The Laguerre spectral method as applied to numerical solution of a two-dimensional linear dynamic seismic problem for porous media

1 Introduction

In order to identify geological structures, the seismic methods based on the analysis of seismic waves propagation in an acoustic or in an ideally elastic medium, were successfully applied to various geophysical problems. In such studies, properties of a pore liquid such as density, module of volumetric deformation, fluid saturation and viscosity were generally neglected. A porous medium, as an elastic deformable matrix filled with a viscous liquid is a more realistic model which allows us to explain observable effects of seismic research into properties of rocks in the presence of a saturating liquid. Recently, the numerical simulation of seismic wave propagation in fluid saturated porous media, has received a special attention because of its practical application in various areas of geophysics, biomechanics and oil reservoir characterization. Generally, the model of the Frenkel-Biot type [1, 2] is used as a basis. A characteristic feature of models of this type, along with transverse and longitudinal (fast) seismic waves, is the presence of slow longitudinal wave. In the models [1, 2], the velocities of propagation of such waves are functions of four elastic parameters for preset values of physical density of a solid matrix, a saturating liquid and porosity. A nonlinear mathematical model for porous media the so-called Continual filtration model, based on the common physical principles, was constructed [3-5]. Just as in the Frenkel-Biot model [1,2], in the model [3] there are three types of the sound oscillations: transverse and two types of longitudinal. As opposed to the models [1,2], in the linearized version of the model [3] a saturated porous medium is described by three elastic parameters [4,5]. These elastic parameters in one-to-one correspondence are expressed by three velocities of seismic wave propagation. This circumstance is important for the numerical modelling of elastic waves in a porous medium when distributions of velocities of acoustic waves and physical densities of the matrix, saturated liquids and porosity are known. The finite difference methods of solving problems for the Frenkel-Biot equations have been for-
mulated in several ways, these are: the central finite difference method in terms of displacement [6,7], the predictor-corrector finite difference method for the velocity-stress system of equations [8]. The semi-analytical method for the Biot equation system in terms of displacement is proposed in [9,10]. In this paper, we discuss the solution of the initial boundary value problem for a hyperbolic system of equations of porous media [4, 5] in reversible approximation. The initial boundary value problem when using the Laguerre transform with respect to \( t \) is reduced to a boundary value problem for a system of equations of first order with respect to the velocities of the elastic matrix, the saturating fluid, the stress tensor, and the pore pressure. The resulting boundary value problem is solved by the finite difference scheme. This method of solving initial boundary value problems for non-stationary problems of elasticity theory was first considered in [11, 12], and then adapted for initial value problems of viscoelasticity [13, 14]. This method can be considered to be an analog to the well-known Fourier-Laplace method for solving evolution problems. In this method, the analog frequency (the dual variable i) takes a parameter of \( n \) · order of the Laguerre polynomial. The use of the Laguerre transform unlike the Fourier-Laplace transform allows us to reduce the original initial boundary value problem to solving a series of boundary value problems for systems of differential equations depending on a parameter recursive \( n \). The presence of the velocity of a slow longitudinal wave increases the volume of required calculations when one uses a finite difference method to design a non-stationary problem of poroelasticity. The following shows the efficiency of the method proposed to solve the initial boundary value problem for the unsteady system of equations of liquid-saturated porous media described by the three elastic parameters.

2 Statement of problem

Let us assume that the half-plane \( x_2 > 0 \) is filled with a fluid-saturated porous medium. Then the propagation of seismic waves for the given medium in the reversible hydrodynamic approximation is described by the following initial boundary value problem [4, 5, 15]:

\[
\begin{align*}
\frac{\partial u_i}{\partial t} + \frac{1}{\rho_{0,s}} \sum_{k=1}^{2} \frac{\partial \sigma_{ik}}{\partial x_k} + \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} &= F_i, \\
\frac{\partial v_i}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} &= F_i,
\end{align*}
\]

The Laguerre spectral method as applied to two-dimensional linear dynamic seismic problem

\[
\frac{\partial \sigma_{ik}}{\partial t} + \mu \left( \frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right) + \left( \frac{\rho_{0,l}}{\rho_0} - \frac{2}{3} \mu \right) \delta_{ik} \nabla v + \frac{\rho_{0,s}}{\rho_0} K \delta_{ik} \nabla v = 0, \\
\frac{\partial p}{\partial t} - \left( K - \alpha \frac{\rho_{0,s}}{\rho_0} \right) \nabla u + \alpha \frac{\rho_{0,l}}{\rho_0} \nabla v = 0,
\]

with the zero initial conditions

\[
u_{|t=0} = u_{|t=0} = \sigma_{ik}|_{t=0} = p|_{t=0} = 0, i, k = 1, 2
\]

and the boundary conditions

\[
\sigma_{22} + p|_{x_2=0} = \sigma_{12}|_{x_2=0} = \frac{\rho_{0,l}}{\rho_0} p|_{x_2=0} = 0,
\]

where \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are the vector of the velocities of the solid matrix with the partial density \( \rho_{0,s} \) and of the liquid with the partial density \( \rho_{0,l} \), respectively, \( p \) is the pore pressure, \( h_{ik} \) is the stress tensor, \( F = (F_1, F_2) \) is the vector of body forces, \( \rho_0 = \rho_{0,l} + \rho_{0,s} = \rho_{0,s}(1 - d_0) \), \( \rho_{0,l} = \rho_{0,s} d_0, \rho_{0,l} \) and \( \rho_{0,l} \) are the physical densities of the solid matrix and the liquid respectively, \( d_0 \) is porosity, \( \delta_{ik} \) is the Kronecker delta, \( K = \lambda + 2 \mu/3 \), \( \lambda > 0, \mu > 0, \gamma > 0 \) are elastic moduli, \( \alpha = \gamma + K \) is the second modulus of volumetric compression of a heterophase medium. Elastic moduli \( K, \mu, \gamma \) are expressed through the velocity of the transverse wave \( c_s \) and two velocities of longitudinal waves \( c_{p1}, c_{p2} \) by the following formulas [16, 17]:

\[
\mu = \rho_{0,s} c_s^2,
\]

\[
K = \frac{\rho_{0,s} (c_{p1}^2 - c_{p2}^2)}{\sigma} = \frac{\rho_{0,s} (c_{p1}^2 - c_{p2}^2)}{\rho_0} = \frac{8 \rho_{0,l}}{9} \frac{\rho_{0,s}}{\rho_0} c_s^2
\]

\[
\gamma = \rho_{0,l} \left( c_{p1}^2 + c_{p2}^2 - \frac{8 \rho_{0,l}}{9} \frac{\rho_{0,s}}{\rho_0} c_s^2 \right) + \frac{\rho_{0,l} (c_{p1}^2 - c_{p2}^2)}{\rho_0} = \frac{64 \rho_{0,l}}{9} \frac{\rho_{0,s}}{\rho_0} c_s^2
\]

These formulas allow carrying out the numerical simulation for fluid-saturated porous media when the velocity model structure of the medium is specified.

3 Algorithm of solution

Applying to both parts of system (1) · (6) the Laguerre transform with respect to \( t \)

\[
\mathcal{F}(\mathcal{W}(x_1, x_2)) = \int_0^\infty \mathcal{W}(x_1, x_2, t)(ht)^{-\frac{1}{2}} \mathcal{F}(ht) d(ht)
\]

\[
\mathcal{W}_m(x_1, x_2) = \int_0^\infty \mathcal{W}(x_1, x_2, t)(ht)^{-\frac{1}{2}} \mathcal{F}(ht) d(ht)
\]
we obtain a series of the following boundary value problems with the parameter $m$:

\[
\frac{h}{2} \bar{u}^m + \frac{1}{\rho_s} \frac{\partial \sigma^m_{ik}}{\partial x_k} + \frac{1}{\rho_0} \frac{\partial P^m}{\partial x_i} = f^m_i - h \sum_{n=0}^{m-1} u^m_n, \tag{8}
\]

\[
\frac{h}{2} v^m_i + \frac{1}{\rho_0} \frac{\partial P^m}{\partial x_i} = f^m_i - h \sum_{n=0}^{m-1} v^m_n, \tag{9}
\]

\[
\frac{h}{2} \sigma^m_{ik} + \mu \left( \frac{\partial u^m_i}{\partial x_j} + \frac{\partial u^m_j}{\partial x_i} \right) + \left( \lambda - \frac{\rho_s}{\rho_0} \right) \delta_{ik} \frac{\partial u^m_i}{\partial x_j} \frac{\partial v^m_j}{\partial x_i} = -h \sum_{n=0}^{m-1} \sigma^m_{ik}, \tag{10}
\]

\[
\frac{h}{2} p^m - \left( K - a \frac{\rho_s}{\rho_0} \right) \frac{\partial u^m_i}{\partial x_j} + \frac{a}{\rho_0} \frac{\partial P^m_i}{\partial x_j} \frac{\partial v^m_j}{\partial x_i} = -h \sum_{n=0}^{m-1} p^m, \tag{11}
\]

where the vector $\vec{W}$ is determined from the formula:

\[
\vec{W}(m) = (\vec{V}_0(m), \vec{V}_1(m), ..., \vec{V}_{M+N}(m))^T,
\]

\[
\vec{V}_{i+j} = (u_{i+j}^0, u_{i+j}^1, ..., u_{i+j}^N, v_{i+j}^0, v_{i+j}^1, ..., v_{i+j}^N, \sigma_{i+j}^{1D}, \sigma_{i+j}^{1D}, ..., \sigma_{i+j}^{1D})^T.
\]

This system of algebraic equations is solved using the iterative methods such as the conjugate gradients, converging to the solution with desired accuracy. At this stage of carrying out calculations, a version of the conjugate gradients method has been parallelized. In terms of the input data, when setting a medium model, it is equivalent to decomposition of the initial domain to a set of subdomains equal to the number of processors. This enables one to distribute memory, both in setting input parameters of the model, and in further numerical realization of the algorithm in subdomains.

\section*{4 Examples of calculations}

Figure 1 shows graphs as a function of the degree of the Laguerre polynomial for finite functions of the probing signal $f(t)$ of the form

\[
f(t) = \exp \left[ -\frac{(2\pi f_0(t - t_0))^2}{\gamma^2} \right] \sin(2\pi f_0(t - t_0)),
\]

where $\gamma = 4, f_0 = 1$ Hz, $t_0 = 1.5$ s.

The results of numerical simulation of seismic wave fields for a test model of medium are represented. This model consists of two homogeneous layers: the upper layer is an elastic medium, the lower one is a fluid-saturated porous medium. Physical characteristics of the layers are the following:

- The elastic layer: $\rho = 1.3$ g/cm$^3$, $c_p = 1.5$ km/s, $c_s = 1$ km/s; The porous layer: $\rho_f' = 1.5$ g/cm$^3$, $\rho_p' = 1$ g/cm$^3$, $c_p = 2.1$ km/s, $c_p = 0.5$ km/s, $c_s = 1.4$ km/s, $d_0 = 0.2$.

The upper layer is 3.5 km thick. The wave field was simulated from a point source such as the vertical force with coordinates $x_0 = 3.5$ km, $z_0 = 1.5$ km, being placed in the upper elastic layer. The parameters of the time signal $f(t)$ in the source took the following values: $\gamma = 4, f_0 = 10$ Hz, $t_0 = 0.15$ s. The results of numerical calculations of the wave field for the given model are represented in figures 2 and 3. In the given figures, snapshots of a wave field for $u_x(x, z, t)$ components of the displacement velocity at fixed moments $t = 1$ and $t = 2.2$ seconds are represented. Figure 2 shows that two waves - longitudinal $P$ and transverse $S$ are propagating from the source. Figure 3 shows that when falling the waves onto the boundary layers the
corresponding types of waves for elastic and porous media are formed. In the elastic layer there occur reflected longitudinal and transverse waves, and in the lower porous layer, there arise two types of longitudinal waves $P_1$ and $P_2$ and the shear wave $S$.

5 Conclusions

The initial boundary value problem of the dynamics of fluid-saturated porous media, described by three elastic parameters in the reversible hydrodynamic approximation, has been numerically solved. The algorithm proposed is an analog to conventional frequency-domain methods of solving dynamic problems. Numerical results of the simulation of seismic wave fields for the test layered models have been obtained on the multiprocessor computer.

Acknowledgement: This work was partially supported the grants of Ministry of Education and Science of the Republic of Kazakhstan (No. 3328/GF4) and RFBR (16-01-00729).

References

[16] Imomnazarov Kh.Kh., Doklady RAS. 2000, 373, 4, 536-537