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Multipliers of sequence spaces

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Abstract: This paper is selective survey on the space ℓ_A^p and its multipliers. It also includes some connections of multipliers to Birkhoff-James orthogonality.

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1 Introduction

Sequence spaces such as the classical spaces ℓ^p play an important role in functional analysis. Indeed, they are often the first Banach spaces covered in a basic functional analysis course. Moreover, they often serve as starting points, or possibly ending points, for various conjectures. Historically, via Beurling's theorem, the unilateral shift S on ℓ^2 was one of the first operators to have a full characterization of its invariant subspaces. They key observation by Beurling was to equivalently recast ℓ^2 from a mere sequence space to a Hilbert space of analytic functions H^2, the Hardy space, where the vast toolbox of function theory comes into play.

This paper is a selective survey of results on the sequence space ℓ^p, indexed by the nonnegative integers, with a special emphasis on the associated Banach space of analytic functions ℓ_A^p. Here we focus on their multipliers. As it turns out, the multipliers of ℓ_A^2 (the Hardy space) are thoroughly understood since they turn our to be just the bounded analytic functions on the open unit disk. When $p \neq 2$, the multipliers are well studied but are still somewhat mysterious, and some basic questions remain open. For example, every inner function is a multiplier of ℓ_A^2; however, the atomic singular inner function is not a multiplier for ℓ_A^p when $p \neq 2$. In fact, it is unknown whether any singular function serves as a multiplier when $p \neq 2$ (though it is known that many do not). Furthermore, since in the $p = 2$ case the multipliers are just the bounded analytic functions, their non-tangential boundary behavior is well understood. When $1 \leqslant p < 2$ the multipliers actually enjoy somewhat better boundary behavior than generic bounded analytic functions. Even more surprising is that the multipliers of ℓ_A^p and those of its dual space ℓ_A^q are the same set – even though the spaces ℓ_A^p and ℓ_A^q are very different in terms of their boundary behavior.

Recent work is beginning to shed some light on the fact that in some Banach spaces of analytic functions, every function can be written as a quotient of two multipliers. This is indeed true for the Hardy space, and for the Dirichlet space, as well as other reproducing kernel Hilbert spaces with a Nevanlinna-Pick kernel. For ℓ_A^p this "quotient of two multipliers" property turns out to be spectacularly false when $p > 2$, but remains an open question when $p < 2$.

We became interested in these sequence spaces through our work in two papers [20] and [5], where we studied various natural function theory questions through the lens of Birkhoff-James orthogonality. In [5] we explored a version of the classical inner-outer factorization and its applications to ARMA processes, while in [20] we revisited some classical estimates of zeros of analytic functions. Work on those papers led us quite naturally to questions

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about multipliers of ℓ^p_A; in fact we continue that discussion in this paper with a multiplier proof of a result in [20], as well as a refinement, via Birkhoff-James, of coefficient estimates of multipliers.

2 Basic properties of ℓ^p_A

For $p \in [1, \infty)$ define ℓ^p to be the set of sequences

$$\mathbf{a} = (a_0, a_1, \dots)$$

of complex numbers for which

$$\|\mathbf{a}\|_p := \left(\sum_{k=0}^{\infty} |a_k|^p\right)^{1/p} < \infty.$$

The quantity $\|\mathbf{a}\|_p$ defines a norm on ℓ^p which makes ℓ^p a Banach space. Furthermore, from Hölder's inequality, we know that $(\ell^p)^*$ the normed dual of ℓ^p is isometrically isomorphic to ℓ^q, where q denotes the usual conjugate index, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{1}$$

via the bi-linear pairing

$$(\mathbf{a}, \mathbf{b}) = \sum_{k=0}^{\infty} a_k b_k, \quad \mathbf{a} \in \ell^p, \mathbf{b} \in \ell^q. \tag{2}$$

Here, in the case $p = 1$, we have $q = \infty$, and the dual space $(\ell^1)^* = \ell^\infty$ is endowed with the norm

$$\|\mathbf{b}\|_\infty := \sup\{|b_k|\}_{k=0}^{\infty}.$$

Throughout this paper, we will always adhere to the notation that q is the Hölder conjugate index to p.

For an $\mathbf{a} \in \ell^p$ we set

$$a(z) = \sum_{k=0}^{\infty} a_k z^k \tag{3}$$

to be the power series whose Taylor coefficients are $\mathbf{a}$. Note the use of $\mathbf{a}$ (bold faced) to represent a sequence and a (not bold faced) to represent the corresponding power series.

Consider the case when $p \in (1, \infty)$. By Hölder's inequality we see that for any $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,

$$\sum_{k=0}^{\infty} |a_k||z^k| \leqslant \left(\sum_{k=0}^{\infty} |a_k|^p\right)^{1/p} \left(\sum_{k=0}^{\infty} |z|^{kq}\right)^{1/q} = \|\mathbf{a}\|_p \left(\frac{1}{1 - |z|^q}\right)^{1/q}.$$

This implies that the above power series used to define the function a in (3) determines an analytic function on $\mathbb{D}$. Let us define

$$\ell^p_A := \{a : \mathbf{a} \in \ell^p\}$$

and endow each $a \in \ell^p_A$ with the norm $\|\mathbf{a}\|_p$. With this, ℓ^p_A becomes a Banach space of analytic functions on $\mathbb{D}$. Furthermore, for each $z \in \mathbb{D}$ and $a \in \ell^p_A$ we have

$$|a(z)| \leqslant \|\mathbf{a}\|_p \left(\frac{1}{1 - |z|^q}\right)^{1/q}. \tag{4}$$

Similarly, if $p = 1$, then

$$|a(z)| = \sum_{k=0}^{\infty} |a_k||z^k| \leqslant \|\mathbf{a}\|_1.$$

Thus if a sequence of functions converges in the norm of ℓ^p_A then it converges uniformly on compact subsets of $\mathbb{D}$.

The following is obvious from the definition of ℓ^p_A, but worth stating here for later use.

Proposition 2.1. *Let* $p \in [1, \infty)$. *If* $a \in \ell^p_A$ *with*

$$a(z) = \sum_{k=0}^{\infty} a_k z^k,$$

then

$$\left\| a - \sum_{k=0}^{K} a_k z^k \right\|_p \to 0, \quad K \to \infty.$$

Corollary 2.2. *If* $p \in [1, \infty)$, *then the analytic polynomials are dense in* ℓ^p_A.

Of special distinction is the *Wiener algebra* ℓ^1_A. Let us recall that $\ell^1 \subseteq \ell^p$ for all $p \in [1, \infty)$. Furthermore, for $a \in \ell^1_A$, the Taylor series converges uniformly on $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and thus ℓ^1_A is contained in $C(\overline{\mathbb{D}})$, the continuous functions on $\overline{\mathbb{D}}$. We now address the "algebra" part of the term Wiener algebra.

For two sequences $\mathbf{a}$ and $\mathbf{b}$, the convolution $\mathbf{a} * \mathbf{b}$ is the sequence

$$\left\{ \sum_{k=0}^{n} a_k b_{n-k} \right\}_{n \geqslant 0}.$$

By multiplying Taylor series coefficients, notice how $\mathbf{a} * \mathbf{b}$ corresponds via (3) to the pointwise product $a(z)b(z)$ of the functions a and b. Young's inequality [26, p. 37]

$$\|\mathbf{a} * \mathbf{b}\|_p \leqslant \|\mathbf{a}\|_p \|\mathbf{b}\|_1, \quad \mathbf{a} \in \ell^p, \mathbf{b} \in \ell^1, \tag{5}$$

shows that ℓ^1 is a convolution algebra (i.e., $\mathbf{a}, \mathbf{b} \in \ell^1 \implies \mathbf{a} * \mathbf{b} \in \ell^1$). Now by the correspondence between convolution of sequences and multiplication of power series, we see that that ℓ^1_A is an algebra of functions (i.e., $a, b \in \ell^1_A \implies ab \in \ell^1_A$).

3 Evaluation functionals and duality

The estimate in (4) says that for each $w \in \mathbb{D}$, the evaluation functional

$$\Lambda_w : \ell^p_A \to \mathbb{C}$$

is continuous for each $p \in (1, \infty)$.

We can even compute its norm

$$\|\Lambda_w\| = \sup\{|f(w)| : f \in \ell^p_A, \|f\|_p \leqslant 1\}.$$

Proposition 3.1. *Let* $p \in [1, \infty)$. *For each* $w \in \mathbb{D}$,

$$\|\Lambda_w\| = \left(\frac{1}{1 - |w|^q} \right)^{1/q}.$$

Proof. From (4) we get

$$\|\Lambda_w\| \leqslant \frac{1}{(1 - |w|^q)^{1/q}}. \tag{6}$$

For fixed $p \in (1, \infty)$, consider the test function

$$f(z) = \frac{1}{1 - |w|^{q-2}\overline{w}z} = \sum_{n=0}^{\infty} (|w|^{q-2}\overline{w})^n z^n$$

and observe that

$$\Lambda_w f = f(w) = \frac{1}{1 - |w|^q}.$$

On the other hand,

$$\|f\|_p^p = \sum_{n=0}^{\infty} ||w|^{q-2}\overline{w}|^{pn} = \sum_{n=0}^{\infty} |w|^{(q-1)pn} = \sum_{n=0}^{\infty} |w|^{qn} = \frac{1}{1-|w|^q}.$$

Hence

$$\|\Lambda_w\| \geqslant \frac{|\Lambda_w f|}{\|f\|_p} = \frac{1}{(1-|w|^q)^{1/q}}. \tag{7}$$

Comparing (6) and (7), we deduce that

$$\|\Lambda_w\| = \frac{1}{(1-|w|^q)^{1/q}}. \tag{8}$$

The $p = 1$ case is similarly handled. □

With our identification of ℓ_A^p with ℓ^p, we can appeal to (2) and see that the norm dual of ℓ_A^p can be isometrically identified with ℓ_A^q via the bi-linear pairing

$$(a, b) = \sum_{k=0}^{\infty} a_k b_k, \qquad a \in \ell_A^p, b \in \ell_A^q.$$

from (2). Since this series converges absolutely, we know from either Abel's Theorem or the Dominated Convergence Theorem that

$$(a, b) = \lim_{r \to 1^-} \sum_{k=0}^{\infty} a_k b_k r^{2k}.$$

Now an integral calculation, and the simple fact that

$$\int_0^{2\pi} e^{ik\theta} \frac{d\theta}{2\pi} = \delta_{k,0},$$

shows that we can write the pairing in terms of the "Cauchy pairing"

$$(a, b) = \lim_{r \to 1^-} \int_0^{2\pi} a(re^{i\theta}) b(re^{-i\theta}) \frac{d\theta}{2\pi}.$$

Using the notion of duality, the following is another useful interpretation of the evaluation functional Λ_w, which also yields a more concise proof of Proposition 3.1. For a fixed $w \in \mathbb{D}$, define

$$k_w(z) := \sum_{n=0}^{\infty} w^n z^n. \tag{9}$$

Clearly $k_w \in \ell_A^q$ and, by (2),

$$\Lambda_w f = (f, k_w). \tag{10}$$

In other words, k_w plays the role of a reproducing kernel.

4 Connection to Hardy spaces

For $p \in [1, \infty)$ the classical *Hardy space* H^p is the space of analytic functions f on $\mathbb{D}$ for which

$$\|f\|_{H^p} = \left(\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty. \tag{11}$$

For $p = \infty$, the class H^∞ is the space of bounded analytic functions on $\mathbb{D}$. Classical theory [7] says that functions in H^p have radial limits

$$f(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

for almost every $\theta \in [0, 2\pi]$ and the corresponding (almost everywhere defined) boundary function $e^{i\theta} \mapsto f(e^{i\theta})$ belongs to $L^p(\mathbb{T}, d\theta)$. In fact,

$$\|f\|_{H^p} = \left(\int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p}.$$

When $p = 2$, the space H^2 can also be described as those analytic functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

on $\mathbb{D}$ for which

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty.$$

Moreover, by Parseval's identity, we have

$$\|f\|_{H^2}^2 = \int_0^{2\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{k=0}^{\infty} |a_k|^2 < \infty.$$

Consequently, $\ell_A^2 = H^2$ with equal norms, and thus H^2 is a Hilbert space.

The Hausdorff-Young inequalities [26, p. 101] show, for $p \in [1, 2]$, that $H^p \subseteq \ell_A^q$ with

$$\|a\|_q \leqslant \|a\|_{H^p} \quad a \in H^p.$$

On the other hand, when $p \in [1, 2]$, we also have $\ell_A^p \subseteq H^q$ with

$$\|a\|_{H^q} \leqslant \|a\|_p \quad a \in \ell_A^p.$$

In particular, when $p \in [1, 2]$, every $a \in \ell_A^p$ has radial boundary values almost everywhere. When $p \in (2, \infty)$ the above containments fail, and, as we will see in the next section, radial limits become a problem as well.

5 Boundary values

From the discussion in the previous discussion, $\ell_A^p \subseteq H^q$, when $p \in [1, 2]$. Furthermore, each function in H^q, and hence ℓ_A^p, has a radial limit almost everywhere on $\mathbb{T}$. When $p > 2$, the boundary behavior can be more complicated. To see this, we bring in a theorem of Littlewood.

Proposition 5.1 (Littlewood [7]). *Assume that $\{a_n\}_{n\geqslant 0}$ is a sequence of complex numbers such that*

$$\sum_{n=0}^{\infty} |a_n|^2 = \infty.$$

Set

$$f(z) = \sum_{n=0}^{\infty} \varepsilon_n a_n z^n,$$

where $\varepsilon_n \in \{-1, 1\}$. Then there are choices of signs ε_n such that the corresponding function f fails to have radial limits almost everywhere on $\mathbb{T}$.

As a matter of fact, Littlewood introduces a probability measure on the family of all possible signs, i.e., $\{-1,1\}^{\aleph_0}$, and then he shows that for almost all such signs the above proposition holds. However, for our discussion below, the existence of even one such function is enough.

Let $\mathcal{N}$ denote the Nevanlinna class of analytic functions on $\mathbb{D}$ which can be written as the quotient of two bounded analytic functions. By well-known theorems of Riesz and Fatou [7], every function in the Nevanlinna class has a radial limit almost everywhere.

Corollary 5.2. *For each $p \in (2,\infty)$ the space ℓ^p_A is not contained in $\mathcal{N}$.*

Proof. Pick any sequence $\{a_n\}_{n\geqslant 0} \in \ell^p$ such that

$$\sum_{n=0}^{\infty} |a_n|^2 = \infty.$$

For example,

$$a_n = \frac{1}{n^{\frac{2}{p+2}}}, \qquad n \geqslant 1,$$

does the job. Then, by Proposition 5.1, there are choices of signs ε_n such that the corresponding function f fails to have radial limits almost everywhere on $\mathbb{T}$. This function certainly belongs to ℓ^p_A but not the Nevanlinna class $\mathcal{N}$. □

6 Operators on ℓ^p_A

We define the *forward shift* operator

$$S : \ell^p \to \ell^p, \quad S\mathbf{a} = (0, a_0, a_1, a_2, \ldots)$$

and observe that S is an isometry on ℓ^p. For $\mathbf{a} \in \ell^p$, let $[\mathbf{a}]$ be the S-invariant subspace generated by $\mathbf{a}$, that is,

$$[\mathbf{a}] := \bigvee \{\mathbf{a}, S\mathbf{a}, S^2\mathbf{a}, \ldots\},$$

where $\bigvee$ denotes the closed linear span in ℓ^p. A vector $\mathbf{a} \in \ell^p$ is said to be *cyclic* if $[\mathbf{a}] = \ell^p$.

Also define the *backward shift* operator

$$S^* : \ell^q \to \ell^q, \quad S^*\mathbf{a} = (a_1, a_2, \ldots)$$

and observe that S^* is a contraction on ℓ^q. If $\mathbf{a} \in \ell^p$ and $\mathbf{b} \in \ell^q$, it is straightforward to see that

$$(S\mathbf{a}, \mathbf{b}) = (\mathbf{a}, S^*\mathbf{b}). \tag{12}$$

This is a fundamental connection between S and S^*.

One can view the shift S on ℓ^p_A as the operator

$$a(z) \mapsto z a(z)$$

of multiplication by the independent variable z on the corresponding function space ℓ^p_A. From this viewpoint, note that for $a \in \ell^p_A$, $[a]$ is the ℓ^p_A-closure of the set of all Pa, where P is an analytic polynomial. We will identify the shift operator on the sequence space ℓ^p with the multiplication (by z) operator on the function space ℓ^p_A, and denote both by S. A similar convention is applied for S^*. For example, considering the definition (9), we see that

$$S^* k_w = w k_w. \tag{13}$$

Again, the reader should be take note of the absence of conjugation in the above formula.

The S-invariant subspaces of ℓ^p_A, i.e., those (closed) subspaces $M \subseteq \ell^p_A$ for which $SM \subseteq M$, are sometimes difficult to describe. When $p = 2$, we have already seen that $\ell^2_A = H^2$ and a well-known theorem of Beurling

[7] says that the (non-trivial) invariant subspaces M of $\ell_A^2 = H^2$ are given by $M = \Theta H^2$, where Θ is an inner function. Notice how this says that the quotient space M/SM is one dimensional. When $p > 2$, the situation is much more complicated. For example, one can show [1] that given any $n \in \mathbb{N} \cup \{\infty\}$ there is an S-invariant subspace M of ℓ_A^p such that the quotient space M/SM is n-dimensional.

Given $w \in \mathbb{D}$, the difference quotient operator $\mathcal{Q}_w$ is defined on the set of analytic functions on $\mathbb{D}$ by

$$(\mathcal{Q}_w f)(z) := \frac{f(z) - f(w)}{z - w}.$$

Proposition 6.1. *Let $w \in \mathbb{D}$. Then*

$$f \in \ell_A^p \implies \mathcal{Q}_w f \in \ell_A^p.$$

Moreover, $\mathcal{Q}_w$ is a bounded operator on ℓ_A^p.

In particular, $\mathcal{Q}_0$ is precisely the backward shift S^*. For the proof of the $p = 2$ case see [11, p. 100]. The general case is substantially similar.

In other words, one can always "divide out" a zero of a and still remain in ℓ_A^p. For many Banach spaces of analytic functions contained in the Nevanlinna class, the most prominent example being the H^p classes, one can divide out any inner factor and still remain in the space. Moreover if Θ is inner then

$$\Theta f \in H^p \implies f \in H^p.$$

Even though ℓ_A^p is contained in H^q for $p \in (1, 2)$, ℓ_A^p does not always have the analogous property. We will discuss this further in the next section about multipliers.

7 Multipliers

An analytic function φ on $\mathbb{D}$ is called a *multiplier* of ℓ_A^p if

$$f \in \ell_A^p \implies \varphi f \in \ell_A^p.$$

The set of multipliers of ℓ_A^p will be denoted by $\mathscr{M}_p$. (One can also consider a multiplier φ from ℓ_A^p to $\ell_A^{p'}$, i.e.,

$$f \in \ell_A^p \implies \varphi f \in \ell_A^{p'},$$

though we will not discuss this in our current paper.)

For $\varphi \in \mathscr{M}_p$, an application of the closed graph theorem shows that the operator

$$M_\varphi : \ell_A^p \to \ell_A^p, \quad M_\varphi f = \varphi f$$

is continuous. Thus we define the *multiplier norm* of φ by

$$\|\varphi\|_{\mathscr{M}_p} := \sup\{\|\varphi f\|_p : f \in \ell_A^p, \|f\|_p \leqslant 1\}.$$

In other words, the multiplier norm of φ coincides with the operator norm of M_φ on ℓ_A^p. It is often customary to equate the multiplier φ with the multiplication operator M_φ.

Proposition 7.1. *Let $p \in (1, \infty)$. If $\varphi \in \mathscr{M}_p$ then φ is a bounded function and*

$$\sup\{|\varphi(z)| : z \in \mathbb{D}\} \leqslant \|\varphi\|_{\mathscr{M}_p}.$$

Proof. Since $\varphi \in \mathscr{M}_p$ then so is φ^n for all $n \in \mathbb{N}$ and thus, for each $z \in \mathbb{D}$, we can use (6) to see that

$$|\varphi(z)|^n = |\Lambda_z \varphi^n| \leqslant \frac{\|\varphi^n\|}{(1 - |z|^q)^{1/q}} = \frac{\|M_\varphi^n(1)\|}{(1 - |z|^q)^{1/q}} = \frac{\|\varphi\|_{\mathscr{M}_p}^n}{(1 - |z|^q)^{1/q}}.$$

Taking the n-th root and letting $n \to \infty$ yields the result. □

The above result, and the fact that the constant functions belong to ℓ^p_A, imply that

$$\mathscr{M}_p \subseteq H^\infty \cap \ell^p_A, \tag{14}$$

whenever $p \in (1, \infty)$. When $p = 1$, Young's inequality (5) ensures that ℓ^1_A, the Wiener algebra, coincides with the algebra of multipliers on ℓ^1_A. In other words,

$$\mathscr{M}_1 = \ell^1_A.$$

When $p = 2$, we use the fact that ℓ^2_A is the Hardy space H^2 and the boundedness of integral means in the definition of H^2 from (11) to show that $H^\infty \subseteq \mathscr{M}_2$. Hence

$$\mathscr{M}_2 = H^\infty.$$

We will see from Corollary 15.7 below that the singular inner function

$$s(z) = \exp(-\frac{1+z}{1-z}),$$

which is certainly bounded on $\mathbb{D}$, is not a member of $\mathscr{M}_p$ for any $p \neq 1, 2$. Thus

$$\mathscr{M}_p \subsetneq \ell^p_A \cap H^\infty, \quad p \neq 1, 2.$$

8 $\mathscr{M}_p$ as the commutanat

Clearly $\varphi(z) = z$ is a multiplier of ℓ^p_A. In fact $M_z = S$ and we have already established that S is an isometric operator. Moreover, since $M_\varphi S = S M_\varphi$ for all $\varphi \in \mathscr{M}_p$, we see that $\mathscr{M}_p$ is a subset of the commutant of S, defined by

$$\{S\}' = \{A \in \mathscr{B}(\ell^p_A) : AS = SA\}.$$

In fact, more can be said.

Proposition 8.1 (Nikolskii [21])**.** *For $p \in [1, \infty)$ we have $\{S\}' = \mathscr{M}_p$.*

Proof. Clearly we have $\mathscr{M}_p \subseteq \{S\}'$. Conversely suppose $A \in \{S\}'$. Then for any analytic polynomial P we have

$$A(P(S1)) = P(S)A(1),$$

equivalently, $A(P) = PA(1)$. By the density of the polynomials in ℓ^p_A (Proposition 2.1) we can, for a given $f \in \ell^p_A$, find a sequence of polynomials $\{P_n\}_{n\geqslant 1}$ such that $P_n \to f$ in the norm of ℓ^p_A. Since point evaluations on $\mathbb{D}$ are continuous in the norm of ℓ^p_A (Proposition 3.1), we get $P_n \to f$ pointwise on $\mathbb{D}$. Since $AP_n \to Af$ both in norm as well as pointwise on $\mathbb{D}$, it follows that $Af = A(1)f$. Thus $\varphi = A(1) \in \mathscr{M}_p$ and $A = M_\varphi$. □

To bring in more operator theory techniques, we now give an equivalent characterization of $\mathscr{M}_p$ explored in [21]. Given a sequence of complex numbers $\{a_n\}_{n\geqslant 0}$, define the infinite (Toeplitz) matrix A by

$$A := \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{15}$$

Proposition 8.2 (Nikolskii [21])**.** *Suppose*

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an analytic function on $\mathbb{D}$. Then $\varphi \in \mathscr{M}_p$ if and only if the infinite matrix A from (15) *defines a bounded operator on ℓ^p. In this case,*

$$\|\varphi\|_{\mathscr{M}_p} = \|A\|_{\ell^p \to \ell^p}.$$

Proof. Let

$$f(z)=\sum_{n=0}^{\infty} b_n z^n \in \ell^p_A$$

and write $\mathbf{b} := \{b_n\}_{n\geqslant 0} \in \ell^p$. Then

$$(\varphi f)(z)=\sum_{n=0}^{\infty} c_n z^n, \quad c_n=\sum_{k=0}^{n} a_k b_{n-k}.$$

Therefore $\varphi \in \mathscr{M}_p$ if and only if

$$\sum_{n=0}^{\infty} |c_n|^p=\sum_{n=0}^{\infty} \left|\sum_{k=0}^{n} a_k b_{n-k}\right|^p<\infty.$$

On the other hand, setting $\mathbf{c} := \{c_n\}_{n\geqslant 0}$, we see that $\mathbf{b}$ and $\mathbf{c}$ are related via the matrix identity

$$A\mathbf{b}=\mathbf{c}. \tag{16}$$

Thus φ is a multiplier for ℓ^p_A if and only if

$$\mathbf{b}\in\ell^p \iff A\mathbf{b}\in\ell^p.$$

By the closed graph theorem, the latter is equivalent to $A \in \mathscr{B}(\ell^p)$.

For the equality of norms, note that by (16),

$$\begin{aligned}\|\varphi\|_{\mathscr{M}_p} &= \sup\{\|\varphi f\|_p : f \in \ell^p_A, \|f\|_p \leqslant 1\}\\ &= \sup\{\|A\mathbf{b}\|_p : \|\mathbf{b}\|_p \leqslant 1\}\\ &= \|A\|_{\ell^p\to\ell^p}.\end{aligned}$$

□

From (2) recall the bi-linear pairing $(\mathbf{a},\mathbf{b})$ between ℓ^p and ℓ^q. Proposition 8.2 implies the useful inequality

$$|(A\mathbf{x},\mathbf{y})| \leqslant \|\varphi\|_{\mathscr{M}_p}\|\mathbf{x}\|_p\|\mathbf{y}\|_q, \quad \mathbf{x}\in\ell^p, \mathbf{y}\in\ell^q. \tag{17}$$

This next result relates the multipliers on the dual spaces and is often an important reduction to our multiplier discussion.

Proposition 8.3 (Nikolskii [21])**.** *For $p \in (1,\infty)$ we have $\mathscr{M}_p=\mathscr{M}_q$ with equal multiplier norms.*

Proof. Extend the definition of $\ell^p=\ell^p(\mathbb{N}_0)$ to

$$\ell^p(\mathbb{Z}) := \left\{\mathbf{b}=\{b_n\}_{n\in\mathbb{Z}} : \|\mathbf{b}\|_p=\left(\sum_{n\in\mathbb{Z}}|b_n|^p\right)^{1/p}<\infty\right\}.$$

Extend the definition of the shift S on ℓ^p to $\ell^p(\mathbb{Z})$ as

$$S\{b_n\}_{n\in\mathbb{Z}}=\{b_{n-1}\}_{n\in\mathbb{Z}}$$

and the backward shift B on $\ell^p(\mathbb{Z})$ as

$$B\{b_n\}_{n\in\mathbb{Z}}=\{b_{n+1}\}_{n\in\mathbb{Z}}.$$

For $p\in(1,\infty)$ the projection $P:\ell^p(\mathbb{Z})\to\ell^p(\mathbb{N}_0)$ defined by

$$P\{b_n\}_{n\in\mathbb{Z}}=\{b_n\}_{n\in\mathbb{N}_0}$$

is continuous (in fact contractive). If $\varphi\in\mathscr{M}_p$ with

$$\varphi(z)=\sum_{n=0}^{\infty} a_n z^n$$

and $A : \ell^p(\mathbb{N}_0) \to \ell^p(\mathbb{N}_0)$ from (15) is the matrix operator formed from the sequence $\{a_n\}_{n\in\mathbb{N}_0}$, one can use the facts that the operators S and B are isometric on $\ell^p(\mathbb{Z})$ and apply the Banach-Steinhaus Theorem to see that the sequence of operators

$$B^N APS^N, \quad N \in \mathbb{N}_0,$$

are uniformly bounded in operator norm. Applying this sequence of operators to the basis vectors $\mathbf{e}_n = \{\delta_{j,n}\}_{j\in\mathbb{Z}}$ one can verify (equating the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ with the sequence $\{a_n\}_{n\in\mathbb{Z}}$, where $a_n = 0$ for $n < 0$) that for any $\{b_n\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$,

$$\lim_{N\to\infty}(B^N APS^N)\{b_n\}_{n\in\mathbb{Z}} = \left\{\sum_{k\in\mathbb{Z}} b_k a_{n-k}\right\}_{n\in\mathbb{Z}}$$

and this operator, which we call L, is continuous on $\ell^p(\mathbb{Z})$. Informally, L is multiplication by φ when equating a sequence in $\ell^p(\mathbb{Z})$ with its corresponding Fourier series. By the dual pairing

$$(\mathbf{c}, \mathbf{d}) = \sum_{k\in\mathbb{Z}}^{\infty} c_k d_k, \quad \mathbf{c} \in \ell^p(\mathbb{Z}), \mathbf{d} \in \ell^q(\mathbb{Z}),$$

one can show that the adjoint L^* of L, which is continuous on $\ell^q(\mathbb{Z})$, turns out to be

$$L^*\{b_n\}_{n\in\mathbb{Z}} = \left\{\sum_{k\in\mathbb{Z}} b_k \widetilde{a}_{n-k}\right\}_{n\in\mathbb{Z}},$$

where

$$\{\widetilde{a}_n\}_{n\in\mathbb{Z}} = \{\dots, a_3, a_2, a_1, a_0, 0, 0, 0, \dots\}.$$

Informally, one can think of L^* is multiplication by $\varphi(e^{-i\theta})$ on the Fourier series formed by sequences from $\ell^q(\mathbb{Z})$.

Restricting L^* to $\ell^q(-\mathbb{N}_0)$ one can see, by reindexing, that φ is a multiplier on ℓ^q_A. Thus we have shown that $\mathscr{M}_p \subseteq \mathscr{M}_q$. Since this argument was symmetric in p and q, we conclude that $\mathscr{M}_p = \mathscr{M}_q$.

The proof also shows that

$$\|M_\varphi\|_{\ell^p_A\to\ell^p_A} = \|M_\varphi\|_{\ell^q_A\to\ell^q_A}.$$ □

Corollary 8.4. *If*

$$\varphi = \sum_{k=0}^{\infty} a_k z^k \in \mathscr{M}_p$$

then

(i) $\varphi \in \ell^p_A \cap \ell^q_A$,

(ii) $\max\{\|\varphi\|_p, \|\varphi\|_q\} \leqslant \|\varphi\|_{\mathscr{M}_p}$,

(iii) $|a_0| + |a_1| + |a_2| + \cdots + |a_n| \leqslant \|\varphi\|_{\mathscr{M}_p}(n+1)^{\frac{1}{q}}$.

Proof. Statement (i) follows from Proposition 8.3. Statement (ii) follows from

$$\|\varphi\|_p = \|\varphi \cdot 1\|_p = \|M_p 1\|_p \leqslant \|\varphi\|_{\mathscr{M}_p}\|1\|_p = \|\varphi\|_{\mathscr{M}_p}$$

and Proposition 8.3.

To prove (iii), apply

$$\mathbf{x} = (1, 0, 0, \dots), \quad \mathbf{y} = (\zeta_0, \dots, \zeta_n, 0, 0, \dots), \quad \zeta_j = e^{-i \arg a_j}$$

to (17) to deduce

$$\begin{aligned} |a_0| + |a_1| + |a_2| + \cdots + |a_n| &= |\zeta_0 a_0 + \cdots + \zeta_n a_n| \\ &= |(A\mathbf{x}, \mathbf{y})| \\ &\leqslant \|\varphi\|_{\mathscr{M}_p}(n+1)^{\frac{1}{q}}. \end{aligned}$$ □

We remark that when examining $\mathscr{M}_p$, we can use Proposition 8.3 to justify focusing our efforts to the study of $\mathscr{M}_p$ for $p \in (1, 2]$.

9 Connection to Fourier multipliers

For $p \in [1, 2]$ let $A_p(\mathbb{T})$ denote the space of all functions f in the Lebesgue space $L^2(\mathbb{T})$ whose Fourier coefficients

$$\widehat{f}(n) := \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}, \quad n \in \mathbb{Z},$$

form a sequence in $\ell^p(\mathbb{Z})$. More explicitly,

$$A_p(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T}) : \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^p < \infty \right\}. \tag{18}$$

Parallel to the definition of the norm in ℓ^p_A, we define a norm

$$\|f\|_{A_p(\mathbb{T})} := \left(\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^p \right)^{\frac{1}{p}}$$

on $A_p(\mathbb{T})$. Use that fact $\ell^p \subseteq \ell^2$ for $p \in [1, 2]$ along with Parseval's Theorem to see that $A_p(\mathbb{T})$ is a Banach space when $p \in [1, 2]$.

The corresponding multiplier space, the so called ℓ^p-*Fourier multipliers*, is defined to be the family of all $\varphi \in L^\infty(\mathbb{T})$ (essentially bounded Lebesgue measurable functions on $\mathbb{T}$) such that $\varphi f \in A_p(\mathbb{T})$ whenever $f \in A_p(\mathbb{T})$, i.e.,

$$M_p(\mathbb{T}) := \{\varphi \in L^\infty(\mathbb{T}) : f \in A_p(\mathbb{T}) : \varphi f \in A_p(\mathbb{T})\}.$$

We naturally define the corresponding multiplier norm by

$$\|\varphi\|_{M_p(\mathbb{T})} := \sup\{\|\varphi f\|_{A_p(\mathbb{T})} : \|f\|_{A_p(\mathbb{T})} \leqslant 1\}.$$

In Proposition 7.1, we saw that $\mathscr{M}_p \subseteq H^\infty$. Therefore each $\varphi \in \mathscr{M}_p$ can be considered either as an analytic function on $\mathbb{D}$ or, via radial boundary values, as a measurable function on $\mathbb{T}$ with a vanishing negative spectrum (i.e., its Fourier coefficients with negative indices are zero). To distinguish between the two interpretations, we denote the latter function, via radial boundary values, by φ^*. It turns out that $\mathscr{M}_p$ naturally sits inside $M_p(\mathbb{T})$.

Proposition 9.1. *Let $p \in [1, 2]$ and $\varphi \in H^\infty$. Then*

$$\varphi \in \mathscr{M}_p \quad \Longleftrightarrow \quad \varphi^* \in M_p(\mathbb{T}).$$

Moreover, $\|\varphi\|_{\mathscr{M}_p} = \|\varphi^*\|_{M_p(\mathbb{T})}$.

Proof. Let $\varphi^* \in M_p(\mathbb{T})$. Since $\ell^p_A \subseteq \ell^2_A = H^2 \subseteq L^2(\mathbb{T})$, we can consider ℓ^p_A as a subclass of $A_p(\mathbb{T})$. In particular, for each $f \in \ell^p_A$ we have $\varphi^* f^* \in A_p(\mathbb{T})$ and

$$\|\varphi^* f^*\|_{A_p(\mathbb{T})} \leqslant \|\varphi^*\|_{M_p(\mathbb{T})} \|f^*\|_{A_p(\mathbb{T})}.$$

Moreover, the negative parts of the spectrum of φ^* and f^* are identically zero, and thus so is that of $\varphi^* f^*$. Therefore, $\varphi^* f^* \in A_p(\mathbb{T})$ implies that $\varphi f \in \ell^p_A$ and the above inequality can be rewritten as

$$\|\varphi f\|_p \leqslant \|\varphi^*\|_{M_p(\mathbb{T})} \|f\|_p.$$

In other words, $\varphi \in \mathscr{M}_p$ and $\|\varphi\|_{\mathscr{M}_p} \leqslant \|\varphi^*\|_{M_p(\mathbb{T})}$.

Now assume that $\varphi \in \mathscr{M}_p$. Let f be a trigonometric polynomial of degree N. Then $e^{iN\theta} f(e^{i\theta})$ is an analytic polynomial and can be considered as an element of ℓ^p_A. Thus, since φ is a multiplier of ℓ^p_A, we have $\varphi e^{iN\theta} f \in \ell^p_A$ and

$$\|\varphi e^{iN\theta} f\|_p \leqslant \|\varphi\|_{\mathscr{M}_p} \|e^{iN\theta} f\|_p.$$

Since the operator $f \mapsto e^{iN\theta} f$ is isometric on $A_p(\mathbb{T})$, we can rewrite the previous inequality as

$$\|\varphi^* f\|_{A_p(\mathbb{T})} \leqslant \|\varphi\|_{\mathscr{M}_p} \|f\|_{A_p(\mathbb{T})}.$$

Since trigonometric polynomials are dense in $A_p(\mathbb{T})$, the estimate above holds for all $f \in A_p(\mathbb{T})$. Therefore $\varphi^* \in M_p(\mathbb{T})$ and $\|\varphi^*\|_{M_p(\mathbb{T})} \leqslant \|\varphi\|_{\mathscr{M}_p}$. Combine this inequality with the reverse of it shown before to complete the proof. □

Though this Fourier multiplier problem might seem like a detour from our main discussion concerning the multipliers of ℓ^p_A, it will surface again later when examining boundary values of multipliers.

10 Quotients of multipliers

Classical factorization results [7] say that any $f \in H^2$ can be written as $f = h_1/h_2$ where h_1 and h_2 are bounded analytic functions and h_2 is zero free. Since the multiplier space of H^2 is precisely H^∞, this result can be stated in the following equivalent form for a Banach space of analytic functions $\mathcal{X}$ and its multiplier space $\mathcal{M}$, i.e.,

$$f = \frac{h_1}{h_2}, \tag{19}$$

where h_1 and h_2 belong to the multiplier space $\mathcal{M}$ and h_2 is zero free. This point of view opens the door for the same question about any Banach space of analytic functions. In some cases the answer is known. Besides the Hardy space H^2, it seems that the answer is affirmative for the classical Dirichlet space as well as reproducing kernel Hilbert spaces with a Nevanlinna-Pick kernel (several unpublished results). We are interested in this question for ℓ^p_A spaces. When $p = 2$, we are in the classical setting of $\ell^2_A = H^2$ and thus, as seen above, the answer is affirmative. The case $p = 1$ is also trivial since ℓ^1_A is itself an algebra, and thus it coincides with its multiplier algebra. For $p \in (1, 2)$, the question is still *open*. Using function theory tools mentioned earlier, we can show that the answer is negative when $p \in (2, \infty)$.

Corollary 10.1. *Let $p \in (2, \infty)$. Then there are functions in ℓ^p_A which cannot be represented as the quotient of two multipliers.*

Proof. By Corollary 5.2, there are functions in ℓ^p_A which are not in the Nevanlinna class $\mathcal{N}$. Such a function cannot be represented as the quotient of two multipliers, since by (14), such a quotient is in $\mathcal{N}$. □

Using a similar technique, one can show that the representation (19) fails in the Bergman space since it is well-known that the Bergman space is not contained in the Nevanlinna class.

11 Isometric multipliers

Which multipliers φ satisfy

$$\|\varphi f\|_p = \|f\|_p, \quad f \in \ell^p_A? \tag{20}$$

These are known as the *isometric multipliers*.

Once again the case $p = 2$ is exceptional. It is well known that (20) holds for $\ell^2_A = H^2$ if and only if φ is an inner function. Indeed, Proposition 7.1 and (20) show that

$$|\varphi(z)| \leqslant 1, \quad z \in \mathbb{D}.$$

Then use Parseval's identity to rewrite (20) in integral form as

$$\int_0^{2\pi} |f(e^{i\theta})|^2 \, (1 - |\varphi(e^{i\theta})|^2) \frac{d\theta}{2\pi} = 0, \quad f \in H^2,$$

which holds if and only if $|\varphi(e^{i\theta})| = 1$ almost everywhere. In other words, φ is inner. When $p \neq 2$ the story is different. Observe that

$$\|z^n f\|_p = \|f\|_p, \qquad f \in \ell^p_A, \tag{21}$$

for all $n \geqslant 0$. In other words, the monomials z^n are isometric multipliers for ℓ^p_A. Are there others?

Theorem 11.1 (Nikolskii [21])**.** *If $p \in (1,\infty)\setminus\{2\}$ and φ is an isometric multiplier for ℓ^p_A, then $\varphi(z) = \gamma z^n$ for some $n \geqslant 0$ and a unimodular constant γ.*

The proof is based on two sets of elementary inequalities due to Bernoulli [22, p. 31]. In the following $x \geqslant 0$, $y \geqslant 0$ and $t \geqslant -1$.

(i) For $0 < \alpha < 1$
 (a) we have
$$(x+y)^\alpha \leqslant x^\alpha + y^\alpha$$
 and equality holds if and only if either $x = 0$ or $y = 0$.
 (b) we have
$$(1+t)^\alpha \leqslant 1 + \alpha t$$
 and equality holds if and only if $t = 0$.

(ii) For $1 < \alpha < \infty$
 (a) we have
$$(x+y)^\alpha \geqslant x^\alpha + y^\alpha$$
 and equality holds if and only if either $x = 0$ or $y = 0$.
 (b) we have
$$(1+t)^\alpha \geqslant 1 + \alpha t$$
 and equality holds if and only if $t = 0$.

Proof of Theorem 11.1. We treat the case $1 < p < 2$ for which the first set of inequalities above are used. The other case is similar.

We may write $\varphi(z) = \gamma z^n \varphi_1(z)$, where γ is unimodular, n is the order of the zero of h at the origin, and φ_1 is such that $\varphi_1(0) > 0$. By (21), we have

$$\|\varphi_1 f\|_p = \|f\|_p, \quad f \in \ell^p_A.$$

We will now show that $\varphi_1 \equiv 1$.

Hence, considering the above reduction, assume that $\varphi(0) > 0$ and $\|\varphi f\|_p = \|f\|_p$ for all $f \in \ell^p_A$. Take

$$f(z) = 1 + e^{i\theta} z,$$

where we treat θ as a free parameter. If

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n,$$

the isometric identity can be rewritten as

$$|a_0|^p + \sum_{n=0}^{\infty} |a_{n+1} + a_n e^{i\theta}|^p = 2. \tag{22}$$

We integrate both sides with respect to $d\theta$. First, due to periodicity, we have

$$\int_0^{2\pi} |a + be^{i\theta}|^p \, d\theta = \int_0^{2\pi} \big||a| + |b|e^{i\theta}\big|^p \, d\theta.$$

Second, an elementary calculation reveals that

$$\big||a| + |b|e^{i\theta}\big|^2 = |a|^2 + |b|^2 + 2|ab|\cos\theta = (|a|^2 + |b|^2)\,(1 + s\cos\theta),$$

where

$$s = \frac{2|ab|}{|a|^2 + |b|^2}.$$

Note that $0 \leqslant s \leqslant 1$ and thus $t := s\cos\theta \geqslant -1$. Third, by the above Bernoulli inequalities we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \big||a| + |b|e^{i\theta}\big|^p\,d\theta &= \frac{(|a|^2+|b|^2)^{\frac{p}{2}}}{2\pi}\int_0^{2\pi} (1 + s\cos\theta)^{\frac{p}{2}}\,d\theta\\ &\leqslant \frac{|a|^p + |b|^p}{2\pi}\int_0^{2\pi}\Big(1 + \frac{ps}{2}\cos\theta\Big)\,d\theta\\ &= |a|^p + |b|^p,\end{aligned}$$

and the equality holds if and only if either $a = 0$ or $b = 0$. Returning to (22), we get

$$2 \leqslant |a_0|^p + \sum_{n=0}^{\infty}(|a_{n+1}|^p + |a_n|^p) = 2\sum_{n=0}^{\infty}|a_n|^p.$$

If we plug $f = 1$ in we see that

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty}|a_n|^p\right)^{1/p} = 1.$$

Hence, in the above relation, equality holds, which in return implies that equality holds in all preceding inequalities. Since $a_0 \neq 0$, we must have

$$a_1 = 0.$$

We now repeat the above procedure with the function

$$f(z) = 1 + e^{i\theta}z^2$$

and deduce that $a_2 = 0$. By induction, we have $a_n = 0$ for all $n \geqslant 1$. Since $a_0 > 0$ and $\sum_{n=0}^{\infty}|a_n|^p = 1$, we conclude that $\varphi \equiv a_0 = 1$. □

12 Smooth multipliers

The family of analytic functions which are defined on a disk larger than the open unit disc is denoted by $\mathrm{Hol}(\overline{\mathbb{D}})$. From Young's Inequality (see (5)) we see that

$$\ell^1_A \subseteq \mathscr{M}_p, \quad p \in [1, \infty)$$

with equality when $p = 1$. Thus certainly we have

$$\mathrm{Hol}(\overline{\mathbb{D}}) \subseteq \mathscr{M}_p.$$

We present below an alternative proof of this fact where we can obtain further information.

Let us recall Schur's test. Let $A = [a_{ij}]$ be an infinite matrix, and let $p \in (1, \infty)$. Assume that there are positive constants α and β and positive sequences $\{p_i\}$ and $\{q_j\}$ such that

$$\sum_i |a_{ij}|\,p_i^p \leqslant \alpha\, q_j^p, \quad j \geqslant 1,$$

and

$$\sum_j |a_{ij}|\, q_j^q \leqslant \beta\, p_i^q, \quad i \geqslant 1.$$

Then A is a bounded operator on ℓ^p and moreover,

$$\|A\|_{\ell^p \to \ell^p} \leqslant \alpha^{\frac{1}{p}} \beta^{\frac{1}{q}}. \tag{23}$$

For each $f \in \mathrm{Hol}(\overline{\mathbb{D}})$ with Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

there is $R > 1$ and a $c > 0$ such that

$$|a_n| \leqslant \frac{c}{R^n}, \quad n \geqslant 0. \tag{24}$$

The constants R and c depend on f, but work uniformly with respect to n. This exponential decay plays the major role in establishing the following result.

Theorem 12.1. *If $p \in (1, \infty)$, then $Hol(\overline{\mathbb{D}}) \subseteq \mathscr{M}_p$.*

Proof. By Proposition 8.2, it is enough to show that the matrix A formed with the coefficients of φ, according to recipe (15), is a bounded operator on ℓ^p. We apply Schur's test with $p_i = q_i = t^i$, where t is a positive parameter to be determined momentarily.

Fixing j, by (24), we have

$$\begin{aligned}
\sum_i |a_{ij}|\, p_i^p &= \sum_{i=j}^{\infty} |a_{i-j}|\, t^{ip} \\
&\leqslant c \sum_{i=j}^{\infty} \frac{t^{ip}}{R^{i-j}} \\
&= c t^{jp} \sum_{i=0}^{\infty} \left(\frac{t^p}{R}\right)^i \\
&= \frac{c}{1 - \frac{t^p}{R}}\, q_j^p, \quad j \geqslant 0.
\end{aligned}$$

Similarly, fixing i, by (24), we have

$$\begin{aligned}
\sum_j |a_{ij}|\, q_j^q &= \sum_{j=0}^{i} |a_{i-j}|\, t^{jq} \\
&\leqslant c \sum_{j=0}^{i} \frac{t^{jq}}{R^{i-j}} \\
&= c t^{iq} \sum_{j=0}^{i} \left(\frac{1}{t^q R}\right)^j \\
&= \frac{c}{1 - \frac{1}{t^q R}}\, p_i^q, \quad i \geqslant 0.
\end{aligned}$$

Schur's test ensures that

$$\|A\|_{\ell^p \to \ell^p} \leqslant \left(\frac{c}{1 - \frac{t^p}{R}}\right)^{\frac{1}{p}} \left(\frac{c}{1 - \frac{1}{t^q R}}\right)^{\frac{1}{q}}.$$

The above geometric series are convergent provided that $t^p < R$ and $t^q > 1/R$. Hence, the acceptable range of t is

$$\frac{1}{R^{1/q}} < t < R^{1/p}.$$

Therefore we can say that

$$\|A\|_{\ell^p \to \ell^p} \leqslant \inf_{t \in (R^{-1/q}, R^{1/p})} \left(\frac{c}{1 - \frac{t^p}{R}} \right)^{\frac{1}{p}} \left(\frac{c}{1 - \frac{1}{t^q R}} \right)^{\frac{1}{q}}. \tag{25}$$

In particular, with $t = 1$ we get

$$\|A\|_{\ell^p \to \ell^p} \leqslant \frac{c}{1 - \frac{1}{R}}, \tag{26}$$

which is enough for our applications. □

In the above proof, we took $t = 1$ in (25). Is it possible to get a better bound by choosing another value of t? In other words, what is the optimal value of t?

The proof of Theorem 12.1 contains more information than presented in the theorem. By a closer look, we obtain the following interesting convergence result.

Corollary 12.2. *For $p \in (1, \infty)$ and $\varphi \in Hol(\overline{\mathbb{D}})$ and denote its Taylor polynomial of degree n by φ_n. Then*

$$\lim_{n \to \infty} \|\varphi_n - \varphi\|_{\mathscr{M}_p} = 0.$$

Moreover, the rate of decay is exponential.

Proof. Since

$$\varphi(z) - \varphi_n(z) = \sum_{k=n+1}^{\infty} a_k z^k = z^{n+1} \sum_{k=0}^{\infty} a_{k+n+1} z^k,$$

by (21), we have

$$\|\varphi_n - \varphi\|_{\mathscr{M}_p} = \left\| \sum_{k=0}^{\infty} a_{k+n+1} z^k \right\|_{\mathscr{M}_p}.$$

Thus, according to Proposition 8.2,

$$\|\varphi_n - \varphi\|_{\mathscr{M}_p} = \|A_n\|_{\ell^p \to \ell^p},$$

where the matrix A_n is given by (15) but with the sequence

$$\alpha_k = a_{k+n+1}, \quad k \geqslant 0.$$

By (24) we have the estimate

$$|\alpha_k| = |a_{k+n+1}| \leqslant \frac{c}{R^{k+n+1}} = \frac{(c/R^{n+1})}{R^k}, \qquad k \geqslant 0.$$

Therefore by (26),

$$\|A_n\|_{\ell^p \to \ell^p} \leqslant \frac{c/R^{n+1}}{1 - \frac{1}{R}},$$

which reveals that $\|\varphi_n - \varphi\|_{\mathscr{M}_p}$ exponentially decreases to zero. □

Corollary 12.2 does not hold for an arbitrary multiplier. For example, we saw that $\mathscr{M}_2 = H^\infty$. However if $\varphi \in \mathscr{M}_2$ satisfies

$$\|\varphi_n - \varphi\|_{\mathscr{M}_2} = \|\varphi_n - \varphi\|_\infty \to 0,$$

then φ is continuous on $\overline{\mathbb{D}}$.

13 ℓ^1_A embeds contractively in $\mathscr{M}_p$

In this Section, we provide a result which contains Theorem 12.1 as a special case. However, we stated that theorem separately since the estimate in (25) should provide a better bound for the norm of the narrower class of multipliers $\mathrm{Hol}(\overline{\mathbb{D}})$.

Theorem 13.1. *For $p \in (1,\infty)$ we have $\ell^1_A \subseteq \mathscr{M}_p$ and*

$$\|h\|_{\mathscr{M}_p} \leqslant \|h\|_1$$

for every $h \in \ell^1_A$.

Proof. Of course the result follows immediately from (5) (and also observed in [21]) but we include a proof using the tools developed above. We again appeal to Proposition 8.2. Hence, it is enough to show that the matrix A formed with the coefficients of h according to recipe (15) is a contraction on ℓ^p. We apply the simplest version of Schur's test, i.e., with $p_i = q_i = 1$.

Fixing j, we have

$$\sum_i |a_{ij}|\, p_i^p = \sum_{i=j}^{\infty} |a_{i-j}| = \sum_{k=0}^{\infty} |a_k| = \|h\|_1.$$

Similarly, fixing i, we obtain

$$\sum_j |a_{ij}|\, q_j^q = \sum_{j=0}^{i} |a_{i-j}| = \sum_{k=0}^{i} |a_k| \leqslant \|h\|_1.$$

Therefore we may take

$$\alpha = \beta = \|h\|_1.$$

Schur's test (23) ensures that

$$\|A\|_{\ell^p \to \ell^p} \leqslant \|h\|_1.$$ □

Corollary 13.2. *Let $p \in (1,\infty)$ and for $h \in \ell^1_A$ denote its Taylor polynomial of degree n by h_n. Then*

$$\lim_{n\to\infty} \|h_n - h\|_{\mathscr{M}_p} = 0.$$

If the coefficients of h are all nonnegative, then Theorem 13.1 is reversible.

Theorem 13.3. *Let $p \in (1,\infty)$. If $h \in \mathscr{M}_p$ and the Taylor coefficients of h are nonnegative, then $h \in \ell^1_A$.*

Proof. In the inequality (17), take

$$\mathbf{x} = \mathbf{y} = (1,1,\dots,1,0,0,\dots)$$

to get

$$\sum_{i=0}^{n} \sum_{j=0}^{i} a_j \leqslant (n+1)\|h\|_{\mathscr{M}_p}.$$

After rearranging the sums we obtain

$$\sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k \leqslant \|h\|_{\mathscr{M}_p}.$$

By the monotone convergence theorem, let $n \to \infty$ to deduce

$$\sum_{k=0}^{\infty} a_k \leqslant \|h\|_{\mathscr{M}_p}.$$ □

14 Boundary properties of multipliers

Since $\mathscr{M}_p \subseteq H^\infty$ each $h \in \mathscr{M}_p$ has a non-tangential limit

$$\sphericalangle \lim_{z \to e^{i\theta}} h(z)$$

for almost every θ. Lebedev and Olevskii [14, 17, 18] observed that more can be said. Their work is based on the following technical discussion.

Recall the definition of $A_p(\mathbb{T})$ for $p \in [1, 2]$ from (18)

$$A_p(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T}) : \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^p < \infty \right\}$$

and its corresponding space of multipliers

$$M_p(\mathbb{T}) := \{\varphi \in L^\infty(\mathbb{T}) : f \in A_p(\mathbb{T}) : \varphi f \in A_p(\mathbb{T})\}.$$

This next result is a bit technical and we refer the reader to the references for the proof.

Theorem 14.1 (Lebedev and Olevskii [14, 17, 18])**.** *If $p \in [1, 2)$ and $\psi \in M_p(\mathbb{T})$, there is a continuous function Ψ on $\mathbb{T}$ such that $\psi = \Psi$ almost everywhere on $\mathbb{T}$.*

Corollary 14.2. *Let $p \in [1, 2)$ and $\varphi \in \mathscr{M}_p$. Then the unrestricted limit*

$$\lim_{z \to e^{i\theta}} \varphi(z)$$

exists for almost all θ.

Proof. From Proposition 9.1 we know that the almost everywhere defined radial boundary function φ^* belongs to $M_p(\mathbb{T})$ and, by Theorem 14.1, there is a continuous function Φ on $\mathbb{T}$ that is equal to φ^* almost everywhere. Since φ^* is the radial boundary function for φ, we have the well-known Poisson integral formula

$$\varphi(z) = \int_0^{2\pi} P_z(e^{i\theta})\varphi^*(e^{i\theta})\frac{d\theta}{2\pi} = \int_0^{2\pi} P_z(e^{i\theta})\Phi(e^{i\theta})\frac{d\theta}{2\pi},$$

where $P_z(e^{i\theta})$ is the Poisson kernel.

If e^{it} is a point of continuity of Ψ, a well-known fact from harmonic analysis says that

$$\lim_{z \to e^{it}} \int_0^{2\pi} P_z(e^{i\theta})\Phi(e^{i\theta})\frac{d\theta}{2\pi} = \Phi(e^{it}).$$

However, almost every point of $\mathbb{T}$ is a point of continuity of Φ and the result now follows. □

The essential feature of Corollary 14.2 is that $z \in \mathbb{D}$ freely tends to the boundary point ζ and it is not obliged to stay in a Stolz domain (non-tangential approach region). We also mention that it is sometimes (often?) the case the set of multipliers for a Banach space of analytic functions are better behaved near the boundary than generic functions in the space.

15 Inner multipliers

A worthy set of functions to test as possible multipliers for $\mathscr{M}_p$ are the inner functions. We know that the monomials $\varphi_n(z) = z^n$ are multipliers, in fact isometric multipliers. Are there any other inner multipliers?

Certainly any finite Blaschke product is an inner multiplier since these are analytic in an open neighborhood of $\overline{\mathbb{D}}$ (see Theorem 12.1). There are some infinite Blaschke products which are multipliers for all of the ℓ^p_A classes.

Theorem 15.1 (Vinogradov [25], Verbitskii [23]). *Let $\{z_n\}_{n\geqslant 1}$ be a Blaschke sequence in $\mathbb{D}$ such that*
(i) $\lim_{n\to\infty} z_n = 1$,
(ii) and

$$\sum_{\{n:|1-z_n|<\varepsilon\}} |1-z_n| = O(\varepsilon), \quad \varepsilon \to 0.$$

Let B be the corresponding Blaschke product. Then

$$B \in \bigcap_{1<p<\infty} \mathscr{M}_p.$$

The Blaschke product presented in Theorem 15.1 is discontinuous at the point $z = 1$. As a matter of fact,

$$\liminf_{z\to 1} |B(z)| = 0 \qquad \text{while} \qquad \limsup_{z\to 1} |B(z)| = 1.$$

The first example of this type was constructed by Vinogradov. He proved Theorem 15.1 for zeros tending nontangentially to 1 in [24] and then generalized it in [25]. The same result was obtained independently by Verbitskii [23]. Finally, Lebedev [15] showed that Theorem 15.1 is partially reversible. See also [16] for a short survey on inner ℓ^p-multipliers.

It is worth mentioning that Vinogradov [24] obtained the weak version of Theorem 15.1 as the corollary of a more general result. To present his result, we need the domain

$$\Omega(r,\alpha) = \{z : |z| < r\} \setminus \{z : |\arg(z-1)| \leqslant \alpha\},$$

where $r > 1$ and $0 \leqslant \alpha < \pi/2$.

Theorem 15.2 (Vinogradov [24]). *For each $r > 1$ and $\alpha \in (0, \frac{\pi}{2})$, we have*

$$H^\infty\big(\Omega(r,\alpha)\big) \subseteq \bigcap_{p\in(1,\infty)} \mathcal{M}_p.$$

A little thought will show that any Blaschke product whose zeros tend non-tangentially to 1 (i.e., lie in a fixed Stolz domain) satisfies the condition of Theorem 15.2 for some $r > 1$ and $\alpha \in [0, \pi, 2)$, and thus the weak version of Theorem 15.1 follows. However, there is a large family of functions which are not inner and still fulfill the requirements of Theorem 15.2. Note that this result does not apply to the singular inner functions

$$s_a(z) := \exp\left(-a\frac{1+z}{1-z}\right), \qquad (a > 0).$$

As a matter of fact, we will see that s_a does not belong to $\mathscr{M}_p$, for any $p \in (1,\infty) \setminus \{2\}$.

We can use Corollary 14.2 to eliminate certain classes of inner functions as multipliers. Let us recall that if h is an inner function, then its boundary spectrum is

$$\sigma(h) := \Big\{\zeta \in \mathbb{T} : \liminf_{z\to\zeta} |h(z)| = 0\Big\}.$$

Equivalently [11, p. 154], if $h = BS$ is the decomposition of h as the product of a Blaschke factor B formed with zeros $\{z_n\}_{n\geqslant 1}$ and a singular inner function S formed with the singular measure ν, then $\sigma(h)$ is precisely the union of the spectrum of ν and the accumulation points of $\{z_n\}_{n\geqslant 1}$ on $\mathbb{T}$. The Lebesgue measure of a measurable set $E \subseteq \mathbb{T}$ is denoted by $|E|$.

Theorem 15.3. *Let h be an inner function such that $|\sigma(h)| > 0$. Then*

$$h \notin \mathscr{M}_p$$

for any $p \in (1,\infty) \setminus \{2\}$.

Proof. Since $\mathscr{M}_p = \mathscr{M}_q$ (Proposition 8.3) we can assume $p \in [1, 2)$. If $h \in \mathcal{M}_p$, then, by Corollary 14.2,

$$h(\zeta) = 0$$

for almost all $\zeta \in \sigma(h)$. Since at the same time $|h| = 1$ almost everywhere on $\mathbb{T}$, we conclude that the Lebesgue measure of $\sigma(h)$ is zero. □

Surprisingly, the mere existence of a *singular* inner function in $\mathscr{M}_p$ is still an open question. According to Theorem 15.3, the condition $|\sigma(h)| = 0$ is necessary for being a multiplier. However, this condition is not sufficient. For example, as we mentioned before, Verbitskii [23] showed that the simplest singular inner function

$$s_a(z) = \exp\left(-a\frac{1+z}{1-z}\right), \quad a > 0.$$

is not in any $\mathscr{M}_p$, for $p \in (1, \infty) \setminus \{2\}$. We provide some further results clarifying the fact that the condition $|\sigma(h)| = 0$ is far from being sufficient for multipliers.

Let $E \subseteq \mathbb{T}$. For an arc $I \subseteq \mathbb{T}$, we define the quantity

$$d_E(I) := \sup\{|J| : J \text{ is an arc}, J \subseteq I, J \cap E = \emptyset\}.$$

In other words, and naively speaking, when we remove the points of E from I, then $d_E(I)$ is the size of largest arc among the remaining pieces. We say that $\zeta \in \mathbb{T}$ is a *point of thickness of* E provided that $d_E(I) = o(|I|)$ for arcs I containing ζ and shrinking to this point. The set of all points of thickness of E is denoted by E^{th}. It is clear that for a closed set E, we have $E^{th} \subseteq E$. The following result establishes the connection between the measures of an arc and $d_E(I)$.

Lemma 15.4. *Let S be a singular inner function, corresponding to the singular measure μ, let E be its support. Suppose that $S \in \mathscr{M}_p$ for some p with $p \in (1, \infty) \setminus \{2\}$. Then for each arc $I \subseteq \mathbb{T}$ with $d_E(I) > 0$ we have*

$$\mu(I) \leqslant c\frac{|I|^4}{d_E^3(I)},$$

where c does not depend on I.

We need a technical result first which uses our previous Fourier multiplier discussion.

Proposition 15.5. *Let $\Theta(t)$ be a real-valued 2π-periodic function on $\mathbb{R}$ such that*

$$\varphi := e^{i\Theta} \in M_p(\mathbb{T})$$

for some $p \in (1, \infty) \setminus \{2\}$. Suppose $I \subseteq \mathbb{R}$ is an interval of length at most 2π and that Θ has a continuous derivative of order $n \geqslant 2$ on I. Then

$$\inf_{t \in I} |\Theta^{(n)}(t)| \leqslant \frac{c}{|I|^n},$$

where the constant $c = c(\Theta, p, n)$ does not depend on I.

Proof. By Corollary 8.3, we have $\mathscr{M}_p = \mathscr{M}_q$. Therefore, without loss of generality, we assume that $p < 2$. Let χ_I denote the characteristic function of I. It is easy to directly verify that $\chi_I \in A_p(\mathbb{T})$ and estimate its norm. In fact, the Fourier coefficient of χ_I are given by

$$\widehat{\chi_I}(0) = \frac{|I|}{2\pi}$$

and

$$\widehat{\chi_I}(k) = \gamma_k \frac{\sin(k|I|/2)}{\pi k}, \qquad k \in \mathbb{Z} \setminus \{0\},$$

where γ_k is a unimodular constant. Hence, $\chi_I \in A_p(\mathbb{T})$ and, moreover,

$$\|\chi_I\|_{A_p(\mathbb{T})} = \|\widehat{\chi_I}\|_{\ell^p(\mathbb{Z})}$$

$$= \left(\sum_{k=-\infty}^{\infty} |\widehat{\chi_I}(k)|^p \right)^{\frac{1}{p}}$$

$$\geqslant \left(\sum_{|k| \leqslant \pi/2|I|} |\widehat{\chi_I}(k)|^p \right)^{\frac{1}{p}}$$

$$\geqslant \left(\sum_{|k| \leqslant \pi/2|I|} (|I|/4\pi)^p \right)^{\frac{1}{p}}$$

$$\geqslant \left(\frac{\pi}{|I|} (\frac{|I|}{4\pi})^p \right)^{\frac{1}{p}} \geqslant \frac{1}{15} |I|^{1-\frac{1}{p}}.$$

Note that the last estimate also holds for $p = \infty$. Since $\varphi \in M_p(\mathbb{T})$, and thus

$$\|\chi_I\|_{A_p(\mathbb{T})} = \|\varphi \times \bar{\varphi}\chi_I\|_{A_p(\mathbb{T})} \leqslant \|\varphi\|_{M_p(\mathbb{T})} \|\bar{\varphi}\chi_I\|_{A_p(\mathbb{T})},$$

we conclude that

$$\|\bar{\varphi}\chi_I\|_{A_p(\mathbb{T})} \geqslant \frac{1}{15\|\varphi\|_{M_p(\mathbb{T})}} |I|^{1-\frac{1}{p}}. \tag{27}$$

Therefore, we need a strategy to deal with the left side of above inequality. To this end, we need the well-known Corput lemma [3, Chapter I]: if a real-valued function g has a continuous derivative of order $n \geqslant 2$ on an interval I, then

$$\left| \int_I e^{ig(t)}\, dt \right| \leqslant \frac{c_n}{\left(\inf_{t \in I} |g^{(n)}(t)| \right)^{\frac{1}{n}}},$$

where c_n is a positive constant that just depends on n. We apply this result to the function $g(t) = \Theta(t) + kt$. Since $n \geqslant 2$, we have

$$\inf_{t \in I} |g^{(n)}(t)| = \inf_{t \in I} |\Theta^{(n)}(t)|.$$

Thus, for all $k \in \mathbb{Z}$,

$$|\widehat{\bar{\varphi}\chi_I}(k)| = \frac{1}{2\pi} \left| \int_I e^{-i(\Theta(t)+kt)}\, dt \right| \leqslant \frac{c_n}{2\pi \left(\inf_{t \in I} |\Theta^{(n)}(t)| \right)^{\frac{1}{n}}}.$$

In short, this means that

$$\|\widehat{\bar{\varphi}\chi_I}\|_{\ell^\infty(\mathbb{Z})} \leqslant \frac{c_n}{2\pi \left(\inf_{t \in I} |\Theta^{(n)}(t)| \right)^{\frac{1}{n}}}.$$

At the same time,

$$\|\widehat{\bar{\varphi}\chi_I}\|_{\ell^2(\mathbb{Z})} = \|\bar{\varphi}\chi_I\|_{\ell^2(\mathbb{Z})} = \left(\frac{|I|}{2\pi} \right)^{\frac{1}{2}}.$$

Now, we can interpolate ℓ^p between ℓ^2 and ℓ^∞ [**?** , Chapter 6] to obtain

$$\begin{aligned} \|\bar{\varphi}\chi_I\|_{A_p(\mathbb{T})} &= \|\widehat{\bar{\varphi}\chi_I}\|_{\ell^p(\mathbb{Z})} \\ &\leqslant \|\widehat{\bar{\varphi}\chi_I}\|_{\ell^\infty(\mathbb{Z})}^{1-\frac{2}{p}} \|\widehat{\bar{\varphi}\chi_I}\|_{\ell^2(\mathbb{Z})}^{\frac{2}{p}} \\ &\leqslant \left(\frac{c_n}{2\pi \left(\inf_{t \in I} |\Theta^{(n)}(t)| \right)^{\frac{1}{n}}} \right)^{1-\frac{2}{p}} \left(\frac{|I|}{2\pi} \right)^{\frac{1}{p}}. \end{aligned}$$

If we plug this estimation into (27) we obtain the required result. □

Proof of Lemma 15.4. Assume that $d_E(I) > 0$ (otherwise the result is trivial). Hence, there is an arc $J \subseteq I$ such that $J \cap E = \emptyset$ and $|J| > d_E(I)/2$. Moreover, we may also assume that $\operatorname{dist}(J, E) > 0$. We consider a copy of J on $\mathbb{R}$ in the following way: let Δ be an interval on $\mathbb{R}$ such that $|\Delta| < 2\pi$ and

$$J = \{e^{it} : t \in \Delta\}.$$

We can write $S(e^{it}) = e^{i\Theta(t)}$, where Θ is a real-valued 2π-periodic function and, for $t \in \Delta$, it is given by

$$\Theta(t) = \int_0^{2\pi} \cot\left(\frac{s-t}{2}\right) d\mu(e^{is}).$$

Since $\operatorname{dist}(J, E) > 0$, the function Θ is infinitely differentiable on Δ and the operator $\partial/\partial t$ can commute with the integral. Since

$$\begin{aligned}\frac{\partial^3}{\partial t^3} \cot\left(\frac{s-t}{2}\right) &= \frac{3\cos^2((s-t)/2) + \sin^2((s-t)/2)}{4\sin^4((s-t)/2)} \\ &\geqslant \frac{1}{4\sin^4((s-t)/2)} = \frac{4}{|e^{is} - e^{it}|^4},\end{aligned}$$

we see that

$$\Theta^{(3)}(t) \geqslant \int_{\mathbb{T}} \frac{4}{|e^{is} - e^{it}|^4} d\mu(e^{is}) \geqslant \int_I \frac{4}{|e^{is} - e^{it}|^4} d\mu(e^{is}) \geqslant \frac{4\mu(I)}{|I|^4}.$$

By Proposition 9.1, the assumption $S \in \mathscr{M}_p$ is equivalent to $e^{i\Theta} \in M_p(\mathbb{T})$. Therefore, applying Proposition 15.5 for the third derivative of Θ, we obtain

$$\frac{4\mu(I)}{|I|^4} \leqslant \frac{c}{|\Delta|^3}.$$

Since $|\Delta| = |J| > d_E(I)/2$, the result follows. □

Now we have all the required tools to establish an important result which shows that the condition $|E| = 0$ alone is not enough to ensure that $S \in \mathscr{M}_p$, for some $p \in (1, \infty) \setminus \{2\}$. The set E is the boundary spectrum of S.

Theorem 15.6 (Lebedev [15]). *Let S be a singular inner function and let E be its boundary spectrum. If $\overline{E^{th}} \neq E$, then $S \notin \mathscr{M}_p$ for any p with $p \in (1, \infty) \setminus \{2\}$.*

Proof. Suppose that, to the contrary, there is a $p \in (1, \infty) \setminus \{2\}$ for which $S \in \mathscr{M}_p$. Then, by Lemma 15.4, for any arc $I \subseteq \mathbb{T}$, we have

$$\mu(I) \leqslant c \frac{|I|^4}{d_E^3(I)},$$

where c does not depend on I. Let us write this inequality as

$$\left(\frac{d_E(I)}{|I|}\right)^3 \leqslant c \frac{|I|}{\mu(I)}.$$

Since μ is singular, for μ-almost all $\zeta \in \mathbb{T}$, we have

$$\frac{\mu(I)}{|I|} \to \infty$$

as I contains ζ and shrinks to this point. Therefore, at all such points,

$$\frac{d_E(I)}{|I|} \to 0,$$

or equivalently, they are points of thickness for E. Hence, we can surely say $\mu(\mathbb{T} \setminus E^{th}) = 0$. As E is closed and $E^{th} \subseteq E$, we must have $\overline{E^{th}} = E$, which contradicts our assumption. □

To effectively use Theorem 15.6, let us introduce the concept of *porous sets*. We say that $E \subseteq \mathbb{T}$ is a porous set if there exists a constant $c > 0$ such that for all arcs $I \subseteq \mathbb{T}$

$$d_E(I) \geqslant c|I|.$$

For example, a singleton is a porous set. But, there are more sophisticated constructions. Clearly, if E is porous, then $E^{th} = \emptyset$. Hence, we immediately deduce the following corollary.

Corollary 15.7 (Lebedev [15])**.** *Let S be a singular inner function whose spectrum is a porous set. Then $S \notin \mathscr{M}_p$ for any $p \in (1, \infty) \setminus \{2\}$.*

In particular,

$$s_a(z) = \exp\left(-a\frac{1+z}{1-z}\right), \qquad a > 0, \tag{28}$$

is not in $\mathscr{M}_p$, for any $p \in (1, \infty) \setminus \{2\}$. Using special techniques from the theory of Bessel functions, this particular result was first obtained by Verbitskii [23]. However, the original proof of Verbitskii has interesting pieces worth mentioning. Bessel functions are an important topic in the theory of special functions [8–10]. A Bessel function of the first kind and order ν is defined by

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}.$$

Fix the constant $\gamma > 0$ and define the kernel

$$k(x) := \frac{1}{\sqrt{x}} J_1(\gamma\sqrt{x}), \qquad x > 0.$$

From here we can define the integral operator

$$(Kf)(x) := \int_0^\infty k(t) f(t-x)\, dt$$

on $L^p(\mathbb{R}^+)$. Using special properties of Bessel functions, we can show that K is unbounded whenever $p \in (2, 4)$. To see this, consider

$$f(x) := x^{\nu/2} J_\nu(\beta\sqrt{x}), \qquad x > 0,$$

where the parameters ν and β will be determined soon. If

$$\nu + \frac{1}{p} > 0 \qquad \text{and} \qquad (\frac{\nu}{2} - \frac{1}{4})p + 1 < 0,$$

then $f \in L^p(\mathbb{R}^+)$. Moreover, one of the Sonine relations for Bessel functions [9] says that

$$(Kf)(x) = \frac{\beta^\nu}{\gamma(\gamma^2-\beta^2)^{\nu/2}} x^{\nu/2} J_{-\nu}\big((\gamma^2-\beta^2)^{1/2}\sqrt{x}\big), \qquad x > 0.$$

Therefore,

$$\begin{aligned}
\frac{\|Kf\|^p_{L^p(\mathbb{R})}}{\|f\|^p_{L^p(\mathbb{R})}} &= \frac{\beta^{\nu p}}{\gamma^p(\gamma^2-\beta^2)^{\nu p/2}} \frac{\int_0^\infty \left|x^{\nu/2} J_{-\nu}\big((\gamma^2-\beta^2)^{1/2}\sqrt{x}\big)\right|^p dx}{\int_0^\infty \left|x^{\nu/2} J_{\nu}\big((\gamma^2-\beta^2)^{1/2}\sqrt{x}\big)\right|^p dx} \\
&= \frac{\beta^{2(\nu p+1)}}{\gamma^p(\gamma^2-\beta^2)^{\nu p+1}} \frac{\int_0^\infty \left|x^{\nu/2} J_{-\nu}\big(\sqrt{x}\big)\right|^p dx}{\int_0^\infty \left|x^{\nu/2} J_{\nu}\big(\sqrt{x}\big)\right|^p dx}.
\end{aligned}$$

The above estimate shows that
$$\frac{\|Kf\|^p_{L^p(\mathbb{R})}}{\|f\|^p_{L^p(\mathbb{R})}} \to \infty$$
as β tends to γ. Hence, K is an unbounded operator on $L^p(\mathbb{R}^+)$ when $p \in (2,4)$. Therefore,
$$\widehat{k} \notin M_p(\mathbb{R}),$$
where $M_p(\mathbb{R})$ is the class of Fourier integrals that is defined similarly to $M_p(\mathbb{T})$ [12]. However, the Fourier transform of the kernel k is
$$\widehat{k}(x) = \frac{2}{\gamma}(1 - e^{-2i\alpha/x}),$$
where $\alpha = \gamma^2/8$ whence
$$e^{-2i\alpha/x} \notin M_p(\mathbb{R}).$$
This assertion is equivalent to
$$e^{-i\alpha/\tan(x/2)} \notin M_p(\mathbb{R}).$$
The advantage of the latter is that it is a 2π-periodic function on $\mathbb{R}$. Thus in light of de Leeuw's theorem [6], the atomic inner function given by the formula (28) is not a multiplier on $\mathscr{M}_p$.

Our second theorem has the same flavor and it leads to interesting corollaries which are easy to verify.

Theorem 15.8 (Lebedev [15]). *Let S be a singular inner function, and let E be its spectrum. Suppose that, for each $\varepsilon > 0$, there are at most countably many arcs I_n such that*
$$E \subseteq \bigcup_{n\geqslant 1} I_n$$
and
$$\sum_{n\geqslant 1} \frac{|I_n|^4}{d_E^3(I_n)} < \varepsilon. \tag{29}$$
Then $S \notin \mathscr{M}_p$ for any $p \in (1,\infty)\setminus\{2\}$.

Proof. Suppose that, to the contrary, there is a $p \in (1,\infty)\setminus\{2\}$ for which $S \in \mathscr{M}_p$. Therefore, for each $\varepsilon > 0$, there is a covering of E by arcs I_n such that (29) holds. In the light of Lemma 15.4, we deduce that
$$\mu(E) \leqslant \sum_{n\geqslant 1} \mu(I_n) \leqslant \sum_{n\geqslant 1} c\frac{|I_n|^4}{d_E^3(I_n)} \leqslant c\varepsilon.$$
Since ε is arbitrary, we must have $\mu = 0$ which is a contradiction. □

The success of Theorem 29 lies on the fact that a covering with the property 29 forces the μ-size of the set to be zero. Hence, if the covering holds just for a smaller subset $F \subseteq E$ for which $\mu(F) > 0$, then the conclusion of theorem is still valid. The condition of Theorem 29 is rather difficult to verify. However, we can establish a link between this result and the length of complementary arcs of E which is way easier to handle.

Corollary 15.9 (Lebedev [15]). *Let $E \subseteq \mathbb{T}$ be a closed subset of Lebesgue measure zero. Let J_n denote the arcs that are complementary to E and ordered such that*
$$|J_n| \geqslant |J_{n+1}|, \qquad n \geqslant 1.$$
Assume that
$$\liminf_{N\to\infty} \frac{1}{|J_N|^{3/4}} \sum_{n>N} |J_n| = 0. \tag{30}$$
If S is a singular inner function whose spectrum is contained in E, then $S \notin \mathscr{M}_p$ for any $p \in (1,\infty)\setminus\{2\}$.

Proof. Fix N. By removing $J_1, \dots, J_N$ from $\mathbb{T}$, we obtain N arcs which we call $K_1, \dots, K_N$. Let I_n be the arc of size $3|K_n|$ and concentric with K_n. Then

$$E \subseteq \bigcup_{n=1}^{N} I_n$$

and

$$d_E(I_n) \geqslant \min\{|K_n|, |J_N|\}.$$

Note that, as an extreme case, it is possible that J_n is a singleton. In this case, to avoid certain technical difficulties, we may replace it with a small arc, e.g., of size $2|J_N|$. Then

$$\begin{aligned}
\sum_{n=1}^{N} \frac{|I_n|^4}{d_E^3(I_n)} &\leqslant \sum_{\substack{1\leqslant n\leqslant N\\ |K_n|\leqslant|J_N|}} \frac{|I_n|^4}{d_E^3(I_n)} + \sum_{\substack{1\leqslant n\leqslant N\\ |K_n|>|J_N|}} \frac{|I_n|^4}{d_E^3(I_n)} \\
&\leqslant \sum_{\substack{1\leqslant n\leqslant N\\ |K_n|\leqslant|J_N|}} \frac{(3|K_n|)^4}{|K_n|^3} + \sum_{\substack{1\leqslant n\leqslant N\\ |K_n|>|J_N|}} \frac{(3|K_n|)^4}{|J_N|^3} \\
&\leqslant 81 \sum_{n=1}^{N} |K_n| + \frac{81}{|J_N|^3} \left(\sum_{n=1}^{N} |K_n| \right)^4 \\
&= 81 \sum_{n>N} |J_n| + \frac{81}{|J_N|^3} \left(\sum_{n>N} |J_n| \right)^4 .
\end{aligned}$$

By hypothesis, the right hand side can be made arbitrarily small. Therefore, the conditions of Theorem 15.8 are fulfilled and S cannot be in any $\mathscr{M}_p$ for any $p \in (1,\infty) \setminus \{2\}$. □

It is easy to see that if

$$\sum_{n\geqslant 1} |J_n| < \infty,$$

then (30) holds. We provide further classes below.

To obtain a similar useful corollary, we need a well-known concept. Given $E \subseteq \mathbb{T}$, its ε-neighborhood is

$$E_\varepsilon := \{\zeta \in \mathbb{T} : \operatorname{dist}(\zeta, E) < \varepsilon\}.$$

To measure the distance, we might use either the arc length or the Euclidian metric. The choice is irrelevant for the following result.

Corollary 15.10 (Lebedev [15]). *Let $E \subseteq \mathbb{T}$ be such that*

$$\lim_{\varepsilon\to 0} \frac{|E_\varepsilon|}{\varepsilon^{3/4}} = 0. \tag{31}$$

If S is a singular inner function whose spectrum is contained in E, then $S \notin \mathscr{M}_p$ for any $p \in (1,\infty) \setminus \{2\}$.

Proof. Fix $\varepsilon > 0$. Without loss of generality, we assume that $E_\varepsilon \neq \mathbb{T}$. Otherwise, we choose a smaller ε. The set E_ε is a disjoint union of a finite number of arcs, e.g., $I_1, \dots, I_N$. Certainly, for each arc,

$$d_E(I_n) \geqslant \varepsilon.$$

Therefore, for the covering I_n of E, we have

$$\sum_{n=1}^{N} \frac{|I_n|^4}{d_E^3(I_n)} \leqslant \frac{1}{\varepsilon^3} \sum_{n=1}^{N} |I_n|^4 \leqslant \frac{1}{\varepsilon^3} \left(\sum_{n=1}^{N} |I_n| \right)^4 = \frac{|E_\varepsilon|^4}{\varepsilon^3}.$$

By assumption, the right hand side can be made arbitrarily small and thus the conditions of Theorem 15.8 are fulfilled. Hence, $S \notin \mathscr{M}_p$ for any p with $p \neq 2$, $1 \leqslant p \leqslant \infty$. □

We end this section by showing that a wide range of generalized Cantor sets fall into the family of sets described above. Let $(\lambda_n)_{n\geqslant 0}$ be a sequence in the interval $(0,1)$. Let I be a closed arc on $\mathbb{T}$ of size $\rho_0 = |I|$. The generalized Cantor set is constructed as follows:

Step 1: from I we remove the concentric open arc of length $\lambda_0\rho_0$. As so, we obtain two arcs of length

$$\rho_1 = \frac{1}{2}(1-\lambda_0)\rho_0.$$

Step 2: from each of the remaining two arcs, we remove an open concentric arc of length $\lambda_1\rho_1$. As so, we obtain four arcs of length

$$\rho_2 = \frac{1}{2}(1-\lambda_1)\rho_1.$$

Step n: we continue the above process and in the n-th step we would have 2^n intervals of length

$$\rho_n = \frac{(1-\lambda_{n-1})\rho_{n-1}}{2} = \frac{(1-\lambda_{n-1})\cdots(1-\lambda_0)\rho_0}{2^n}.$$

The remaining set, which can be written as the intersection of the union of the above 2^n arcs, is called the generalized Cantor set. It is rather straightforward to see that if

$$\lim_{N\to\infty} \frac{1}{\lambda_N^3} \prod_{n=1}^{N-1} (1-\lambda_n) = 0$$

then the conditions of Theorem 15.8 are fulfilled. Hence, S is any singular inner function whose spectrum is contained in E then $S \notin \mathscr{M}_p$ for any $p \in (1,\infty)\setminus\{2\}$.

16 Orthogonality

The notion of Birkhoff-James orthogonality [2, 13] extends the concept of orthogonality from an inner product space to a more general normed linear space. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors belonging to a normed linear space $\mathcal{X}$. We say that $\mathbf{x}$ is *orthogonal* to $\mathbf{y}$ in the Birkhoff-James sense if

$$\|\mathbf{x} + \beta\mathbf{y}\|_{\mathcal{X}} \geqslant \|\mathbf{x}\|_{\mathcal{X}} \tag{32}$$

for all scalars β. In this situation we write $\mathbf{x} \perp_{\mathcal{X}} \mathbf{y}$. It is straightforward to show that when $\mathcal{X}$ is a Hilbert space, then $\mathbf{x} \perp \mathbf{y}$ is equivalent to $\mathbf{x} \perp_{\mathcal{X}} \mathbf{y}$. The relation $\perp_{\mathcal{X}}$ is generally neither symmetric nor linear. When $\mathcal{X} = \ell^p$, let us write $\perp_p$ in place of the more cumbersome $\perp_{\ell^p}$. Of particular importance here is the following explicit criterion for the relation $\perp_p$ when $p \in (1,\infty)$.

Theorem 16.1 (James [13])**.**

$$\mathbf{a} \perp_p \mathbf{b} \iff \sum_{k=0}^{\infty} |a_k|^{p-2}\overline{a}_k b_k = 0, \tag{33}$$

where any occurrence of "$|0|^{p-2}0$" in the sum above is interpreted as zero.

Borrowing from (33) we define, for a complex number $\alpha = re^{i\theta}$, and any $s > 0$, the quantity

$$\alpha^{\langle s\rangle} = (re^{i\theta})^{\langle s\rangle} := r^s e^{-i\theta}. \tag{34}$$

It is easy to verify that for any complex numbers α and β, real exponent $s > 0$, and integer $n \geqslant 0$, we have

$$(\alpha\beta)^{\langle s\rangle} = \alpha^{\langle s\rangle}\beta^{\langle s\rangle}$$
$$|\alpha^{\langle s\rangle}| = |\alpha|^s$$

$$\alpha^{\langle s\rangle}\alpha = |\alpha|^{s+1}$$
$$(\alpha^{\langle s\rangle})^n = (\alpha^n)^{\langle s\rangle}$$
$$(\alpha^{\langle p-1\rangle})^{\langle q-1\rangle} = \alpha.$$

In light of the definition (34), for $\mathbf{a} = (a_k)_{k\geqslant 0}$, let us write

$$\mathbf{a}^{\langle p-1\rangle} := (a_k^{\langle p-1\rangle})_{k\geqslant 0}. \tag{35}$$

If $\mathbf{a} \in \ell^p$, it is easy to see that $\mathbf{a}^{\langle p-1\rangle} \in \ell^q$ and thus from (2) and (33),

$$\mathbf{a} \perp_p \mathbf{b} \iff (\mathbf{b}, \mathbf{a}^{\langle p-1\rangle}) = 0. \tag{36}$$

Note that $\perp_p$ is therefore linear in its second argument, when $p \in (1,\infty)$, and it then makes sense to speak of a vector being orthogonal to a subspace of ℓ^p.

Due to the isometry between ℓ^p and ℓ_A^p, we can pass the Birkhoff-James orthogonality from ℓ^p to ℓ_A^p. More explicitly, if

$$a(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \text{and} \quad b(z) = \sum_{k=0}^{\infty} b_k z^k$$

are in ℓ_A^p, then

$$a \perp_p b \iff \mathbf{a} \perp_p \mathbf{b}.$$

Similarly, we define

$$a^{\langle p-1\rangle}(z) = \sum_{k=0}^{\infty} a_k^{\langle p-1\rangle} z^k.$$

Note that $a^{\langle p-1\rangle} \in \ell_A^q$ and (36) is rewritten as

$$a \perp_p b \iff (b, a^{\langle p-1\rangle}) = 0. \tag{37}$$

17 An application to zeros of analytic functions

In this section we discuss an application of multipliers and Birkhoff-James orthogonality to estimating zeros of polynomials from [20].

We first introduce a special function that enables us to connect an analytic function to an orthogonality condition. For $p \in (1,\infty)$ and $w \in \mathbb{D}\setminus\{0\}$, define

$$B_{p,w}(z) := \frac{1 - z/w}{1 - w^{\langle q-1\rangle} z}.$$

Since $|w^{\langle q-1\rangle}| = |w|^{q-1} < 1$, the function $B_{p,w}$ is analytic in $\mathbb{D}$. When $p = 2$ observe that $w^{\langle 2-1\rangle} = \overline{w}$ and so

$$B_{2,w} = \frac{1}{w}\frac{w - z}{1 - \overline{w}z}, \tag{38}$$

which is just a constant multiple of a Blaschke factor. Using the fact that $|B_{2,w}(e^{i\theta})| = |w|^{-1}$ for all θ, we see that

$$\int_0^{2\pi} \overline{B_{2,w}(e^{i\theta})} S^k B_{2,w}(e^{i\theta}) \frac{d\theta}{2\pi} = \frac{1}{|w|^2}\int_0^{2\pi} e^{ik\theta}\frac{d\theta}{2\pi} = 0, \quad k \geqslant 1.$$

Thus $B_{2,w} \perp_2 S^k B_{2,w}$ for all $k \geqslant 1$. It turns out that something analogous holds when $p \in (1,\infty)$. We refer the reader to [20] for the proof.

Lemma 17.1. *For each $p \in (1,\infty)$ and $w \in \mathbb{D}\setminus\{0\}$ we have*

(i) $B_{p,w} \perp_p B_{p,w} f$ for all $f \in \ell^p_A$ with $f(0) = 0$;

(ii) $\|B_{p,w}\|_p = \Big[1 + \dfrac{(1-|w|^q)^{p-1}}{|w|^p}\Big]^{1/p}$.

It could be said that $B_{p,w}$ plays a role in ℓ^p_A analogous to that of a Blaschke factor in the Hardy space H^2. However, here the situation is more complicated. See [5] for an exploration of this idea.

Here is our application of ℓ^p_A multipliers to obtain another proof of a set of classical bounds for the zeros of an analytic function. These bounds are tied to the well-known estimates for polynomial roots by Cauchy, Lagrange, and others (see, for example, [19]). Yet another proof of this result appears in [20], along with some extensions, based on using Birkhoff-James orthogonality more directly.

Theorem 17.2. *Suppose that*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is analytic in $\mathbb{D}$, *and* $a_0 \neq 0$. *If* $w \in \mathbb{D}$ *is a zero of* f, *then*

$$|w| \geqslant \Big[\Big|\frac{a_1}{a_0}\Big| + \Big|\frac{a_2}{a_0}\Big| + \Big|\frac{a_3}{a_0}\Big| + \cdots\Big]^{-1} \tag{39}$$

$$|w| \geqslant \Big[1 + \sup\Big\{\Big|\frac{a_1}{a_0}\Big|, \Big|\frac{a_2}{a_0}\Big|, \Big|\frac{a_3}{a_0}\Big|, \cdots\Big\}\Big]^{-1} \tag{40}$$

and

$$|w| \geqslant \Big[1 + \Big(\Big|\frac{a_1}{a_0}\Big|^p + \Big|\frac{a_2}{a_0}\Big|^p + \Big|\frac{a_3}{a_0}\Big|^p + \cdots\Big)^{q/p}\Big]^{-1/q} \tag{41}$$

for all $p \in (1, \infty)$.

Proof. Fix $p \in (1, \infty)$ and assume that $f \in \ell^p_A$. Otherwise the right hand side is zero and the inequality is automatically true. Recall that

$$[f] = \bigvee\{z^k f : k \geqslant 0\}$$

and note that $S[f]$ is a (closed) subspace of ℓ^p_A since S is an isometry. Write $\widetilde{f}$ for the metric projection of f onto $S[f]$. Since ℓ^p_A is uniformly convex [4], this metric projection is the unique function $\widetilde{f} \in \ell^p_A$ satisfying

$$\inf\{\|f - g\|_p : g \in S[f]\} = \|f - \widetilde{f}\|_p. \tag{42}$$

Let

$$f_1(z) := \frac{f(z)}{1 - z/w} = -w(Q_w f)(z)$$

and note, by hypothesis, that f_1 analytic in $\mathbb{D}$ and

$$f_1(0) = f(0) = a_0. \tag{43}$$

Furthermore, by Proposition 6.1, we have $f_1 \in \ell^p_A$. If $\mathcal{P}$ denotes the analytic polynomials, we have

$$\begin{aligned} \|f\|_p &\geqslant \|f - \widetilde{f}\|_p \quad \text{(by (42))} && (44)\\ &= \inf\{\|f + zqf\|_p : q \in \mathcal{P}\}\\ &= \inf\Big\{\Big\|f_1\Big(1 - \frac{z}{w}\Big) + zq\Big(1 - \frac{z}{w}\Big)f_1\Big\|_p : q \in \mathcal{P}\Big\}\\ &= \inf\Big\{\Big\|f_1\Big(1 - \frac{z}{w}\Big)(1 + zq)\Big\|_p : q \in \mathcal{P}\Big\}\\ &= \inf\Big\{\Big\|f_1\Big(1 - \frac{z}{w}\Big)Q\Big\|_p : Q \in \mathcal{P}, Q(0) = 1\Big\}. && (45) \end{aligned}$$

For any $Q \in \mathcal{P}$ with $Q(0) = 1$ we have the identity

$$f_1 B_{p,w}(1 - w^{\langle q-1 \rangle} z) Q = f_1 \Big(1 - \frac{z}{w}\Big) Q. \tag{46}$$

Furthermore, if

$$\frac{Q}{1 - w^{\langle q-1 \rangle} z} = \sum_{k=0}^{\infty} d_k z^k$$

and

$$Q_n = \sum_{k=0}^{n} d_k z^k,$$

then $Q_n \in \mathcal{P}$ with $Q_n(0) = 1$ and by Corollary 12.2, we get

$$\Big\| Q_n - \frac{Q}{1 - w^{\langle q-1 \rangle} z} \Big\|_{\mathscr{M}_p} \to 0,$$

From here we see that

$$\Big\| Q_n f_1 \Big(1 - \frac{z}{w}\Big) - f_1 B_{p,w} Q \Big\|_p \leqslant \Big\| 1 - \frac{z}{w} \Big\|_{\mathscr{M}_p} \Big\| Q_n - \frac{Q}{1 - w^{\langle q-1 \rangle} z} \Big\|_{\mathscr{M}_p} \| f_1 \|_p$$

which goes to zero as $n \to \infty$. Combine this with (46) to see that the sets

$$\{ f_1 B_{p,w} Q : Q \in \mathcal{P}, Q(0) = 1 \} \quad \text{and} \quad \{ f_1 (1 - \frac{z}{w}) Q : Q \in \mathcal{P}, Q(0) = 1 \}$$

have the same closure in ℓ^p_A. Thus (45) is equal to

$$\inf\{ \| f_1 B_{p,w} Q \|_p : Q \in \mathcal{P}, Q(0) = 1 \}. \tag{47}$$

In a somewhat similar way, let $Q \in \mathcal{P}$ with $Q(0) = 1$. If

$$f_1 Q = \sum_{k=0}^{\infty} c_k z^k$$

and

$$Q_n = \frac{1}{a_0} \sum_{k=0}^{n} c_k z^k,$$

then $Q_n \in \mathcal{P}$ with $Q_n(0) = 1$ (see (43)) and

$$\| a_0 Q_n - f_1 Q \|_p = \left(\sum_{k=n+1}^{\infty} |c_k|^p \right)^{1/p} \to 0.$$

Thus

$$\| a_0 B_{p,w} Q_n - f_1 B_{p,w} Q \|_p \leqslant \| B_{p,w} \|_{\mathscr{M}_p} \| a_0 Q_n - f_1 Q \|_p \to 0.$$

This says that the closure of

$$\{ a_0 B_{p,w} Q : Q \in \mathcal{P}, Q(0) = 1 \}$$

in ℓ^p_A is contained in the closure of

$$\{ f_1 B_{p,w} Q : Q \in \mathcal{P}, Q(0) = 1 \}.$$

From this containment of closures, we see that (47) is bounded below by

$$|a_0| \inf\{ \| B_{p,w} Q \|_p : Q \in \mathcal{P}, Q(0) = 1 \}. \tag{48}$$

Using the fact that $Q(0) = 1$, write

$$B_{p,w} Q = B_{p,w} + (Q - Q(0)) B_{p,w}$$

and use Lemma 17.1 and (32) to see that (48) is bounded below by

$$|a_0| \cdot \|B_{p,w}\|_p.$$

By Lemma 17.1, the above is equal to

$$|a_0| \cdot \Big[1 + \frac{(1-|w|^q)^{p-1}}{|w|^p}\Big]^{1/p}.$$

Following this all the way back to (44) yields the inequality

$$\|f\|_p \geqslant |a_0| \cdot \Big[1 + \frac{(1-|w|^q)^{p-1}}{|w|^p}\Big]^{1/p}$$

from which it follows

$$\frac{(1-|w|^q)^{p-1}}{|w|^p} \leqslant \Big|\frac{a_1}{a_0}\Big|^p + \Big|\frac{a_2}{a_0}\Big|^p + \Big|\frac{a_3}{a_0}\Big|^p + \cdots.$$

Writing

$$M := \Big\{\Big|\frac{a_1}{a_0}\Big|^p + \Big|\frac{a_2}{a_0}\Big|^p + \Big|\frac{a_3}{a_0}\Big|^p + \cdots\Big\}^{1/p},$$

we have

$$\begin{aligned}
\frac{(1-|w|^q)^{p-1}}{|w|^p} &\leqslant M^p \\
(1-|w|^q)^{p-1} &\leqslant |w|^p M^p \\
(1-|w|^q)^{1/q} &\leqslant |w| M \\
(1-|w|^q) &\leqslant |w|^q M^q \\
\frac{1}{(M^q+1)^{1/q}} &\leqslant |w|.
\end{aligned}$$

This proves (41). The bounds in (39) and (40) are obtained by taking the limits $p \to 1$ and $p \to \infty$, respectively. □

18 Coefficient estimates

We know from Corollary 8.4 that if $h \in \mathcal{M}_p$ then in fact $h \in \ell^p_A$ and

$$\|h\|_p \leqslant \|h\|_{\mathcal{M}_p}. \tag{49}$$

If we use the $h^{\langle p-1\rangle}$ idea via Birkhoff-James orthogonality, we can sharpen this inequality considerably.

Proposition 18.1. *Suppose that $p \in (1,\infty)$ and $h \in \mathscr{M}_p$. Then*

$$\begin{aligned}
\|h\|_p^{p-1} \cdot \|h\|_{\mathscr{M}_p} \geqslant \Big[&\Big(|a_0|^p + |a_1|^p + |a_2|^p + \cdots\Big)^q \\
&+ \Big|a_1^{\langle p-1\rangle} a_0 + a_2^{\langle p-1\rangle} a_1 + a_3^{\langle p-1\rangle} a_2 + \cdots\Big|^q \\
&+ \Big|a_2^{\langle p-1\rangle} a_0 + a_3^{\langle p-1\rangle} a_1 + a_4^{\langle p-1\rangle} a_2 + \cdots\Big|^q \\
&+ \Big|a_3^{\langle p-1\rangle} a_0 + a_4^{\langle p-1\rangle} a_1 + a_5^{\langle p-1\rangle} a_2 + \cdots\Big|^q \\
&+ \cdots\Big]^{1/q}
\end{aligned} \tag{50}$$

Proof. Let $f(z) = \sum_{k=0}^{\infty} b_k z^k$ belong to ℓ^p_A. Let us write $\mathbf{f}$ for the isomorph of f in the sequence space ℓ^p and $\mathbf{hf}$ for the corresponding sequence for hf. Then for any $\mathbf{c} \in \ell^q$, the linear functional

$$f \mapsto (\mathbf{hf}, \mathbf{c})$$

is continuous on ℓ^p_A with norm no greater than $\|h\|_{\mathscr{M}_p}\|c\|_q$. In particular, we could consider the basic vectors

$$\mathbf{e}_0 := (1, 0, 0, 0, \ldots),$$
$$\mathbf{e}_1 := (0, 1, 0, 0, \ldots),$$
$$\mathbf{e}_2 := (0, 0, 1, 0, \ldots),$$
$$\vdots$$

for ℓ^q, where we find that

$$\begin{aligned}\langle \mathbf{hf}, \mathbf{e}_k\rangle &= \sum_{j=0}^{k} a_{k-j} b_j \\ &= (\mathbf{f}, (a_k, a_{k-1}, \ldots, a_1, a_0, 0, 0, \ldots)).\end{aligned}$$

Set

$$\mathbf{u}^{(k)} := (a_k, a_{k-1}, \ldots, a_1, a_0, 0, 0, \ldots).$$

It follows that for any $\mathbf{c} := (c_0, c_1, c_2, \ldots) \in \ell^q$, we have

$$(\mathbf{hf}, \mathbf{c}) = (\mathbf{f}, \sum_{k=0}^{\infty} c_k \mathbf{h}^{(k)})$$

If we interpret this operation

$$\mathbf{f} \mapsto (\mathbf{f}, \sum_{k=0}^{\infty} c_k \mathbf{u}^{(k)})$$

as a bounded linear functional on ℓ^p, the Riesz Representation Theorem says that the sequence

$$\sum_{k=0}^{\infty} c_k \mathbf{u}^{(k)}$$

must belong to ℓ^q. Let us write out what that means term by term:

$$\begin{aligned}&\Big|a_0c_0 + a_1c_1 + a_2c_2 + \cdots\Big|^q \\ &+ \Big|a_0c_1 + a_1c_2 + a_2c_3 + \cdots\Big|^q \\ &+ \Big|a_0c_2 + a_1c_3 + a_2c_4 + \cdots\Big|^q \\ &+ \cdots \\ &\leqslant \|h\|_{\mathscr{M}_q} \cdot \|c\|^q_{\ell^q}.\end{aligned} \tag{51}$$

Now it is a simple matter to substitute

$$\mathbf{c} = (a_0^{\langle p-1\rangle}, a_1^{\langle p-1\rangle}, a_2^{\langle p-1\rangle}, \ldots),$$

which belongs to ℓ^q, to obtain (50). □

Remark 18.2. *Notice that if you drop all but the first line in the right side of (50), then you obtain (49), and so Proposition 18.1 does indeed represent a dramatic sharpening of (49). Also notice that if h is Birkhoff-James orthogonal in ℓ^p_A to $z^k h$ for all positive integers k, then in fact all but the first line in the right side of (50) is zero. This is the case when $h = B_{p,w}$.*

19 Hadamard multipliers

Let f and g be two analytic functions on $\mathbb{D}$ with Taylor series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The *Hadamard product* of f and g is defined by

$$(f \diamond g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Since f and g are analytic on $\mathbb{D}$, it is easy to verify that $f \diamond g$ is also analytic on $\mathbb{D}$. Note that the disk of convergence might be larger, e.g., consider odd and even functions. We say that h is a *Hadamard multiplier* of ℓ_A^p if the operator

$$\mathfrak{M}_h : \ell_A^p \to \ell_A^p, \quad M_h f = h \diamond f.$$

is well-defined and continuous. As a rule of thumb, if the norm is defined via Taylor coefficients, e.g., Dirichlet space, Hardy space H^2, Bergman space A^2, and ℓ_A^p spaces, then determining the Hadamard multipliers if usually not a difficult task. However, if the norm is defined differently, e.g., via an integral as in the Hardy spaces H^p, the Bergman spaces A^p, and the harmonically weighted Dirichlet spaces D_μ, determining the Hadamard multipliers is non-trivial.

In [7] one can find a discussion of the Hadamard multipliers between various H^p classes. Here is a characterization of the Hadamard multipliers for ℓ_A^p.

Theorem 19.1. *Let $p \in (1, \infty)$. Then the Hadamard multiplier space of ℓ_A^p is isometrically isomorphic to ℓ_A^∞. More explicitly, for each $h \in \ell_A^\infty$, we have*

$$\|\mathfrak{M}_h\| = \|h\|_{\ell_A^\infty}.$$

Conversely, if $\|\mathfrak{M}_h\| < \infty$, then $h \in \ell_A^\infty$.

Proof. First suppose that $h(z) = \sum_{n=0}^{\infty} a_n z^n \in \ell_A^\infty$. Then for each $f(z) = \sum_{n=0}^{\infty} b_n z^n \in \ell_A^p$, we have

$$\begin{aligned}
\|\mathfrak{M}_h f\|_{\ell_A^p} &= \left(\sum_{n=0}^{\infty} |a_n b_n|^p \right)^{\frac{1}{p}} \\
&\leqslant \left(\sup_{n \geqslant 0} |a_n| \right) \left(\sum_{n=0}^{\infty} |b_n|^p \right)^{\frac{1}{p}} \\
&= \|h\|_{\ell_A^\infty} \, \|f\|_{\ell_A^p}.
\end{aligned}$$

Thus h is in fact a Hadamard multiplier and moreover,

$$\|\mathfrak{M}_h\| \leqslant \|h\|_{\ell_A^\infty}.$$

To show that equality holds, consider the monomials $f_m(z) = z^m$, $m \geqslant 0$. Then $\|f_m\|_{\ell_A^p} = 1$ and

$$\|\mathfrak{M}_h f\|_{\ell_A^p} = \left(\sum_{n=0}^{\infty} |a_n b_n|^p \right)^{\frac{1}{p}} = |a_m|.$$

Thus

$$\|\mathfrak{M}_h\| \geqslant |a_m| \qquad m \geqslant 0,$$

which implies the reverse inequality

$$\|\mathfrak{M}_h\| \geqslant \|h\|_{\ell_A^\infty}.$$

By a similar argument, we can show that if $\|\mathfrak{M}_h\| < \infty$, then $h \in \ell_A^\infty$. In fact, we must have

$$\|\mathfrak{M}_h f\|_{\ell_A^p} \leqslant \|\mathfrak{M}_h\| \, \|f\|_{\ell_A^p}$$

for all $f \in \ell_A^p$. In particular, if we apply the inequality above to f_m, we obtain

$$|a_m| \leqslant \|\mathfrak{M}_h\|, \qquad m \geqslant 0.$$

Taking supremum with respect to m implies $h \in \ell_A^\infty$. □

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