DE GRUYTER OPEN Concr. Oper. 2017; 4: 7–17

Concrete Operators Open Access

Research Article

Gajath Gunatillake*

Hermitian composition operators on Hardy-Smirnov spaces

DOI 10.1515/conop-2017-0002
Received June 13, 2016; accepted December 6, 2016.

Abstract: Let Ω be an open simply connected proper subset of the complex plane and ϕ an analytic self map of Ω. If f is in the Hardy-Smirnov space defined on Ω, then the operator that takes f to $f \circ \phi$ is a composition operator. We show that for any Ω, analytic self maps that induce bounded Hermitian composition operators are of the form $\phi(w) = aw + b$ where a is a real number. For ceratin Ω, we completely describe values of a and b that induce bounded Hermitian composition operators.

Keywords: Composition operator, Hermitian operator, Hardy-Smirnov space

MSC: 47B33, 30H10

1 Introduction

Let Ω be an open simply connected proper subset of the complex plane and γ be a Riemann map from the open unit disc onto Ω. The Hardy-Smirnov space $H^2(\Omega)$ is the Hilbert space of functions F analytic on Ω such that the integrals of $|F|^2$ over the images of the circles $|z| = r$, $0 < r < 1$, under γ are uniformly bounded; see [5, Chapter 10] or [10, p. 63]. We will refer to the map γ as the underlying Riemann map of $H^2(\Omega)$.

If $f \in H^2(\Omega)$, and ϕ is an analytic self map of Ω, then the composition operator induced by ϕ on $H^2(\Omega)$, denoted by C_ϕ, is the linear operator defined by

$$C_\phi(f) = f \circ \phi.$$

Such operators on $H^2(\Omega)$ are studied in [10]. Also, composition operators on a Hardy space of a half-plane are studied in [7],[8], and [9].

In this paper we study bounded Hermitian composition operators on $H^2(\Omega)$. In Theorem 3.3 we show that if C_ϕ is bounded and Hermitian, then $\phi(w) = aw + b$ where $a \in \mathcal{R}$. However, the self map ϕ having the form $aw + b, a \in \mathcal{R}$, is not always sufficient for C_ϕ to be Hermitian. Some sufficient conditions are given in Theorem 3.3 in order for C_ϕ to be Hermitian. We also show that if a constant map induces a Hermitian composition operator, then Ω is a disc.

The sufficient condition in Theorem 3.3 requires the computation of $\gamma^{-1} \circ \phi \circ \gamma$ which could be complicated for some γ. Therefore in section 4, for some domains, we provide sufficient conditions that do not require the computation of $\gamma^{-1} \circ \phi \circ \gamma$. In Lemma 4.7 it is proved that if $\phi(w) = -w + b$ is a self map of Ω, then C_ϕ is bounded and Hermitian on $H^2(\Omega)$. In Theorem 4.10 we show that if the boundary of Ω has infinite length, ϕ has a fixed point in Ω and $a^2 \neq 1$, then C_ϕ is not Hermitian on $H^2(\Omega)$.

Some concrete examples are provided in section 5. First we consider the strip $\Omega = \{z : -1 < Im(z) < 1\}$, where $Im(z)$ denotes the imaginary part of z. It is proved in Theorem 5.2 that the composition operator C_ϕ is

***Corresponding Author: Gajath Gunatillake:** American University of Sharjah, UAE,
E-mail: mgunatillake@aus.edu, gajathg@gmail.com

bounded, non-trivial and Hermitian on $H^2(\Omega)$ if and only if $\phi(w) = -w + b$ where $b \in \mathcal{R}$. Next we consider $\Omega = \{z : -1 < Im(z) < 1\} \cup \{z : -1 < Re(z) < 1\}$, where the real part of z is denoted by $Re(z)$. In Theorem 5.3 it is proved that the only non-trivial bounded Hermitian composition operator C_ϕ on $H^2(\Omega)$ is induced by $\phi(w) = -w$.

2 Background material

Throughout this paper Ω will represent an open simply connected proper subset of the complex plane. The underlying Riemann map associated with $H^2(\Omega)$ is denoted by γ.

Notation

- The set of real numbers is denoted by $\mathcal{R}$.
- The complex plane is denoted by $\mathcal{C}$.
- The open unit disc $\{z : |z| < 1\}$ is denoted by $\mathcal{D}$.
- The real part of the complex number z is denoted by $Re(z)$.
- The imaginary part of the complex number z is denoted by $Im(z)$.

Hardy-Smirnov spaces

Let γ be a Riemann map that takes $\mathcal{D}$ onto Ω. For $0 < r < 1$, let Γ_r be the curve in Ω defined by $\Gamma_r = \gamma(\{|z| = r\})$. The set of functions analytic on Ω for which

$$\sup_{0<r<1} \int_{\Gamma_r} |f(w)|^2 |dw| < \infty$$

is a Hardy-Smirnov space on Ω. We denote this space by $H^2(\Omega)$ and refer to it simply as a Hardy space on Ω. The functional $\|\cdot\|_\Omega$ defined on $H^2(\Omega)$ by $\|f\|_\Omega = \left(\sup_{0<r<1}(1/2\pi)\int_{\Gamma_r} |f(w)|^2|dw|\right)^{1/2}$ is a norm on $H^2(\Omega)$ (see [10, p. 63]).

Hardy space of the unit disc H^2

The classical Hardy space H^2 of analytic functions on the open unit disc corresponds to the choice $\Omega = \mathcal{D}$ and γ the identity map. Thus H^2 is the set of analytic functions on the open unit disc for which

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

For an arbitrary Ω it turns out that $H^2(\Omega)$ is isometrically isomorphic to H^2 and the details of the isomorphism are given in Theorem A below.

Theorem A. *Suppose f is holomorphic on Ω. Then $f \in H^2(\Omega)$ if and only if $(f \circ \gamma)(\gamma')^{1/2} \in H^2$. The map V given by $V(f) = (f \circ \gamma)(\gamma')^{1/2}$ is a linear isometry from $H^2(\Omega)$ onto H^2.*

For a proof see [10, p. 63] or [5, p. 169]. Using V, we can define a function $\langle\cdot,\cdot\rangle_\Omega : H^2(\Omega) \times H^2(\Omega) \to \mathcal{C}$ by

$$\langle f, g\rangle_\Omega = \langle V(f), V(g)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in H^2. Since V is an isometry it follows that $\langle f, f\rangle_\Omega = \|V(f)\|^2_{H^2} = \|f\|^2_\Omega$. Thus $H^2(\Omega)$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_\Omega$. If $g \in H^2$, then it is easy to see that

$$V^{-1}(g) = \frac{1}{(\gamma' \circ \gamma^{-1})^{1/2}} g \circ \gamma^{-1}. \tag{1}$$

Suppose that C_ϕ is a composition operator on $H^2(\Omega)$. If $g \in H^2$ and $z \in \mathcal{D}$, then

$$VC_\phi V^{-1}(g)(z) = \left(\frac{\gamma'(z)}{\gamma'(\varphi(z))}\right)^{1/2} g(\varphi(z)), \tag{2}$$

where $\varphi = \gamma^{-1} \circ \phi \circ \gamma$. Let $\psi = (\gamma'/\gamma' \circ \varphi)^{1/2}$. Then

$$VC_\phi V^{-1}(g) = C_{\psi,\varphi}(g), \tag{3}$$

where $C_{\psi,\varphi}(g) = \psi \cdot g \circ \varphi$. Such an operator is called a weighted composition operator. From the discussion above, C_ϕ on $H^2(\Omega)$ is isometrically similar to the weighted composition operator $C_{\psi,\varphi}$ on H^2 (see [10, p. 66]).

Next we cite Theorem 2.1 of [3] which characterizes maps ψ and φ when $C_{\psi,\varphi}$ is Hermitian on H^2. The following theorem can also be deduced from Theorem 6 of [4].

Theorem B. *If the weighted composition operator $C_{\psi,\varphi}$ is bounded and Hermitian on H^2, then*

$$\varphi(z) = a_0 + \frac{a_1 z}{1-\overline{a}_0 z} \quad \text{and} \quad \psi(z) = \frac{c}{1-\overline{a}_0 z}$$

where a_1 and c are real numbers.

Conversely, suppose that $a_0 \in \mathcal{D}$ and $c, a_1 \in \mathcal{R}$. If $\varphi(z) = a_0 + a_1 z/(1-\overline{a}_0 z)$ maps the unit disc into itself and $\psi(z) = c/(1-\overline{a}_0 z)$, then the weighted composition operator $C_{\psi,\varphi}$ is Hermitian and bounded on H^2.

We cite the following theorem from [6] that will be used later.

Theorem C. *Let ϕ be an analytic self map of Ω. The composition operator C_ϕ is unitary on $H^2(\Omega)$ if and only if ϕ is an automorphism of Ω that takes the form $\phi(w) = e^{i\theta}w + \lambda$ for some real number θ and a complex number λ.*

3 Hermitian operators

We begin our work by studying bounded Hermitian composition operators induced by constant maps. Notice that it is elementary to prove that a Riemann map from the unit disc onto another disc is a linear fractional transformation.

Lemma 3.1. *Let ϕ be a constant self map of Ω. If C_ϕ is a bounded Hermitian operator on $H^2(\Omega)$, then Ω is a disc.*

Conversely, let Ω be an open disc and $\gamma(z) = (az+b)/(cz+d)$, where $a, b, c, d \in \mathcal{C}$, be the underlying Riemann map of $H^2(\Omega)$. If ϕ is a constant map that takes the value $(b\overline{d} - a\overline{c})/(|d|^2 - |c|^2)$, then C_ϕ is bounded and Hermitian.

Proof. First assume that ϕ is constant and C_ϕ is a bounded Hermitian operator. For $z \in \mathcal{D}$ let

$$\varphi(z) = \gamma^{-1} \circ \phi \circ \gamma(z) \text{ and } \psi(z) = \left(\frac{\gamma'(z)}{\gamma'(\varphi(z))}\right)^{1/2}. \tag{4}$$

Next let $V : H^2(\Omega) \to H^2$ be given by $V(f) = (f \circ \gamma)(\gamma')^{1/2}$ (see Theorem A). From (3) it follows that $VC_\phi V^{-1}$ is the weighted composition operator $C_{\psi,\varphi}$. Since V is a linear isometry $C_{\psi,\varphi}$ is bounded and Hermitian on H^2. Therefore from Theorem B it follows that

$$\varphi(z) = a_0 + \frac{a_1 z}{1-\overline{a}_0 z} \text{ and } \psi(z) = \frac{k}{1-\overline{a}_0 z}$$

where $k, a_1 \in \mathcal{R}$. Since ϕ is a constant map φ is a constant self map of $\mathcal{D}$. Hence $a_1 = 0$ and $|a_0| < 1$. Then $\varphi(z) = a_0$ and from (4) it follows that

$$\frac{\gamma'(z)}{\gamma'(a_0)} = \frac{k^2}{(1-\overline{a}_0 z)^2}. \tag{5}$$

Since $\Omega = \gamma(\mathcal{D})$ and Ω is an open set γ cannot be a constant map. Thus, $k \neq 0$.

Next we consider the two cases $a_0 = 0$ and $a_0 \neq 0$.

First assume that $a_0 = 0$. Then, $\gamma'(z) = k^2\gamma'(0)$. Thus $\gamma(z) = k^2\gamma'(0)z + j$ for some $j \in \mathcal{C}$. Since $k^2\gamma'(0)$ is nonzero $\gamma(\mathcal{D})$ is a disc.

Next assume that $a_0 \neq 0$. Then, $\gamma'(z) = k^2\gamma'(a_0)/(1-\overline{a}_0 z)^2$ and it easily follows that

$$\gamma(z) = \frac{k^2\gamma'(a_0)}{\overline{a}_0(1-\overline{a}_0 z)} + j$$

for some $j \in \mathcal{C}$. Therefore γ is a linear fractional map without any poles on the closed unit disc. Thus $\gamma(\mathcal{D})$ is a disc.

Next we prove the converse;

From direct computations we get $\gamma^{-1} \circ \phi \circ \gamma(z) = -\overline{c/d}$ and

$$\frac{\gamma'(z)}{\gamma'(-\overline{c/d})} = \frac{(|d|^2 - |c|^2)^2}{|d|^4(1+(c/d)z)^2}.$$

Now let $\varphi(z) = \gamma^{-1} \circ \phi \circ \gamma(z)$ and $\psi(z) = (\gamma'(z)/\gamma'(\varphi(z)))^{1/2}$. Since $\gamma(\mathcal{D})$ is a bounded set, $|c/d| < 1$. From Theorem B it follows that $C_{\psi,\varphi}$ is bounded and Hermitian on H^2. Since $C_\phi = V^{-1}C_{\psi,\varphi}V$ and V is an isometry C_ϕ is bounded and Hermitian on $H^2(\Omega)$. □

The domain Ω is said to be symmetric about the point p if $-(\Omega - p) = \Omega - p$.

Lemma 3.2. *Let ϕ be an analytic self map of Ω which is neither a constant nor the identity. Suppose that C_ϕ is bounded and Hermitian on $H^2(\Omega)$ and $\gamma^{-1} \circ \phi \circ \gamma(0) = 0$. Then the following are true.*
(1) *If $(\gamma^{-1} \circ \phi \circ \gamma)'(0) = -1$, then Ω is symmetric about $\gamma(0)$.*
(2) *If $(\gamma^{-1} \circ \phi \circ \gamma)'(0) \neq -1$, then Ω is a disc.*

Proof. Let $\varphi(z) = \gamma^{-1} \circ \phi \circ \gamma(z)$ and $\psi(z) = \left(\gamma'(z)/\gamma'(\varphi(z))\right)^{1/2}$, $z \in \mathcal{D}$. Let $V : H^2(\Omega) \to H^2$ be given by $V(f) = (f \circ \gamma)(\gamma')^{1/2}$ (see Theorem A). From (3) it follows that $VC_\phi V^{-1}$ is the weighted composition operator $C_{\psi,\varphi}$. Since V is a linear isometry $C_{\psi,\varphi}$ is bounded and Hermitian on H^2. Therefore from Theorem B it follows that

$$\varphi(z) = a_0 + \frac{a_1 z}{1-\overline{a}_0 z} \text{ and } \psi(z) = \frac{c}{1-\overline{a}_0 z}$$

where $c, a_1 \in \mathcal{R}$. Since $a_0 = \varphi(0)$, and $\varphi = \gamma^{-1} \circ \phi \circ \gamma$ it is easy to see that $\varphi(z) = a_1 z$, $\psi(z) = c$. Therefore

$$\gamma'(z) = \gamma'(a_1 z)c^2. \tag{6}$$

By letting $z = 0$ in the equation above it can be readily seen that $c^2 = 1$. Notice that $a_1 = (\gamma^{-1} \circ \phi \circ \gamma)'(0)$.

(1) If $a_1 = -1$, from equation (6) it easily follows that $\gamma(z) = -\gamma(-z) + 2\gamma(0)$. Thus $\gamma(z) - \gamma(0) = -(\gamma(-z) - \gamma(0))$. Since $\{\gamma(z) : z \in \mathcal{D}\} = \{\gamma(-z) : z \in \mathcal{D}\}$, it can be easily seen that Ω is symmetric about $\gamma(0)$.

(2) Let $\gamma^{(n)}$ denote the derivative of order n of γ. It follows from equation (6) that $\gamma^{(n)}(0) = a_1^{n-1}\gamma^{(n)}(0)$ where $n \geq 2$. Since ϕ is not the identity map φ is also not the identity thus, $a_1 \neq 1$. Since a_1 is real and $|a_1| \neq 1$, it easily follows that $\gamma^{(n)}(0) = 0$ for $n \geq 2$. Since Ω is a nonempty open set γ cannot be a constant thus γ is a polynomial of degree 1. Therefore $\gamma(\mathcal{D})$ is a disc.

□

Theorem 3.3. *Let ϕ be an analytic self map of Ω.*
(1) *Let ϕ be a nonconstant map. If the composition operator C_ϕ is bounded and Hermitian on $H^2(\Omega)$, then $\phi(w) = aw + b$ for some $a \in \mathcal{R} \setminus \{0\}$, $b \in \mathcal{C}$.*
(2) *Suppose that $\phi(w) = aw + b$, $a \in \mathcal{R}$, $b \in \mathcal{C}$. Let $\varphi = \gamma^{-1} \circ \phi \circ \gamma$. If*

$$\varphi(z) = a_0 + \frac{a_1 z}{1-\overline{a}_0 z}, \quad \left(\frac{\gamma'(z)}{\gamma'(\varphi(z))}\right)^{1/2} = \frac{c}{1-\overline{a}_0 z}$$

with $c, a_1 \in \mathcal{R}$, $a_0 \in \mathcal{D}$, and $-1 + |a_0|^2 \leq a_1 \leq (1-|a_0|)^2$, then C_ϕ is a bounded Hermitian operator on $H^2(\Omega)$.

Proof. Let $\varphi(z) = \gamma^{-1} \circ \phi \circ \gamma(z)$ and $\psi(z) = \left(\gamma'(z)/\gamma'(\varphi(z))\right)^{1/2}$, $z \in \mathcal{D}$. Let $V : H^2(\Omega) \to H^2$ be given by $V(f) = (f \circ \gamma)(\gamma')^{1/2}$ (see Theorem A). From (3) it follows that $VC_\phi V^{-1}$ is the weighted composition operator $C_{\psi,\varphi}$. Since V is a linear isometry $C_{\psi,\varphi}$ is bounded and Hermitian on H^2. Therefore from Theorem B it follows that

$$\varphi(z) = a_0 + \frac{a_1 z}{1 - \overline{a}_0 z} \text{ and } \psi(z) = \frac{c}{1 - \overline{a}_0 z}$$

where $c, a_1 \in \mathcal{R}$. Thus,

$$\frac{\gamma'(z)}{\gamma'(\varphi(z))} = \frac{c^2}{(1 - \overline{a}_0 z)^2}.$$

Since Ω is an open set γ cannot be a constant map. Thus $c \neq 0$. The map ϕ is nonconstant therefore φ is a nonconstant map. Hence $a_1 \neq 0$. Since $\varphi'(z) = a_1/(1 - \overline{a}_0 z)^2$ it follows that

$$\gamma'(z) = \frac{c^2}{a_1}\gamma'(\varphi(z))\varphi'(z).$$

Thus,

$$\gamma(z) = \frac{c^2}{a_1}\gamma(\varphi(z)) + p$$

for some constant p. But $\gamma(\varphi(z)) = \phi(\gamma(z))$, hence $\gamma(z) = (c^2/a_1)\phi(\gamma(z)) + p$. Now let $w = \gamma(z)$. Then,

$$\phi(w) = \frac{a_1}{c^2} w - \frac{a_1}{c^2} p$$

Proof of (2):

Since $-1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2$, from Corollary 2.3 of [3] we get that φ maps the open unit disc into itself. Then from Theorem B it follows that $C_{\psi,\varphi}$ is bounded and Hermitian on H^2. Since $VC_\phi V^{-1} = C_{\psi,\varphi}$ and V is an isometry, C_ϕ is bounded and Hermitian on $H^2(\Omega)$. □

Let Π denote the upper half plane and take $\gamma(z) = i(1 + z)/(1 - z)$ to be the underlying Riemann map. The following result, which is also in [9] can be obtained using the theorem above.

The composition operator C_ϕ is bounded and Hermitian on $H^2(\Pi)$ if and only if

$$\phi(w) = w + ik, k \geq 0.$$

4 Geometry of Ω

Throughout Section 4 the symbol Ω will represent an open, simply connected proper subset of the complex plane. The underlying Riemann map of $H^2(\Omega)$ from $\mathcal{D}$ onto Ω is denoted by γ.

There are domains with simple geometric descriptions whose Riemann maps are complicated. For such domains computing $\gamma^{-1} \circ \phi \circ \gamma$ and $\gamma'/\gamma' \circ \varphi$ as required by Theorem 3.3 could be difficult. Thus in this section we find some conditions for C_ϕ to be Hermitian that does not involve the computation of $\gamma^{-1} \circ \phi \circ \gamma$ and $\gamma'/\gamma' \circ \varphi$.

The following lemma describes some geometric properties of the linear fractional transformation φ when $C_{\psi,\varphi}$ is Hermitian on H^2.

Lemma 4.1. *Let* $\varphi(z) = a_0 + \dfrac{a_1 z}{1 - \overline{a_0} z}$ *where* $0 < |a_0| < 1$ *and* $a_1 \in \mathcal{R}$. *Then the following are true.*

(1) $a_0 = \varphi(0)$ *and* $a_1 = \varphi'(0)$.
(2) φ *maps the unit disc into itself if and only if* $-1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2$.
(3) *If* $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$, *then the closure of* $\varphi(\mathcal{D})$ *is contained in* $\mathcal{D}$, *and* φ *has a fixed point in* $\mathcal{D}$.
(4) *If* r_1, r_2 *are the fixed points of* φ *in* $\mathcal{C}$, *then* $|r_1 r_2| = 1$.
(5) *If* $-1 + |a_0|^2 = a_1$, *then* $\varphi(z) = \dfrac{a_0 - z}{1 - \overline{a_0} z}$ *and the fixed point of* φ *inside* $\mathcal{D}$ *is* $(1 - \sqrt{1 - |a_0|^2})/\overline{a}_0$.
(6) *If* $a_1 = (1 - |a_0|)^2$, *then* φ *has no fixed points inside* $\mathcal{D}$ *and has only one fixed point on the unit circle. The fixed point on the unit circle is* $|a_0|/\overline{a}_0$ *and* $\varphi'(|a_0|/\overline{a}_0) = 1$.

Proof. (1) A routine computation shows that $a_0 = \varphi(0)$ and $a_1 = \varphi'(0)$.

(2) See Corollary 2.3 of [3].

(3) See the first paragraph on page 5778 of [3].

To prove the rest fixed points of φ must be investigated. Solutions of $\varphi(z) = z$ are the fixed points of φ. It is easy to see that the solutions of $\varphi(z) = z$ are the roots of the quadratic equation

$$\overline{a}_0 z^2 + (a_1 - 1 - |a_0|^2)z + a_0 = 0. \tag{7}$$

(4) If r_1, r_2 are the roots of the equation (7), then $r_1 r_2 = a_0/\overline{a}_0$.

(5) If $-1 + |a_0|^2 = a_1$, then a routine computation yields that $\varphi(z) = \dfrac{a_0 - z}{1 - \overline{a}_0 z}$. It is easy to see that $r_1 = (1 - \sqrt{1 - |a_0|^2})/\overline{a}_0$ is a root of the equation (7) and $|r_1| < 1$. From part (4) it follows that the other fixed point must lie outside the closed unit disc.

(6) If $a_1 = (1 - |a_0|)^2$, then the equation (7) has only one root and it is readily seen that the root is $|a_0|/\overline{a}_0$. A routine computation shows that $\varphi'(z) = (1 - |a_0|)^2/(1 - \overline{a}_0 z)^2$, hence $\varphi'(|a_0|/\overline{a}_0) = 1$. □

The following lemma describes how the different values of $\varphi(0)$ and $\varphi'(0)$ affect the operator theoretic properties of the Hermitian operator $C_{\psi,\varphi}$.

Lemma 4.2. *Suppose that $C_{\psi,\varphi}$ is bounded and Hermitian on H^2. Then the following are true,*

(1) *If $-1 + |\varphi(0)|^2 < \varphi'(0) < (1 - |\varphi(0)|)^2$, then $C_{\psi,\varphi}$ is compact and φ has a fixed point inside the open unit disc.*

(2) *If $-1 + |\varphi(0)|^2 = \varphi'(0)$, then $C_{\psi,\varphi}$ is an isometry and φ has a fixed point inside the open unit disc.*

For a proof of the lemma above see the third paragraph on page 5778 of [3].

Lemma 4.3. *Suppose that $\phi(w) = rw + k$, where $r \in \mathcal{R}$ and $k \in \mathcal{C}$ maps Ω into itself. Assume that C_ϕ is bounded and Hermitian on $H^2(\Omega)$. Let $\varphi = \gamma^{-1} \circ \phi \circ \gamma$. Then the following are true.*

(1) *$\varphi'(0) = r\gamma'(0)/\gamma'(\varphi(0))$.*

(2) *If $\varphi(p) = p$ and $p \in \mathcal{D}$, then $r = \varphi'(p)$.*

(3) *Suppose that $\varphi(p) = p$ and $|p| = 1$. If γ' exists at p and $\gamma'(p) \neq 0$, then $r = 1$.*

Proof. Let $\psi = (\gamma'/\gamma' \circ \varphi)^{1/2}$. From (3) it follows that $C_{\psi,\varphi} = VC_\phi V^{-1}$. Let $z \in \mathcal{D}$. Since $\gamma \circ \varphi = \phi \circ \gamma$, we have, $\gamma(\varphi(z)) = r\gamma(z) + k$. Now it follows that

$$\gamma'(\varphi(z))\varphi'(z) = r\gamma'(z). \tag{8}$$

To prove (1) let $z = 0$ in (8).

To prove (2) let $z = p$ in (8).

Proof of (3); since C_ϕ is bounded and Hermitian φ is continuous on the closed unit disc (see Theorem B). Since γ' exist at p we let $z = p$ in (8). Then $r = \varphi'(p)$. From part (6) of Lemma 4.1 it follows that $\varphi'(p) = 1$. □

Fixed points

Suppose that $\phi(w) = aw + b, a \in \mathcal{R} \setminus \{1\}, b \in \mathcal{C}$ is a self map of Ω. It is easy to see that ϕ has a natural extension $\tilde{\phi}$ to the whole complex plane. Since $\tilde{\phi}(b/(1-a)) = b/(1-a)$, and $\tilde{\phi}$ clearly does not have more than one fixed point in $\mathcal{C}$ it can be readily seen that ϕ has a fixed point in Ω if and only if $b/(1-a) \in \Omega$.

Proposition 4.4. *Suppose that Ω is unbounded. Let $\phi(w) = aw+b, a \in \mathcal{R}, b \in \mathcal{C}$ be a nonautormorphic self map of Ω without any fixed points in Ω. Further suppose that C_ϕ is bounded and Hermitian on $H^2(\Omega)$. Let $\varphi = \gamma^{-1} \circ \phi \circ \gamma$. Then, φ has a fixed point ζ on the unit circle and $\lim_{z \to \zeta} \gamma(z)$ does not exist.*

Proof. From Lemma 3.1 and Theorem B it follows that φ is a nonconstant linear fractional self map of $\mathcal{D}$. Hence $\varphi(\mathcal{D})$ must be a disc. Since the map ϕ is not an automorphism of Ω the map φ is not an automorphism of $\mathcal{D}$. Thus, $\varphi(\mathcal{D})$ must be properly contained in $\mathcal{D}$.

Since ϕ does not possess a fixed point in Ω it follows that φ does not fix any points in $\mathcal{D}$. But, φ is an analytic self map of the unit disc, therefore it has a fixed point on the unit circle (see page 59 of [2]). Since $\varphi(\mathcal{D})$ is a disc properly contained in $\mathcal{D}$ it follows that φ has exactly one fixed point on the unit circle. Let ζ be the fixed point of φ.

To prove the remaining part suppose that $\lim_{z\to\zeta}\gamma(z)$ exist finitely. Then, there is a disc U centered at ζ such that $\gamma(\mathcal{D}\cap U)$ is a bounded set. The closure of $\varphi(\mathcal{D})\setminus U$ is contained in $\mathcal{D}$. Therefore, the set $\gamma(\varphi(\mathcal{D})\setminus U)$ is bounded. Since

$$\gamma(\varphi(\mathcal{D})) = (\gamma(\varphi(\mathcal{D})\setminus U)) \cup (\gamma(\varphi(\mathcal{D})\cap U))$$

it easily follows that the set $\gamma(\varphi(\mathcal{D}))$ is bounded. Clearly, $\gamma\circ\varphi(z) = \phi\circ\gamma(z)$, for $z\in\mathcal{D}$, hence

$$\gamma(z) = \frac{1}{a}(\gamma(\varphi(z)) - b).$$

Note that since φ is a nonconstant a is nonzero. Since $\gamma(\varphi(\mathcal{D}))$ is a bounded set, from the equation above it follows that the set $\gamma(\mathcal{D})$ is bounded. But Ω is an unbounded set, hence our assumption that $\lim_{z\to\zeta}\gamma(z)$ exist finitely must be false. □

Bergman space of the unit disc A^2

The set of analytic functions on the open unit disc for which

$$\int_{\mathcal{D}} |f(w)|^2 dA < \infty$$

where dA is the Lebesgue area measure is known as the Bergman space of the unit disc. This space is denoted by A^2.

Let $w\in\mathcal{D}$. If $K_w(z) = 1/(1-\overline{w}z)^2$ for $z\in\mathcal{D}$, then K_w is the reproducing kernel function at w in A^2 and $\int f\overline{K_w}dA = f(w)$, for any $f\in A^2$.

In the following result we look at γ whose derivative is in the Bergman space.

Proposition 4.5. *Suppose that $\phi(w) = aw+b, a\in\mathcal{R}, b\in\mathcal{C}$, is a self map of Ω and ϕ has no fixed points in Ω. If γ' is contained in the Bergman space of the unit disc A^2, then C_ϕ is not a bounded Hermitian operator on $H^2(\Omega)$.*

Proof. We prove this by contradiction. Assume that C_ϕ is bounded and Hermitian on $H^2(\Omega)$. Let $\varphi = \gamma^{-1}\circ\phi\circ\gamma$ and $\psi = (\gamma'/\gamma'\circ\varphi)^{1/2}$. Then $C_{\psi,\varphi} = VC_\phi V^{-1}$ (see (3)). Since V is an isometry $C_{\psi,\varphi}$ is bounded and Hermitian on H^2, hence

$$\varphi(z) = a_0 + \frac{a_1 z}{1-\overline{a}_0 z} \text{ and } \psi(z) = \frac{c}{1-\overline{a}_0 z}$$

for some $a_1, c\in\mathcal{R}$ (see Theorem B). Clearly $\gamma(\varphi(z)) = \phi(\gamma(z))$. Hence

$$\gamma(\varphi(z)) = a\gamma(z) + b.$$

Now it follows that

$$\gamma'(\varphi(z))\varphi'(z) = a\gamma'(z). \tag{9}$$

Notice that $\varphi'(z) = a_1/(1-\overline{a}_0 z)^2$ is a scalar multiple of the reproducing kernel function at a_0 in A^2. Thus, from Theorem 6 of [4] it follows that the weighted composition operator $C_{\varphi',\varphi}$ is Hermitian on A^2. Equation (9) yields that

$$C_{\varphi',\varphi}(\gamma') = a\gamma'.$$

Therefore, γ' is an eigenvector of $C_{\varphi',\varphi}$. Since φ is a self map of the unit disc $-1+|a_0|^2 \le a_1 \le (1-|a_0|)^2$, (see part (2) of Lemma 4.1). Hence one of the following 3 cases must be true;

(1) $-1+|a_0|^2 < a_1 < (1-|a_0|)^2$
(2) $a_1 = -1+|a_0|^2$
(3) $a_1 = (1-|a_0|)^2$

Since the map ϕ does not have any fixed points in Ω the map φ does not have any fixed points in $\mathcal{D}$. If the case (1) or case (2) above is true, then φ has a fixed point in the open unit disc (see parts (3) and (5) of Lemma 4.1). Therefore, $a_1 = (1 - |a_0|)^2$. Now, from corollaries 2 and 15 of [4] it follows that $C_{\varphi',\varphi}$ does not have any eigenvectors. This is the desired contradiction. □

If C_φ is the identity operator on the Hardy space of the unit disc H^2, then it is very easy to see that φ is the identity function on the unit disc. Next we prove for that for any Ω, if C_ϕ is the identity on $H^2(\Omega)$, then ϕ is the identity map on Ω.

Lemma 4.6. *Let τ be an analytic self map of Ω. If the composition operator C_τ is the identity operator on $H^2(\Omega)$, then $\tau(w) = w$ for all $w \in \Omega$.*

Proof. If $\varphi = \gamma^{-1} \circ \tau \circ \gamma$ and $\psi = (\gamma'/\gamma' \circ \varphi)^{1/2}$, from (2) it follows that

$$C_{\psi,\varphi} = VC_\tau V^{-1}.$$

Therefore, if C_τ is the identity operator on $H^2(\Omega)$ it easily follows that $C_{\psi,\varphi}$ is the identity operator on H^2. If $f \in H^2$, then

$$C_{\psi,\varphi}(f) = \psi \cdot f \circ \varphi \tag{10}$$

Substitute the constant function $f(z) = 1$ in (10) and it follows immediately that $\psi(z) = 1$ for all $z \in \mathcal{D}$. Thus, for all $f \in H^2$,

$$C_{\psi,\varphi}(f) = f \circ \varphi. \tag{11}$$

Next substitute the function $f(z) = z$ in (11) and it yields $\varphi(z) = z$ for $z \in \mathcal{D}$. Since $\varphi = \gamma^{-1} \circ \phi \circ \gamma$, it follows that $\phi(w) = w$ for all $w \in \Omega$. □

For self maps $\phi(w) = aw + b$ of Ω with $a \in \mathcal{R}, b \in \mathcal{C}$, next we look at $a = -1, a = 1$ and $|a| \neq 1$ separately.

Lemma 4.7. *If $\phi(w) = -w + r, r \in \mathcal{C}$ maps Ω into itself, then C_ϕ is bounded and Hermitian on $H^2(\Omega)$.*

Proof. If $w \in \Omega$, then it is easy to see that

$$\phi \circ \phi(w) = -(-w + r) + r = w.$$

Thus ϕ is an automorphism of Ω. From Theorem C it follows that C_ϕ is unitary. Hence

$$C_\phi^{-1} = C_\phi^*.$$

If $f \in H^2(\Omega)$, then $C_\phi(f) = f \circ \phi$ and it follows that $C_\phi C_\phi(f) = f \circ \phi \circ \phi$. Hence $C_\phi C_\phi$ is the identity operator on $H^2(\Omega)$. Therefore $C_\phi = C_\phi^{-1}$ and now it follows easily that

$$C_\phi = C_\phi^*.$$

□

Next we look at Hermitian C_ϕ where $\phi(w) = w + r$.

Lemma 4.8. *Suppose that $\phi(w) = w + b, b \in \mathcal{C} \setminus \{0\}$ is a self map of Ω. If ϕ is an automorphism of Ω, then C_ϕ is not a bounded Hermitian operator on $H^2(\Omega)$.*

Proof. Suppose that C_ϕ is bounded and Hermitian. Since ϕ is an automorphism of Ω, from Theorem C it follows that C_ϕ is unitary. Now, C_ϕ is both Hermitian and unitary therefore, $C_\phi C_\phi$ is the identity operator on $H^2(\Omega)$. Let $f \in H^2(\Omega)$. Since $C_\phi(f) = f \circ \phi$, it is easy to see that $C_\phi C_\phi(f) = f \circ \phi \circ \phi$. Thus,

$$C_\phi C_\phi = C_{\phi\circ\phi}.$$

Now, from Lemma 4.6 it follows that $\phi \circ \phi(w) = w$ for $w \in \Omega$. But, $\phi \circ \phi(w) = w + 2b$, and this leads to a contradiction since $b \neq 0$. □

Next we look at domains with infinitely long boundaries.

Lemma 4.9. *Suppose that one-dimensional Hausdorff measure of the boundary of Ω is infinite. Let $\phi(w) = aw + b, a \in \mathcal{R}, b \in \mathcal{C}$ be a self map of Ω. If $a^2 \neq 1$ and $b/(1-a) \in \Omega$, then C_ϕ is not a bounded Hermitian operator.*

Proof. Suppose that C_ϕ is bounded and Hermitian on $H^2(\Omega)$. For $z \in \mathcal{D}$, let $\varphi(z) = \gamma^{-1} \circ \phi \circ \gamma(z)$ and $\psi(z) = (\gamma'(z)/\gamma'(\varphi(z)))^{1/2}$. Then $VC_\phi V^{-1} = C_{\psi,\varphi}$ (see (3)). Since V is an isometry $C_{\psi,\varphi}$ is bounded and Hermitian on H^2. Now from Theorem B it follows that $\varphi(z) = a_0 + a_1 z/(1 - \overline{a}_0 z)$ where $a_1 \in \mathcal{R}$. Clearly φ is a self map of $\mathcal{D}$, hence from Lemma 4.1 it follows that

$$-1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2.$$

Therefore, one of the following 3 cases must be true;

(1) $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$
(2) $a_1 = -1 + |a_0|^2$
(3) $a_1 = (1 - |a_0|)^2$

Next we will show that none of these cases are possible.

Case (1): if $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$, then $C_{\psi,\varphi}$ is compact (Lemma 4.2). Thus C_ϕ is compact. Since the length of the boundary of Ω is infinite, from Theorem 1.5 of [10] it follows that $H^2(\Omega)$ does not possess any compact composition operators. Thus, case (1) is not possible.

Case (2): if $-1 + |a_0|^2 = a_1$, then $C_{\psi,\varphi}$ is an isometry (Lemma 4.2). Since V is an isometry C_ϕ is also an isometry. Thus,

$$C_\phi^* C_\phi = I$$

where I is the identity operator on $H^2(\Omega)$. Since C_ϕ is Hermitian we get $C_\phi C_\phi = I$. For g in $H^2(\Omega)$ it is readily seen that $C_\phi C_\phi(g) = g \circ \phi \circ \phi$. Hence $C_\phi C_\phi = C_{\phi\circ\phi}$. Thus, $C_{\phi\circ\phi}$ is the identity operator on $H^2(\Omega)$ and from Lemma 4.6 it follows that $\phi \circ \phi(w) = w$ for $w \in \Omega$. A direct computation shows that for $w \in \Omega$,

$$\phi \circ \phi(w) = a^2 w + (ab + b)$$

and since $a^2 \neq 1$ it is clear that $\phi \circ \phi(w) \neq w$. Therefore case(2) is not possible.

Case (3): if $a_1 = (1 - |a_0|)^2$, then φ does not have any fixed points inside the open unit disc (part 6 of Lemma 4.1). Let $w_0 = b/(1-a)$. It can be readily seen that ϕ fixes w_0. Therefore, φ fixes $\gamma^{-1}(w_0)$ which is inside $\mathcal{D}$. This shows that case(3) is not possible.

Since none of the three cases are possible it follows that our assumption, C_ϕ is bounded and Hermitian is false. □

The next theorem summarizes results above when boundary of Ω is infinite and ϕ has a fixed point inside Ω.

Theorem 4.10. *Suppose that one-dimensional Hausdorff measure of the boundary of Ω is infinite. Let $\phi(w) = aw + b, a \in \mathcal{R}, b \in \mathcal{C}$ be a self map of Ω which is neither the identity nor a constant. If ϕ has a fixed point in Ω, then the following are true.*

(1) *If $a^2 \neq 1$, then C_ϕ is not a bounded Hermitian operator on $H^2(\Omega)$.*
(2) *If $a = -1$, then C_ϕ is a bounded Hermitian operator on $H^2(\Omega)$.*

The proof of the theorem above easily follows from Lemma 4.7 and Lemma 4.9.

5 Examples

5.1 Strip

Throughout the subsection 5.1 the set $\{z : -1 < Im(z) < 1\}$ is denoted by Ω and the underlying Riemann map of $H^2(\Omega)$ is denoted by γ.

Suppose that $\phi(w) = aw + b,\ a \in \mathcal{R}$ and $b \in \mathcal{C}$ is a self map of Ω. It is easy to see that if $|a| > 1$, then $\phi(\Omega) \nsubseteq \Omega$. Thus $-1 \leq a \leq 1$.

Lemma 5.1. *Let $\phi(w) = aw + b,\ -1 < a < 1,\ b \in \mathcal{C}$ be a self map of Ω. If C_ϕ is a bounded Hermitian operator on $H^2(\Omega)$, then ϕ has a fixed point in Ω.*

Proof. Recall that ϕ has a fixed point in Ω if and only if $b/(1-a)$ is in Ω. From Lemma 3.1 it follows that $a \neq 0$.

First consider $-1 < a < 0$. Since $\phi(0) = b$ it follows that $b \in \Omega$. Clearly $1 - a > 1$, and Ω is a convex region that contains 0 therefore, $b/(1-a) \in \Omega$.

Now consider $0 < a < 1$. Let $b = \alpha + i\beta$. If $w \in \Omega$, then $Im(\phi(w)) = aIm(w) + \beta$. Thus

$$-1 < aIm(w) + \beta < 1. \tag{12}$$

Since $Im(w)$ can take any value in the interval $(-1, 1)$ it follows that

$$a + \beta \leq 1.$$

We will next show that the assumption $a + \beta = 1$, leads to a contradiction. If $a + \beta = 1$, then $\beta/(1-a) = 1$. Since the imaginary part of $b/(1-a)$ is $\beta/(1-a)$ it follows that $b/(1-a)$ is on the boundary of Ω. Let

$$\varphi = \gamma^{-1} \circ \phi \circ \gamma. \tag{13}$$

Since C_ϕ is a bounded Hermitian operator $\varphi(z) = a_0 + (a_1 z)/(1 - \overline{a}_0 z)$ where $a_1 \in \mathcal{R}$ and $a_0 \in \mathcal{D}$ (see (3) and Theorem B). Clearly φ is a self map of $\mathcal{D}$, hence from Lemma 4.1 it follows that $-1 + |a_0|^2 \leq a_1 \leq (1 - |a_0|)^2$. Therefore, one of the following 3 cases must be true;

(1) $-1 + |a_0|^2 < a_1 < (1 - |a_0|)^2$
(2) $a_1 = -1 + |a_0|^2$
(3) $a_1 = (1 - |a_0|)^2$

If case (1) or (2) above is true, then from parts (3) and (5) of Lemma 4.1 it follows that φ has a fixed point in $\mathcal{D}$. Since the point $b/(1-a)$ is on the boundary of Ω the map ϕ does not fix any points in Ω. Thus φ does not have any fixed points in the open unit disc. Therefore cases (1) or (2) above cannot be true: hence

$$a_1 = (1 - |a_0|)^2.$$

Then from part (6) of Lemma 4.1 it follows that φ has exactly one fixed point ζ on the unit circle. The map φ is not an automorphism of the unit disc (see the second paragraph of page 5778 of [3]). Therefore ϕ cannot be an automorphism of Ω.

Let $\{w_m\}$ be a sequence in Ω that converges to $b/(1-a)$. Clearly $\{\gamma^{-1}(w_m)\}$ has a subsequence $\{\gamma^{-1}(w_{m_n})\}$ that converges to some μ in the closed unit disc. From Theorem 2 of chapter 6 in [1] it follows that μ is on the unit circle. Let $z_n = \gamma^{-1}(w_{m_n})$. If $z_n \to \zeta$, from Proposition 4.4 it follows that $\{\gamma(z_n)\}$ diverges. Thus $\mu \neq \zeta$. Let $y_0 = \varphi(\mu)$. Since φ is a linear fractional self map of $\mathcal{D}$ which is not an automorphism it is easy to see that $y_0 \in \mathcal{D}$. The continuity of φ at μ yields that $\varphi(z_n) \to y_0$. From equation (13) it follows that

$$\gamma(\varphi(z_n)) = a\gamma(z_n) + b. \tag{14}$$

Letting n tend to infinity in equation (14) we get

$$\gamma(y_0) = \frac{b}{1-a}.$$

But $\gamma(y_0) \in \Omega$ and $b/(1-a) \notin \Omega$ hence our assumption that $a + \beta = 1$, cannot be true. Thus $a + \beta < 1$.

From inequality (12) it also follows that $-1 \leq -a + \beta$. Therefore, using a method similar to the one used above it can be proved that $-1 < -a + \beta$. Therefore

$$-1 < \beta/(1-a) < 1$$

Since the imaginary part of $b/(1-a)$ is $\beta/(1-a)$ it follows that Ω contains the point $b/(1-a)$. Hence ϕ has a fixed point in Ω. □

The next theorem characterizes bounded Hermitian operators on $H^2(\Omega)$ when Ω is the strip $\{x+iy : -1 < y < 1\}$.

Theorem 5.2. *Let $\Omega = \{z : -1 < Im(z) < 1\}$. The composition operator C_ϕ is bounded, non-trivial and Hermitian on $H^2(\Omega)$ if and only if $\phi(w) = -w + b$ where $b \in \mathcal{R}$.*

Proof. First assume that C_ϕ is bounded and Hermitian on $H^2(\Omega)$. From Theorem 3.3 it follows that $\phi(w) = aw+b$ where $a \in \mathcal{R}$ and $b \in \mathcal{C}$. Since ϕ is a self map of Ω it is easy to see that $-1 \leq a \leq 1$.

If $-1 < a < 1$, then from Lemma 5.1 it follows that ϕ has a fixed point in Ω. Then, the part 1 of Theorem 4.10 says that C_ϕ is not a bounded Hermitian operator. Thus $a = \pm 1$.

Since ϕ maps Ω to itself, if $a = \pm 1$, then it is clear that $Im(b) = 0$.

If $a = 1$, and b is a real number, then ϕ is an automorphism of Ω. Now from Lemma 4.8 it follows that $b = 0$. Since C_ϕ is not the identity operator it can now be concluded that $a = -1$. Thus, $\phi(w) = -w + b$ for some $b \in \mathcal{R}$.

Now assume that $\phi(w) = -w + b$ where b is a real number. It is readily seen that ϕ is a self map of Ω. Then from Lemma 4.7 it follows that C_ϕ is bounded and Hermitian on $H^2(\Omega)$. □

5.2 Cross

Throughout the subsection 5.2 the set $\Omega = \{z : -1 < Im(z) < 1\} \cup \{z : -1 < Re(z) < 1\}$ is denoted by Ω and the underlying Riemann map of $H^2(\Omega)$ is denoted by γ.

Theorem 5.3. *The only non-trivial bounded Hermitian composition operator C_ϕ on $H^2(\Omega)$ is induced by $\phi(w) = -w$.*

The proof of Theorem 5.3 is similar to the proof of Theorem 5.2 therefore we will only provide an outline.

The map $\phi(w) = -w$ is a self map of Ω, therefore from Lemma 4.7 it follows that C_ϕ is bounded and Hermitian on $H^2(\Omega)$.

If C_ϕ is bounded and Hermitian on $H^2(\Omega)$, then $\phi(w) = aw + b$ where $a \in \mathcal{R}, b \in \mathcal{C}$ (see Theorem 3.3). If $|a| > 1$, then Ω cannot contain $\phi(\Omega)$. Thus $|a| \leq 1$. Using a proof similar to the proof of Lemma 5.1 it can be proved that $a \notin (-1, 1)$. Hence $a = \pm 1$. If $\phi(w) = w + b$ maps Ω into itself then it is clear that $b = 0$. Thus $a = 1$ results in the identity operator. If $\phi(w) = -w + b$ is a self map of Ω it is not difficult to see that $b = 0$. Therefore $\phi(w) = -w$.

References

[1] L. Ahlfors, *Complex Analysis*, McGraw-Hill 1979.

[2] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[3] C. Cowen and E. Ko, *Hermitian Weighted Composition Operators on H^2*, Trans. Amer. Math. Soc.,362 (2010), 5771-5801.

[4] C. Cowen, G. Gunatillake and E. Ko, *Hermitian Weighted Composition Operators and Bergman Extremal Functions,* Complex Analysis and Operator Theory, 7(2013), 69-99.

[5] P. L. Duren, *Theory of H^p Spaces*, Academic Press 1970.

[6] G. Gunatillake, M. Jovovic and W. Smith, *Groups of Unitary Composition Operators on Hardy-Smirnov Spaces,* Proc. Amer. Math. Soc, 143(2015), 2439-2449.

[7] S. Elliott and M. Jury, *Composition Operators on Hardy Spaces of a Half Plane,* Bull. London Math. Soc., 44 No. 3(2012), 489-495.

[8] V. Matache, *Composition Operators on Hardy Spaces of a Half Plane,* Proc. Amer. Math. Soc., 127(1999), 1483-1491.

[9] V. Matache, *Invertible and Normal Composition Operators on the Hilbert Hardy Space of a Half -Plane,* Concr. Oper., 3(2016), 77-84.

[10] J. Shapiro and W. Smith, *Hardy Spaces That Support No Compact Composition Operators,* J. Funct. Anal., 205(2003), 62-89.