_3^3\left[\lambda_2^2\,\lambda_3^3 - (\lambda_1^1)^2\right]v^{14} + \lambda_2^2\,\lambda_3^3\left(\lambda_3^3 - \lambda_2^2\right)v^{25} + \xi,$$

where $\xi \in \langle v^{14}, v^{25}\rangle^{\perp} \subset \bigwedge^2\langle v^1, \dots, v^8\rangle$. Since $\lambda_2^2\,\lambda_3^3 \neq 0$, from the condition (7) for $i = 6$ it follows that $\lambda_3^3 = \lambda_2^2$ and $(\lambda_2^2)^2 = (\lambda_1^1)^2$, that is,

$$\lambda_3^3 = \lambda_2^2 = \pm\lambda_1^1. \tag{16}$$

Let us remark that this greatly simplifies (15), which reduces to

$$\lambda_4^4 = \pm(\lambda_1^1)^2, \qquad \lambda_5^5 = (\lambda_1^1)^2, \qquad \lambda_6^6 = \pm(\lambda_1^1)^3, \qquad \lambda_7^7 = (\lambda_1^1)^3. \tag{17}$$

Furthermore, the conditions (7) for $i = 6, 7$ become

$$\begin{aligned}
0 = d(F(v'^6)) - F(dv'^6) &= -\lambda_1^1\left[\lambda_2^4 \mp \lambda_1^5\right] v^{12} + \left[\lambda_4^6 - \lambda_1^1\,(\lambda_3^4 \pm \lambda_1^5)\right] v^{13} + \left[\lambda_5^6 \mp \lambda_1^1\,(\lambda_2^5 + \lambda_3^5)\right] v^{23},\\
0 = d(F(v'^7)) - F(dv'^7) &= -\lambda_1^1\left[\lambda_2^5 \mp \lambda_1^4 - \lambda_1^1\,(\alpha\,\lambda_1^1 \mp \alpha')\right] v^{12} + \left[\lambda_4^7 - \lambda_1^1\,(\lambda_3^5 \mp \lambda_1^4)\right] v^{13}\\
&\quad + \left[\lambda_5^7 \pm \lambda_1^1\,(\lambda_2^4 - \lambda_3^4)\right] v^{23}.
\end{aligned}$$

Hence, it is easy to see that

$$\begin{aligned}
&\lambda_1^5 = \pm\lambda_2^4, && \lambda_4^6 = \lambda_1^1\,(\lambda_3^4 + \lambda_2^4), && \lambda_4^7 = \lambda_1^1\,(\lambda_3^5 \mp \lambda_1^4),\\
&\lambda_2^5 = \pm\lambda_1^4 + \lambda_1^1\,(\alpha\,\lambda_1^1 \mp \alpha'), && \lambda_5^6 = \lambda_1^1\left[\lambda_1^4 \pm \lambda_3^5 - \lambda_1^1(\alpha' \mp \alpha\,\lambda_1^1)\right], && \lambda_5^7 = \pm\lambda_1^1\,(\lambda_3^4 - \lambda_2^4).
\end{aligned} \tag{18}$$

Let us note that this makes the conditions (7) to be satisfied for every $1 \le i \le 7$, so we must turn our attention to $i = 8$. Using (16) and (17), we compute

$$\begin{aligned}
d(F(v'^8)) - F(dv'^8) = &\left[\lambda_6^8 - \lambda_1^1\,(\lambda_4^6 \pm \lambda_1^1\,\lambda_1^5)\right] v^{14} + \left[\lambda_7^8 \mp \lambda_1^1\,(\lambda_4^7 + \lambda_1^1\,\lambda_2^5)\right] v^{24}\\
&+ \left[\lambda_6^8 - 2\,\beta\,\lambda_8^8 + \lambda_1^1\left(\mp\lambda_5^7 + \lambda_1^1\,(\lambda_2^4 \pm 2\,\beta'\,\lambda_1^1)\right)\right] v^{25} + \left[\lambda_7^8 \mp \lambda_3^5\,(\lambda_1^1)^2\right] v^{34}\\
&- \left[\lambda_6^8 + \beta\,\lambda_8^8 - (\lambda_1^1)^2\,(\lambda_3^4 \pm \beta'\,\lambda_1^1)\right] v^{35} - \left[\lambda_8^8 \mp (\lambda_1^1)^4\right] v^{45} + \vartheta,
\end{aligned}$$

where $\vartheta \in \langle v^{14}, v^{24}, v^{25}, v^{34}, v^{35}, v^{45}\rangle^{\perp} \subset \bigwedge^2\langle v^1, \dots, v^8\rangle$. Equalling to zero the coefficients in v^{14}, v^{34}, and v^{45}, and bearing in mind (18), one has

$$\lambda_6^8 = (\lambda_1^1)^2\,(\lambda_3^4 + 2\,\lambda_2^4), \qquad \lambda_7^8 = \pm\lambda_3^5\,(\lambda_1^1)^2, \qquad \lambda_8^8 = \pm(\lambda_1^1)^4.$$

Moreover, from these equalities together with the vanishing of the coefficient in v^{35}, it is easy to see that

$$\lambda_2^4 = \pm\frac{\lambda_1^1}{2}\,(\beta' - \beta\,\lambda_1^1).$$

We now replace all these values in the coefficients of v^{24} and v^{25}. Their annihilation gives us the desired relation between (α, β) and (α', β'). □

As we observed above, Lemma 3.6 implies Proposition 3.1 with $\rho = \lambda_1^1 \neq 0$, because a, b, a', b' are related to $\alpha, \beta, \alpha', \beta'$ by $\alpha = \frac{4\,a}{\sqrt{3}}$, $\beta = \frac{2\,b}{3}$, $\alpha' = \frac{4\,a'}{\sqrt{3}}$ and $\beta' = \frac{2\,b'}{3}$. □

4 An infinite family of nilmanifolds N_a with complex structures

In this section we show how the results of the previous section allow us to conclude that in dimension eight there exist infinitely many real homotopy types of nilmanifolds admitting complex structures.

Let us start reviewing some results about minimal models and homotopy theory of nilmanifolds. Sullivan showed in [26] that it is possible to associate to any nilpotent CW-complex X a minimal model, i.e. a commutative differential graded algebra, cdga, $(\bigwedge V_X, d)$ defined over the rational numbers $\mathbb{Q}$ satisfying a certain minimality condition, which encodes the rational homotopy type of X [12].

Recall that a space X is nilpotent if its fundamental group $\pi_1(X)$ is a nilpotent group and acts in a nilpotent way on the higher homotopy groups $\pi_k(X)$ for $k > 1$. These conditions are fulfilled when the space

is a nilmanifold N, i.e. $N = \Gamma\backslash G$ is a compact quotient of a connected, simply connected, nilpotent Lie group G by a lattice Γ of maximal rank. Indeed, for any nilmanifold N one has $\pi_1(N) = \Gamma$, hence nilpotent, and $\pi_k(N) = 0$ for every $k \geq 2$.

A commutative differential graded algebra $(\bigwedge V, d)$ defined over $\mathbb{K}$ ($= \mathbb{Q}$ or $\mathbb{R}$) is said to be minimal if the following conditions are satisfied:

(i) $\bigwedge V$ is the free commutative algebra generated by the graded vector space $V = \oplus V^k$;
(ii) there exists a basis $\{x_j\}_{j\in J}$, for some well-ordered index set J, such that $\deg(x_i) \leq \deg(x_j)$ if $i < j$, and each dx_j is expressed in terms of preceding x_i $(i < j)$.

Given a differentiable manifold M, a $\mathbb{K}$-minimal model of M is a minimal cdga $(\bigwedge V, d)$ over $\mathbb{K}$ together with a quasi-isomorphism ϕ from $(\bigwedge V, d)$ to the $\mathbb{K}$-de Rham complex of M, i.e. a morphism ϕ inducing an isomorphism on cohomology. Note that the $\mathbb{K}$-de Rham complex of M is the usual de Rham complex of differential forms when $\mathbb{K} = \mathbb{R}$, whereas for $\mathbb{K} = \mathbb{Q}$ one considers the $\mathbb{Q}$-polynomial forms instead. Two manifolds M_1 and M_2 have the same $\mathbb{K}$-homotopy type if and only if their $\mathbb{K}$-minimal models are isomorphic [4, 26]. Clearly, if M_1 and M_2 have different real homotopy types, then M_1 and M_2 also have different rational homotopy types.

Let $N = \Gamma\backslash G$ be a nilmanifold of dimension n and denote by $\mathfrak{g}$ the Lie algebra of G. The minimal model of $N = \Gamma\backslash G$ is given by the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, d)$ of the nilpotent Lie algebra $\mathfrak{g}$. Indeed, according to Mal'cev [19], the existence of the lattice Γ of maximal rank in G is equivalent to the nilpotent Lie algebra $\mathfrak{g}$ being rational. The latter condition is in turn equivalent to the existence of a basis $\{e^1, \dots, e^n\}$ for the dual $\mathfrak{g}^*$ for which the structure constants are rational numbers. Since the Lie algebra $\mathfrak{g}$ is nilpotent, one can take a basis $\{e^1, \dots, e^n\}$ for $\mathfrak{g}^*$ as above satisfying

$$de^1 = de^2 = 0, \quad de^j = \sum_{i,k<j} a^j_{ik}\, e^i \wedge e^k \quad \text{for } j = 3, \dots, n,$$

where $a^j_{ik} \in \mathbb{Q}$. Therefore, the cdga $(\bigwedge \mathfrak{g}^*, d)$ over $\mathbb{Q}$ is minimal, because it satisfies *(i)* and *(ii)* just taking $J = \{1, \dots, n\}$ and $V = V^1 = \langle x_1, \dots, x_n\rangle = \sum_{j=1}^n \mathbb{Q}x_j$, where $x_j = e^j$ for $1 \leq j \leq n$. That is to say, there are n generators $x_1, \dots, x_n$ of degree 1, and one has the $\mathbb{Q}$-minimal cdga

$$\Big(\bigwedge\langle x_1, \dots, x_n\rangle,\ d\Big). \tag{19}$$

Now, the cdga $(\bigwedge \mathfrak{g}^*, d)$ over $\mathbb{R}$ is also minimal, since it is given by

$$\Big(\bigwedge\langle x_1, \dots, x_n\rangle \otimes \mathbb{R},\ d\Big). \tag{20}$$

Nomizu proved in [21] that the canonical morphism ϕ from the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^*, d)$ to the de Rham complex $(\Omega^*(\Gamma\backslash G), d)$ induces an isomorphism on cohomology. Therefore, the $\mathbb{R}$-minimal model of the nilmanifold $N = \Gamma\backslash G$ is given by (20). As it was observed by Hasegawa in [13], the cdga (19) is also the $\mathbb{Q}$-minimal model of N. Moreover, by using a result in [4] asserting that a $\mathbb{K}$-minimal model, $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$, of a compact birational Kähler manifold is formal, Hasegawa proved that an even-dimensional nilmanifold does not admit any Kähler metric unless it is a torus, by showing that the minimal model (19) is never formal. (For more results on rational homotopy theory see [8], and for more applications to symplectic geometry see [22].)

It is proved in [1] that, up to dimension 5, the number of rational homotopy types of nilmanifolds is finite. However, in six dimensions the following result holds:

Theorem 4.1 ([1, Theorem 2])**.** *There are* 34 *real homotopy types of* 6*-dimensional nilmanifolds, and infinitely many rational homotopy types of* 6*-dimensional nilmanifolds.*

As an obvious consequence, there is a finite number of real homotopy types of 6-dimensional nilmanifolds admitting a complex structure. In what follows, we will show that for nilmanifolds with complex structures in eight dimensions there are infinitely many real homotopy types. To see this, let us take a non-negative

rational number a and consider the connected, simply connected, nilpotent Lie group G_a corresponding to the nilpotent Lie algebra $\mathfrak{g}_{a,1}$ given in Section 3. It follows from (4) that the algebra $\mathfrak{g}_{a,1}$ is rational, hence by Mal'cev [19], there exists a lattice Γ_a of maximal rank in G_a. We denote by $N_a = \Gamma_a \backslash G_a$ the corresponding compact quotient. The nilmanifold N_a admits complex structures, for instance, the strongly non-nilpotent complex structure $J_{a,1}$ defined in (5) for $b = 1$.

Now, as we explained above, the $\mathbb{R}$-minimal model of the nilmanifold N_a is given by the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}_{a,1}^*, d)$ of the nilpotent Lie algebra $\mathfrak{g}_{a,1}$. As a consequence of Theorem 3.2, one has that two such $\mathbb{R}$-minimal models are isomorphic if and only if $a = a'$. Therefore, we have proved the following result:

Theorem 4.2. *There are infinitely many real homotopy types of 8-dimensional nilmanifolds admitting a complex structure.*

A direct calculation using Nomizu's theorem [21] allows to explicitly compute the de Rham cohomology groups of any nilmanifold N_a. In our case, we have:

$$
\begin{aligned}
H^1_{\mathrm{dR}}(N_a) &= \langle\, [e^1],\, [e^2],\, [e^3]\,\rangle,\\
H^2_{\mathrm{dR}}(N_a) &= \langle\, [e^{12}],\, [e^{14}],\, [e^{25}],\, [e^{34}]\,\rangle,\\
H^3_{\mathrm{dR}}(N_a) &= \langle\, [e^{147}],\, [e^{257} - 2\,e^{346}],\, [3\,e^{146} - e^{256} + 2\,e^{356}],\, [e^{156} + e^{246} - 2\,e^{346}],\\
&\qquad [3\,e^{157} + 3\,e^{247} + 2\,e^{256} + 2\,e^{356}],\, [2\,e^{126} + 3\,e^{128} + 4\,e^{237} - 2\,e^{256}],\, [2a\,e^{237} - e^{346} - e^{357}]\,\rangle.
\end{aligned}
$$

By Poincaré duality, the following Betti numbers for N_a are obtained:

$$
b_0(N_a) = b_8(N_a) = 1, \quad b_1(N_a) = b_7(N_a) = 3, \quad b_2(N_a) = b_6(N_a) = 4, \quad b_3(N_a) = b_5(N_a) = 7.
$$

The Betti number $b_4(N_a)$ can be computed by taking into account that the Euler characteristic χ of a nilmanifold always vanishes. Hence,

$$
0 = \chi(N_a) = \sum_{k=0}^{8} (-1)^k b_k(N_a) = b_4(N_a) + 2\left(b_0(N_a) - b_1(N_a) + b_2(N_a) - b_3(N_a)\right),
$$

which implies $b_4(N_a) = 10$.

The second de Rham cohomology group provides the following additional geometric information on the nilmanifolds N_a.

Proposition 4.3. *For each non-negative rational number a, the nilmanifold N_a does not admit any symplectic structure.*

Proof. It is enough to see that any de Rham cohomology class $\mathfrak{a}$ in $H^2_{\mathrm{dR}}(N_a)$ is degenerate. Indeed, since $\mathfrak{a} = \lambda_1[e^{12}] + \lambda_2[e^{14}] + \lambda_3[e^{25}] + \lambda_4[e^{34}]$ it is clear that the cup product $\mathfrak{a} \cup \mathfrak{a} \cup \mathfrak{a}$ vanishes, so $\mathfrak{a}^4 \equiv 0$ and $\mathfrak{a}$ is degenerate. □

In contrast to this result, in the following section we will show that these nilmanifolds have many special Hermitian metrics.

5 Special metrics on the nilmanifolds N_a

We here study the existence of special Hermitian metrics on the nilmanifolds N_a endowed with the strongly non-nilpotent complex structures given in the previous sections. More concretely, we find many generalized Gauduchon metrics as well as balanced Hermitian metrics.

Let us start by recalling the definition and the main properties of the k-th Gauduchon metrics studied in [10].

Definition 5.1 ([10]). *Let $X = (M, J)$ be a compact complex manifold of complex dimension n, and let k be an integer such that $1 \leq k \leq n-1$. A J-Hermitian metric g on X is called* k-th Gauduchon *if its fundamental 2-form F satisfies the condition $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$.*

Clearly, any SKT metric (a J-Hermitian metric g with fundamental form F satisfying $\partial\bar{\partial}F = 0$) is a 1-st Gauduchon metric, and any astheno-Kähler metric (i.e. a J-Hermitian metric g such that its fundamental form F satisfies the condition $\partial\bar{\partial}F^{n-2} = 0$) is an $(n-2)$-th Gauduchon metric. SKT, resp. astheno-Kähler, metrics were first introduced by Bismut in [2], resp. by Jost and Yau in [15], and they have been further studied by many authors. Notice that for $k = n-1$ one gets the standard metrics found in [11].

In [10] an extension of Gauduchon's result for standard metrics is proved. More concretely, it is shown in [10, Corollary 4] that if (M, J, F) is an n-dimensional compact Hermitian manifold, then for any integer $1 \leq k \leq n-1$, there exists a unique constant $\gamma_k(F)$ and a (unique up to a constant) function $v \in C^\infty(M)$ such that

$$\frac{i}{2}\partial\bar{\partial}(e^v F^k) \wedge F^{n-k-1} = \gamma_k(F) e^v F^n.$$

If (M, J, F) is Kähler, then $\gamma_k(F) = 0$ and v is a constant function for any $1 \leq k \leq n-1$. Moreover, for any Hermitian metric on a compact complex manifold one has $\gamma_{n-1}(F) = 0$.

The constant $\gamma_k(F)$ depends smoothly on F (see [10, Proposition 9]). Furthermore, it is proved in [10, Proposition 8] that $\gamma_k(F) = 0$ if and only if there exists a k-th Gauduchon metric in the conformal class of F.

Some compact complex manifolds with generalized Gauduchon metrics are constructed in [9, 10, 14, 17] by different methods. In the following result we find many generalized Gauduchon nilmanifolds of complex dimension 4.

As in Section 4, for each non-negative rational number a, we consider the nilmanifold N_a endowed with the complex structure given by (5) for $b = 1$, i.e. with the complex structure $J_{a,1}$. Hence, $(N_a, J_{a,1})$ is a compact complex manifold of complex dimension 4. We will denote this complex nilmanifold by X_a, i.e. $X_a = (N_a, J_{a,1})$.

Theorem 5.2. *For each non-negative rational number a, the complex nilmanifold X_a has a k-th Gauduchon metric, for every $1 \leq k \leq 3$. Therefore, there are infinitely many real homotopy types of 8-dimensional k-th Gauduchon nilmanifolds, for every $1 \leq k \leq 3$.*

Proof. By Theorem 4.2, we know that the nilmanifolds N_a and $N_{a'}$ have different real homotopy types for $a \neq a'$. The complex structure equations for $X_a = (N_a, J_{a,1})$ are given by (3) with $b = 1$, i.e.

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = -\omega^{14} + \omega^{1\bar{4}}, \\ d\omega^3 = a\,\omega^{1\bar{1}} - i\,(\omega^{12} - \omega^{1\bar{2}} - \omega^{2\bar{1}}) + i\,(\omega^{24} - \omega^{2\bar{4}}), \\ d\omega^4 = i\,(\omega^{1\bar{2}} + \omega^{2\bar{1}}) + \omega^{1\bar{3}} + i\,\omega^{2\bar{2}} - \omega^{3\bar{1}}. \end{cases} \tag{21}$$

We define the $(1,1)$-form F on X_a given by

$$F = \frac{101}{4}\,i\,\omega^{1\bar{1}} + i\,\omega^{2\bar{2}} + i\,\omega^{3\bar{3}} + \frac{1}{2}\,i\,\omega^{4\bar{4}} - 5\,\omega^{1\bar{3}} + 5\,\omega^{3\bar{1}}.$$

It is easy to check that F defines a positive-definite metric on X_a, hence a Hermitian metric on X_a. Furthermore, a direct calculation using the structure equations (21) shows that

$$\begin{aligned} \partial\bar{\partial}F = &-4\,i\,\omega^{12\bar{1}\bar{2}} - (1-6\,i)\,\omega^{12\bar{1}\bar{3}} + a\,\omega^{12\bar{1}\bar{4}} - (1-i)\,\omega^{12\bar{2}\bar{3}} + (1+6\,i)\,\omega^{13\bar{1}\bar{2}} - i\,\omega^{13\bar{1}\bar{3}} \\ &- \omega^{13\bar{2}\bar{3}} - a\,\omega^{14\bar{1}\bar{2}} - 12\,i\,\omega^{14\bar{1}\bar{4}} + \omega^{14\bar{3}\bar{4}} + (1+i)\,\omega^{23\bar{1}\bar{2}} + \omega^{23\bar{1}\bar{3}} - 2\,i\,\omega^{24\bar{2}\bar{4}} - \omega^{34\bar{1}\bar{4}}. \end{aligned}$$

Therefore, one has $\partial\bar{\partial}F \wedge F^2 = 0$ for any a, i.e. F is a 1-st Gauduchon metric on the complex nilmanifold X_a for any a.

By [16, Proposition 2.2], if an invariant Hermitian metric F on a complex nilmanifold X of complex dimension $n \geq 4$ is k-th Gauduchon for some $1 \leq k \leq n-2$, then it is k-th Gauduchon for any other k.

Since any invariant Hermitian metric is $(n-1)$-Gauduchon, we conclude that the 1-st Gauduchon metric on X_a constructed above are also k-th Gauduchon for $k = 2, 3$. □

By a direct calculation it can be proved that the compact complex manifolds X_a are not SKT (see [7] for general results) and, moreover, that the manifolds X_a do not admit any *invariant* astheno-Kähler metric.

A particularly interesting class of standard metrics is the one given by *balanced* Hermitian metrics, defined by the condition $dF^{n-1} = 0$. Important aspects of these metrics were first investigated by Michelsohn in [20], and many authors have constructed balanced manifolds and studied their properties since then. In the following result we find many balanced nilmanifolds of complex dimension 4.

Theorem 5.3. *For each non-negative rational number a, the complex nilmanifold X_a has a balanced Hermitian metric. Therefore, there are infinitely many real homotopy types of 8-dimensional balanced nilmanifolds.*

Proof. We will define a balanced Hermitian metric on $X_a = (N_a, J_{a,1})$ depending on the values of the rational number a.

For $a = 0$, we consider the $(1,1)$-form F on X_0 given by

$$2F_0 = i\,(2\,\omega^{1\bar{1}} + \omega^{2\bar{2}} + 4\,\omega^{3\bar{3}} + \omega^{4\bar{4}}) + 2\,\omega^{1\bar{3}} - 2\,\omega^{3\bar{1}}.$$

Hence, F_0 defines a positive-definite metric on X_0. Since F_0 is real, the closedness of F_0^3 is equivalent to the condition

$$\partial F_0 \wedge F_0^2 = 0.$$

Using the structure equations (21) for $a = 0$, we get

$$\begin{aligned} 2\,\partial F_0 = {} & 4\,\omega^{12\bar{3}} - 4\,\omega^{13\bar{2}} + (i-1)\omega^{14\bar{2}} + i\,\omega^{14\bar{3}} - 4\,\omega^{23\bar{1}} \\ & -(i+1)\omega^{24\bar{1}} - \omega^{24\bar{2}} - 4\,\omega^{24\bar{3}} - i\,\omega^{34\bar{1}} - 4\,\omega^{34\bar{2}}. \end{aligned}$$

One can check that $\partial F_0 \wedge F_0^2 = 0$, i.e. $dF_0^3 = 0$, so F_0 is a balanced Hermitian metric on the complex nilmanifold X_0.

For any rational number $a > 0$, we define the $(1,1)$-form F_a on X_a given by

$$2F_a = i\left(a(a+1)\omega^{1\bar{1}} + \omega^{2\bar{2}} + \omega^{3\bar{3}} + 2\,\omega^{4\bar{4}}\right) + a\,\omega^{1\bar{3}} - a\,\omega^{3\bar{1}} + \omega^{2\bar{4}} - \omega^{4\bar{2}}.$$

It is clear that F_a defines a Hermitian metric on X_a $(a > 0)$. A direct calculation using the structure equations (21) shows that

$$\begin{aligned} 2\,\partial F_a = {} & i\,\omega^{12\bar{2}} + 2\,\omega^{12\bar{3}} - ai\,\omega^{13\bar{1}} - \omega^{13\bar{2}} - (2 - i(a-1))\,\omega^{14\bar{2}} + 2i\,\omega^{14\bar{3}} \\ & -\omega^{14\bar{4}} - (2 + i(a-1))\,\omega^{24\bar{1}} - 2\,\omega^{24\bar{2}} - \omega^{24\bar{3}} - 2i\,\omega^{34\bar{1}} - \omega^{34\bar{2}}. \end{aligned}$$

Moreover, one can check that $\partial F_a \wedge F_a^2 = 0$, and consequently $dF_a^3 = 0$. Therefore, F_a is a balanced metric on the complex nilmanifold X_a for any positive rational number a. □

In [23, 24] Popovici has introduced and investigated the class of *strongly Gauduchon* metrics F, which are defined by the condition $\partial F^{n-1} = \bar{\partial}\gamma$, for some complex form γ of bidegree $(n, n-2)$ on a compact complex manifold X of complex dimension n. It is clear by definition that any balanced metric is strongly Gauduchon, and any strongly Gauduchon metric is standard.

Hence, as an immediate consequence of Theorem 5.3 we get:

Corollary 5.4. *There are infinitely many real homotopy types of 8-dimensional strongly Gauduchon nilmanifolds.*

It is well known that the existence of a Kähler metric on an even dimensional nilmanifold reduces the nilmanifold to be a torus, so the underlying Lie algebra is abelian and the minimal model is formal [13]. More generally, it seems natural to ask if the existence of other geometric structures on nilmanifolds imposes

some restriction on their real homotopy type. For instance, as mentioned in the introduction, it is an interesting question if the number of real homotopy types of 8-dimensional nilmanifolds admitting an invariant hypercomplex structure is finite or not. In addition to this problem, the answers to the following questions are, to our knowledge, unknown:

Question A. *Are there infinitely many real homotopy types of* 8*-dimensional SKT nilmanifolds?*

Question B. *Are there infinitely many real homotopy types of* 8*-dimensional astheno-Kähler nilmanifolds?*

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