



Concrete Operators

Research Article

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Toeplitz operators and Wiener-Hopf factorisation: an introduction

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Abstract: Wiener-Hopf factorisation plays an important role in the theory of Toeplitz operators. We consider here Toeplitz operators in the Hardy spaces H^p of the upper half-plane and we review how their Fredholm properties can be studied in terms of a Wiener-Hopf factorisation of their symbols, obtaining necessary and sufficient conditions for the operator to be Fredholm or invertible, as well as formulae for their inverses or one-sided inverses when these exist. The results are applied to a class of singular integral equations in $L^p(\mathbb{R})$.

Keywords: Toeplitz operator, Wiener-Hopf factorisation, Singular integral equations

MSC: 45E10, 47A68, 47B35

1 Introduction

Wiener-Hopf factorisation, which was developed mainly in connection with the study of singular integral equations, convolution type operators and Riemann-Hilbert problems that are important in a variety of areas in Mathematics, Physics and Engineering, plays a prominent role in the theory of Toeplitz operators. Indeed many spectral properties, Fredholmness, invertibility and formulae for the inverses, when they exist, can be expressed in terms of a Wiener-Hopf (WH for short) factorisation of their symbols.

Several monographs have appeared in the last four decades ([3, 8, 10, 12, 13, 15] for instance) intending to present, as systematically and completely as possible, the myriad of results that kept appearing on WH factorisation and its relations with singular integral equations, boundary value problems and Toeplitz operators. However, the enormous quantity of results and the extension of most monographs make it difficult for someone not familiar with this topic to come to grips with the vast amount of information contained there and quickly cut through to get to the heart of the matter.

The present article is based on a mini-course given in the context of the 13th Advanced Course in Operator Theory and Complex Analysis held in Lyon in 2016. Its aim is to introduce the reader to a set of concepts and results enabling one to rapidly understand the essential ideas behind the notion of WH factorisation and its usefulness in the study of Toeplitz operators.

It would be clearly impossible to cover in a single article this vast domain, so the purpose of this paper required choices regarding which results to present, their setting, how to show their interrelations and how they build upon each other. This was done based on the author's own experience in learning the subject, as well as the author's research interests.

First, it should be explicitly noticed that only scalar-valued symbols will be considered. The factorisation of matrix-valued functions and the study of Toeplitz operators with matrix symbols involve considerably more difficulties that would overshadow the exposition in a first approach.

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Secondly, we study here Toeplitz operators defined in the Hardy spaces of the (upper) half-plane, $H^p(\mathbb{C}^+)$, with $1 < p < \infty$. The existing literature overwhelmingly puts its emphasis on factorisation relative to a closed contour such as the unit circle and on Toeplitz operators defined in the Hardy spaces of the disk, whether in the context of H^2 or H^p , $1 < p < \infty$. However, the natural setting in which many problems appear, as for instance those formulated in terms of finite-interval convolution equations, is the real line. Translating the results presented here from the context of the unit circle to that of the real line, for $p \neq 2$, would require considering weighted Hardy spaces of the disk (see [4, 13], for instance). By considering the half-plane setting from the start, on the one hand, we avoid this; on the other hand, it allows for a better understanding of the reasoning and the techniques involved when the real line is considered, provides a direct approach to problems that are naturally formulated in the context of \mathbb{R} , and allows the use of tools such as the Fourier transform.

The paper is organised as follows. In Section 2 the fundamental function spaces used in the paper and their main properties are described. Toeplitz operators and some of their basic properties are presented in Section 3, and their relation to paired operators is described in Section 4. In Section 5 the Fredholm properties and invertibility of a Toeplitz operator are characterised in terms of a WH factorisation of its symbol. Piecewise continuous symbols are studied in Section 6, where criteria for existence of a WH factorisation of those functions are presented and explicit formulae for the factors are obtained in a particular case. The results of the previous sections are used in Section 7 to study the spectrum of the singular integral operator S_J in three cases, $J = \mathbb{R}$, $J = \mathbb{R}^+$, $J = [0, 1]$, and to obtain an expression for the resolvent operator $(S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+})^{-1}$ with $\lambda \in \mathbb{C} \setminus [-1, 1]$.

The main references for Section 2 are [6, 7, 14, 17]; for Section 3, [3, 10, 17]; for Section 4, [3, 9, 15]; for Section 5, [6, 13]; for Sections 6 and 7, [6, 17].

2 Spaces of functions

For $0 < p < \infty$, let L_p denote the Lebesgue space of all complex Lebesgue measurable functions f which are p -integrable on \mathbb{R} , with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{1/p}.$$

We denote by H_p^+ the Hardy space of all functions f which are analytic in the upper half plane \mathbb{C}^+ such that, for all $y > 0$, $|f(x + iy)|^p$ is integrable over \mathbb{R} and there is a constant $M \in \mathbb{R}^+$ such that

$$\int_{\mathbb{R}} |f(x + iy)|^p dx < M \quad \text{for all } y > 0,$$

with the norm

$$\|f\|_{H_p^+} = \sup_{y \in \mathbb{R}^+} \left(\int_{\mathbb{R}} |f(x + iy)|^p dx \right)^{1/p}$$

([7]). If $f_+ \in H_p^+$ then the nontangential boundary function

$$\tilde{f}_+(x) = \lim_{z \rightarrow x, z \in \mathbb{C}^+} f_+(z)$$

is defined a.e. on \mathbb{R} and belongs to L_p , with $\|\tilde{f}_+\|_p = \|f_+\|_{H_p^+}$.

In what follows, we assume that $1 \leq p < \infty$. We have

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\tilde{f}_+(t)}{t - z} dt = \begin{cases} f_+(z), & \text{if } z \in \mathbb{C}^+ \\ 0, & \text{if } z \in \mathbb{C}^-. \end{cases} \tag{1}$$

Conversely, if $\tilde{f}_+ \in L_p$ and $\int_{\mathbb{R}} \frac{\tilde{f}_+(t)}{t - z} dt = 0$ for all $z \in \mathbb{C}^+$, then f_+ defined by

$$f_+(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\tilde{f}_+(t)}{t - z} dt \quad \text{for } z \in \mathbb{C}^+$$

belongs to H_p^+ and its boundary value on \mathbb{R} is \tilde{f}_+ .

We define H_p^- similarly for the lower half plane \mathbb{C}^- and analogous results hold for $f_- \in H_p^-$. In particular, if \tilde{f}_- is the (nontangential) boundary value of f_- on \mathbb{R} then

$$-\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\tilde{f}_-(t)}{t-z} dt = \begin{cases} f_-(z), & \text{if } z \in \mathbb{C}^- \\ 0, & \text{if } z \in \mathbb{C}^+. \end{cases} \quad (2)$$

Identifying each function in H_p^\pm with its boundary value on \mathbb{R} , we have that H_p^+ and H_p^- are closed subspaces of L_p and the following holds:

$$H_p^+ \cap H_p^- = \{0\} \quad (3)$$

$$\overline{H_p^+} = H_p^- \quad (4)$$

and for $p, r \in]1, \infty[$

$$f \in H_p^\pm, g \in H_r^\pm \Rightarrow fg \in H_s^\pm \text{ with } \frac{1}{s} = \frac{1}{p} + \frac{1}{r}. \quad (5)$$

By L_∞ we denote the space of all essentially bounded functions on \mathbb{R} , with the norm

$$\|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}} |f(x)|,$$

and by H_∞^\pm the space of all functions which are analytic and bounded in \mathbb{C}^\pm . We identify H_∞^\pm with the subspaces of L_∞ consisting of their (nontangential) boundary functions on \mathbb{R} . We have

$$a_\pm \in H_\infty^\pm, f_\pm \in H_p^\pm \Rightarrow a_\pm f_\pm \in H_p^\pm, \text{ for } 1 \leq p \leq \infty, \quad (6)$$

$$H_\infty^+ \cap H_\infty^- = \mathbb{C}. \quad (7)$$

We also define, for $1 \leq p < \infty$,

$$\mathcal{H}_p^\pm = \lambda_\pm H_p^\pm, \text{ where } \lambda_\pm(x) = x \pm i \quad (8)$$

([17]) and we have

$$H_p^\pm \subset \mathcal{H}_p^\pm, H_p^\pm \subset \mathcal{H}_1^\pm, H_\infty^\pm \subset \mathcal{H}_p^\pm \text{ for all } p > 1; \quad (9)$$

$$\overline{\mathcal{H}_p^+} = \mathcal{H}_p^- \text{ for all } p \geq 1; \quad (10)$$

$$\mathcal{H}_p^+ \cap \mathcal{H}_p^- = \mathbb{C} \text{ for all } p > 1; \quad (11)$$

$$\mathcal{H}_1^+ \cap \mathcal{H}_1^- = \{0\}, \lambda_+ \mathcal{H}_1^+ \cap \lambda_- \mathcal{H}_1^- = \mathbb{C}; \quad (12)$$

$$H_p^\pm \mathcal{H}_{p'}^\pm \subset \mathcal{H}_1^\pm \text{ where } \frac{1}{p} + \frac{1}{p'} = 1, p > 1. \quad (13)$$

By the Luzin-Privalov theorem, if f is meromorphic in \mathbb{C}^+ (resp. \mathbb{C}^-) and has nontangential boundary value 0 on a set $E \subset \mathbb{R}$ with positive measure, then $f(z) = 0$ for all $z \in \mathbb{C}^+$ (resp. \mathbb{C}^-). Thus if a function f belonging to H_p^\pm or to \mathcal{H}_p^\pm ($1 \leq p \leq \infty$) vanishes on a set of positive measure $E \subset \mathbb{R}$ then f is identically zero.

By $C(\mathbb{R}_\infty)$ we denote the space of all continuous functions on $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$, with the supremum norm, and by \mathcal{R} the set of all rational functions without poles on \mathbb{R}_∞ . The set of all piecewise continuous functions on \mathbb{R}_∞ , i.e., $f \in L_\infty(\mathbb{R})$ such that f is continuous on \mathbb{R}_∞ with the possible exception of finitely many points and with finite limits $f(\pm\infty)$, $f(x^\pm)$ for all $x \in \mathbb{R}$, is denoted by $PC(\mathbb{R}_\infty)$.

If \mathcal{A} is an algebra, we denote by \mathcal{GA} the group of all invertible elements in \mathcal{A} .

3 Toeplitz operators in H_p^+

Let now $p \in]1, \infty[$. For $f \in L_p$, the equality

$$S_{\mathbb{R}} f(x) = \frac{1}{i\pi} \int_{\mathbb{R}} \frac{f(t)}{t-x} dt, \tag{14}$$

where the integral is understood as a Cauchy principal value, defines, by a classical theorem of M. Riesz ([16]), a bounded operator in L_p . We have

$$S_{\mathbb{R}}^* = S_{\mathbb{R}}, \quad S_{\mathbb{R}}^2 = I \tag{15}$$

$$\overline{S_{\mathbb{R}} f} = -S_{\mathbb{R}} \bar{f}, \quad f \in L_p \tag{16}$$

$$S_{\mathbb{R}} a - a S_{\mathbb{R}} \text{ is compact if } a \in C(\mathbb{R}_{\infty}). \tag{17}$$

We associate with $S_{\mathbb{R}}$ two complementary projections

$$P^{\pm} = \frac{I \pm S_{\mathbb{R}}}{2}. \tag{18}$$

By the Sokhotski-Plemelj formulae, $P^{\pm} f$ are the nontangential boundary values on \mathbb{R} of the analytic functions

$$\pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z} dt, \quad z \in \mathbb{C}^{\pm}, \tag{19}$$

respectively. Thus $P^{\pm} L_p = H_p^{\pm}$ and $\ker P^{\pm} = H_p^{\mp}$. In terms of these projections, (6) can be expressed by

$$P^{\pm} a_{\pm} P^{\pm} = a_{\pm} P^{\pm}, \quad \text{if } a_{\pm} \in H_{\infty}^{\pm}. \tag{20}$$

It follows from (16) and (4) that, if C denotes complex conjugation ($C f = \bar{f}$) then

$$C P^{\pm} = P^{\mp} C, \quad \text{i.e., } C P^{\pm} C = P^{\pm} \tag{21}$$

and, from (17), that $P^{\pm} a - a P^{\pm}$ is compact in L_p if $a \in C(\mathbb{R}_{\infty})$.

The Toeplitz operator with symbol $g \in L_{\infty}$, T_g , is defined in H_p^+ by

$$T_g f_+ = P_+ g f_+, \quad f_+ \in H_p^+. \tag{22}$$

Whenever we want to make the domain H_p^+ explicit, we will use the notation $T_{g,p}$. We have $\|g\|_{\infty} \leq \|T_g\| \leq C \|g\|_{\infty}$ ([6]), where $C = 1$ if $p = 2$. Thus $T_g = 0$ if and only if $g = 0$.

The following properties hold:

- (P1) $(T_{g,p})^* = T_{\bar{g},p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$;
- (P2) if $a_{\pm} \in H_{\infty}^{\pm}$ and $g \in L_{\infty}$, then $T_{a_-} T_g = T_{g a_-}$, $T_g T_{a_+} = T_{g a_+}$;
- (P3) if $a_{\pm} \in \mathcal{G} H_{\infty}^{\pm}$ then $T_{a_{\pm}}$ is invertible and $(T_{a_{\pm}})^{-1} = T_{(a_{\pm})^{-1}}$.

Note that in general $(T_g)^{-1} \neq T_{g^{-1}}$.

For $f_+ \in H_p^+$ we have that

$$f_+ \in \ker T_g \Leftrightarrow g f_+ = f_- \in H_p^- \tag{23}$$

where the equality holds on \mathbb{R} . Defining r by

$$r(x) = \frac{x-i}{x+i}, \quad x \in \mathbb{R}, \tag{24}$$

it follows from (23) that, for $k \geq 0$,

$$\ker T_{r,k} = \{0\} \text{ , } \dim \ker T_{r,-k} = k. \tag{25}$$

On the other hand from (P2) we have, for $g \in L_\infty$, $k \in \mathbb{Z}$,

$$T_g r^k = T_g T_{r^k} \text{ , if } k \geq 0 \tag{26}$$

$$T_g r^k = T_{r^k} T_g \text{ , if } k \leq 0. \tag{27}$$

From (23)-(27) we conclude the following.

Proposition 3.1. *Let $k \in \mathbb{Z}$ and let r be given by (24).*

- (i) $T_{r,k}$ is left invertible if $k \geq 0$, with left inverse $(T_{r,k})_l^{-1} = T_{r,-k}$; $\dim \ker T_{r,k} = 0$, $\dim \ker T_{r^k}^* = k$.
- (ii) $T_{r,k}$ is right invertible if $k \leq 0$, with right inverse $(T_{r,k})_r^{-1} = T_{r,-k}$; $\dim \ker T_{r,k} = |k|$, $\dim \ker T_{r^k}^* = 0$.
- (iii) $T_{r,k}$ is Fredholm with index $-k$, and it is invertible if and only if $k = 0$.

Recall that an operator T is Fredholm if and only if its kernel and the kernel of its adjoint are finite dimensional, and the range of T is closed; the Fredholm index of T is $Ind T = \dim \ker T - \dim \ker T^*$. We will see in the next section that, in a certain sense, all Toeplitz operators which are Fredholm can be reduced, through an appropriate factorisation of their symbol, to a Toeplitz operator with a simple rational symbol such as the one considered in Proposition 3.1. From the latter we see moreover that: (i) either $\dim \ker T_{r,k} = 0$ or $\dim \ker T_{r^k}^* = 0$; (ii) $T_{r,k}$ is always one sided invertible. We will also show in the next theorem that property (i) is shared by all non-zero Toeplitz operators, while (ii) holds for all Toeplitz operators which are Fredholm.

We have:

Theorem 3.2. (Coburn’s Lemma) *If $g \in L_\infty$, $g \neq 0$, then either $\ker T_g = \{0\}$ (in H_p^+) or $\ker T_g^* = \{0\}$ (in $H_{p'}^+$).*

Proof. Suppose that there are $f_+ \in H_p^+$, $h_+ \in H_{p'}^+$, different from 0, such that $f_+ \in \ker T_g$, $h_+ \in \ker T_g^* = \ker T_{\bar{g}}$. This means that there are $f_- \in H_p^- \setminus \{0\}$, $h_- \in H_{p'}^- \setminus \{0\}$ such that

$$g f_+ = f_- \text{ , } \bar{g} h_+ = h_-.$$

Thus

$$f_- \bar{h}_+ = g f_+ \bar{h}_+ = f_+ g \bar{h}_+ = f_+ \bar{h}_-. \tag{28}$$

Since the left hand side of (28) represents a function in H_1^- and the right hand side represents a function in H_1^+ , we conclude that both are zero. By the Luzin-Privalov theorem either f_+ or \bar{h}_- must be zero, which is impossible since f_+ , $h_+ \neq 0$. □

Coburn’s Lemma can also be stated as follows: a non-zero Toeplitz operator has a trivial kernel or a dense range.

A necessary and sufficient condition for an operator $A \in \mathcal{L}(X, Y)$, where X and Y are Banach spaces, to have a left (resp. right) inverse is that $\ker A = \{0\}$ and $\text{Im } A$ is closed and complemented in Y (resp., $\text{Im } A = Y$ and $\ker A$ is complemented in X) ([9]). Therefore we have:

Corollary 3.3. *Let T_g be a Toeplitz operator in H_2^+ . If $\text{Im } T_g$ is closed, then T_g is (at least) one sided invertible.*

Corollary 3.4. *T_g is invertible if and only if it is Fredholm with index 0.*

By the Hartman-Wintner theorem ([3]), if T_g is semi-Fredholm, i.e., $\text{Im } T_g$ is closed and $\ker T_g$ or $\ker T_g^*$ are finite-dimensional, then $g \in \mathcal{GL}_\infty$. Therefore we also have:

Corollary 3.5. *If $\text{Im } T_g$ is closed, then $g \in \mathcal{GL}_\infty$.*

4 Toeplitz operators and paired operators

It is clear that

$$T_g = P^+ g I_{|H_P^+} = P^+ g P^+_{|H_P^+}.$$

Therefore Toeplitz operators belong to the class of operators of the form

$$P A|_{\text{Im } P} = P A P|_{\text{Im } P}$$

where $P \in \mathcal{L}(X)$ is a projection in the Banach space X and $A \in \mathcal{L}(X)$. These operators, which are called *operators of Wiener-Hopf (WH) type*, are closely connected with operators in X of the form $PAP + Q$ (where $Q = I - P$) and to the *paired operators* $AP + Q$ and $PA + Q$ as follows:

Proposition 4.1. *Let X be a Banach space, $A \in \mathcal{L}(X)$, and let $P, Q \in \mathcal{L}(X)$ be complementary projections, i.e., $P + Q = I$. Then*

$$AP + Q = (PAP + Q)(I + QAP)$$

$$PA + Q = (I + PAQ)(PAP + Q)$$

where $I + QAP$ and $I + PAQ$ are invertible operators with inverses $I - QAP$ and $I - PAQ$, respectively.

If $X, Y, \tilde{X}, \tilde{Y}$ are Banach spaces, we say that two operators $T : X \rightarrow \tilde{X}$ and $S : Y \rightarrow \tilde{Y}$ are *equivalent* ($T \sim S$) if and only if $T = ESF$ where E, F are invertible operators. More generally, T and S are *equivalent after extension* if and only if there exist (possibly trivial) Banach spaces X_0, Y_0 , called *extension spaces*, and invertible bounded linear operators $E : \tilde{Y} \oplus Y_0 \rightarrow \tilde{X} \oplus X_0$ and $F : X \oplus X_0 \rightarrow Y \oplus Y_0$, such that

$$\begin{pmatrix} T & 0 \\ 0 & I_{X_0} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Y_0} \end{pmatrix} F. \quad (29)$$

In this case we say that $T \overset{*}{\sim} S$ ([2]).

From Proposition 4.1 we see that

$$AP + Q \sim PA + Q \sim PAP + Q. \quad (30)$$

On the other hand, taking for instance the operator $AP + Q$, we can write

$$\begin{pmatrix} AP + Q & 0 \\ 0 & I_{\{0\}} \end{pmatrix} = E \begin{pmatrix} P A P|_{\text{Im } P} & 0 \\ 0 & Q|_{\text{Im } Q} \end{pmatrix} F$$

where

$$F = \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix} \begin{pmatrix} I + QAP & 0 \\ 0 & I_{\{0\}} \end{pmatrix} : X \oplus \{0\} \rightarrow \text{Im } P \oplus \text{Im } Q$$

and

$$E = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix} : \text{Im } P \oplus \text{Im } Q \rightarrow X \oplus \{0\}$$

are invertible; thus each one of the three operators in (30) is equivalent after extension to $P A P|_{\text{Im } P}$. In particular for $P = P^+, Q = P^-$ we see that

$$T_g = P^+ g P^+_{|H_P^+} \overset{*}{\sim} P^+ g I + P^- \sim g P^+ + P^-. \quad (31)$$

Operators which are equivalent after extension share many properties, namely the following ([2]).

Theorem 4.2. *Let $T : X \rightarrow \tilde{X}, S : Y \rightarrow \tilde{Y}$ be operators and assume that $T \overset{*}{\sim} S$. Then*

1. $\ker T \simeq \ker S$;

2. $\text{Im } T$ is closed if and only if $\text{Im } S$ is closed and, in that case, $\widetilde{X}/\text{Im } T \simeq \widetilde{Y}/\text{Im } S$;
3. if one of the operators T , S is generalised (left, right) invertible, then the other is generalised (left, right) invertible too;
4. T is Fredholm if and only if S is Fredholm and in that case $\dim \ker T = \dim \ker S$, $\text{codim Im } T = \text{codim Im } S$.

It follows from (31) and from Theorem 4.2 that we can reduce the study of many properties of Toeplitz operators to the corresponding study for paired operators. We make use of this in the following theorem, which will be used later.

Theorem 4.3. *Let $g \in \mathcal{GL}_\infty$. Then T_g is invertible in H_p^+ if and only if $T_{g^{-1}}$ is invertible in $H_{p'}^+$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

Proof. Taking Theorem 4.2 into account, it is enough to prove that $(P^+g + P^-)^*$ is invertible in $H_{p'}^+$ if and only if $g^{-1}P^+ + P^-$ is invertible in H_p^+ . Indeed we have, by (21),

$$g^{-1}(P^+ + gP^-) = g^{-1}(CP^-C + gP^-) = g^{-1}C(P^- + \bar{g}P^+)C = g^{-1}C(P^+g + P^-)^*C. \quad \square$$

5 Fredholmness, invertibility and Wiener-Hopf factorisation

It is easy to see, using properties (P1)-(P3) in Section 3, that the invertibility properties of T_{r^k} (cf. Proposition 3.1) also hold for any Toeplitz operator whose symbol g can be represented as a product

$$g = g_- r^k g_+, \quad \text{with } g_\pm \in \mathcal{GH}_\infty^\pm, \quad k \in \mathbb{Z}. \quad (32)$$

In that case we say that (32) is a *bounded WH factorisation* and we have a corresponding factorisation for the Toeplitz operator T_g :

$$T_g = T_{g_-} T_{r^k} T_{g_+}. \quad (33)$$

The integer k is called the *index* of the factorisation (32) and the latter is said to be *canonical* when $k = 0$. Since T_{g_\pm} are invertible, and taking Proposition 3.1 into account, we have the following.

Theorem 5.1. *Let g admit a bounded WH factorisation (32).*

- (i) *If $k = 0$ then T_g is invertible with inverse*

$$(T_g)^{-1} = T_{g_+^{-1}} T_{g_-^{-1}}. \quad (34)$$

- (ii) *If $k > 0$, then T_g is left invertible, $\dim \ker T_g^* = k$ and a left inverse is*

$$(T_g)_l^{-1} = T_{r^{-k}} T_{g_+^{-1}} T_{g_-^{-1}}. \quad (35)$$

- (iii) *If $k < 0$, then T_g is right invertible, $\dim \ker T_g = |k|$ and a right inverse is*

$$(T_g)_r^{-1} = T_{g_-^{-1}} T_{g_+^{-1}} T_{r^{-k}}. \quad (36)$$

Proof. If $g = g_- g_+$ then (34) holds by (P2) in Section 3. If $k > 0$, then $r^k \in H_\infty^+$, $r^{-k} \in H_\infty^-$ and we have

$$(T_{r^{-k}} T_{g_+^{-1}} T_{g_-^{-1}}) T_g = T_{r^{-k}} T_{g_+^{-1}} T_{g_-^{-1}} T_{g_-} T_{r^k} T_{g_+} = T_{r^{-k} g_+^{-1} r^k g_+} = I_+.$$

We can prove (iii) analogously. □

Left and right inverses are not unique; for $k \neq 0$ another one sided inverse (left or right depending on whether $k > 0$ or $k < 0$) is given by $T_{g_+^{-1}} T_{r^{-k}} T_{g_-^{-1}}$.

Definition 5.2 ([6, 15]). *Whenever we associate an integer k to an operator A , and A is invertible if $k = 0$, only left invertible if $k \in \mathbb{Z}^+$ and only right invertible when $k \in \mathbb{Z}^-$, we say that the invertibility of A corresponds to the integer k .*

A class of symbols that always admit a bounded WH factorisation is \mathcal{GR} , the set of all rational functions that belong to \mathcal{GL}_∞ . It is easy to see that, for $R \in \mathcal{GR}$, we have $R = R_- r^k R_+$ where $R_\pm^{\pm 1} \in \mathcal{R} \cap H_\infty^\pm$, $R_\pm^{\pm 1} \in \mathcal{R} \cap H_\infty^\pm$ and $k \in \mathbb{Z}$ is the winding number of R around the origin, $\text{ind } R$, also called the index of R with respect to the origin ([1]), which is given by the difference between the number of zeroes and the number of poles of R in \mathbb{C}^+ .

Thus, for example,

$$\frac{(x+2i)^2}{x^2+1} = \frac{x+i}{x-i} \left(\frac{x+2i}{x+i} \right)^2 \quad (R_- = 1, k = -1, R_+ = \left(\frac{x+2i}{x+i} \right)^2).$$

As a consequence of Theorem 5.1 and Corollary 3.5 we have thus:

Corollary 5.3. *Let $R \in \mathcal{R}$. A necessary and sufficient condition for T_R to be Fredholm in H_p^+ is that $\inf_{x \in \mathbb{R}} |R(x)| > 0$. In that case the invertibility of T_R corresponds to $k = \text{ind } R$ and $\dim \ker T_R^* = k$ if $k > 0$, $\dim \ker T_R = |k|$ if $k < 0$.*

Other classes of continuous functions, called decomposing algebras with the factorisation property ([5, 10, 13]), such as the Wiener algebra on the real line $W(\mathbb{R}_\infty) = \mathbb{C} + \mathcal{FL}_1$ and the algebra $C_\mu(\mathbb{R}_\infty)$ of Hölder continuous functions with exponent $\mu \in]0, 1[$, also possess the property that every invertible element of the algebra admits a bounded WH factorisation (32). As a consequence, the result of Corollary 5.3 also holds if we replace \mathcal{R} by $W(\mathbb{R}_\infty)$ or $C_\mu(\mathbb{R}_\infty)$ and Theorem 32 provides formulae for the inverses, or one sided inverses.

Using rational approximation, we can show that the results of Corollary 5.3 still hold when we replace \mathcal{R} by $C(\mathbb{R}_\infty)$ and we can also obtain formulae for the inverses, or one sided inverses, of the operator T_g . However, in contrast to (34)-(36), these formulae now involve infinite series of operators ([5, 10]). Extending the previous results to more general classes of symbols in such a way that we can obtain similarly simple explicit formulae requires generalising the concept of factorisation. This is done in such a way that the main properties of the factorisation which are used to guarantee invertibility of T_g , when g admits a canonical bounded WH factorisation $g = g_- g_+$, hold at least in a dense subset of H_p^+ . We choose the latter to be \mathcal{R}_0^+ , consisting of the rational functions belonging to $\mathcal{R} \cap H_p^+$. Thus we look for a factorisation of the form $g = g_- r^k g_+$, where the factors are no longer required to be bounded but the equalities

$$P^+ g_+^{\pm 1} P^+ = g_+^{\pm 1} P^+, \quad P^- g_-^{\pm 1} P^- = g_-^{\pm 1} P^-$$

still hold in \mathcal{R}_0^+ (cf. (6)). However, unlike the case when $g_+^{\pm 1}$ and $g_-^{\pm 1}$ are bounded in \mathbb{C}^+ and \mathbb{C}^- respectively, this is not enough to guarantee the invertibility of T_g when $g = g_- g_+$. Having in mind that $(T_{g_- g_+})^{-1} = T_{g_+^{-1}} T_{g_-^{-1}} = g_+^{-1} P^+ g_-^{-1} I_+ : H_p^+ \rightarrow H_p^+$ when $g = g_- g_+$ is a bounded factorisation, we will also require now that the operator $g_+^{-1} P^+ g_-^{-1} I_+ : \mathcal{R}_0^+ \rightarrow H_p^+$ is well defined and admits a bounded extension to H_p^+ .

Definition 5.4. *A Wiener-Hopf (or generalised) factorisation of a function $g \in L_\infty$ with respect to L_p (WH p -factorisation for short) is a representation $g = g_- r^k g_+$ with $k \in \mathbb{Z}$ and*

$$g_+ \in \mathcal{H}_{p'}^+, \quad g_- \in \mathcal{H}_p^- \quad (37)$$

$$(g_+)^{-1} \in \mathcal{H}_{p'}^+, \quad (g_-)^{-1} \in \mathcal{H}_p^- \quad (38)$$

$$g_+^{-1} P^+ g_-^{-1} I_+ : \mathcal{R}_0^+ = \mathcal{R} \cap H_p^+ \rightarrow H_p^+ \text{ is a bounded operator.} \quad (39)$$

The integer k is called the *index* of the factorisation, and the latter is said to be *canonical* when $k = 0$. If a WH p -factorisation exists, then it is unique up to a constant factor:

Theorem 5.5. *Let $g \in L_\infty$ and let $g = g_- r^k g_+$ and $g = \tilde{g}_- r^{\tilde{k}} \tilde{g}_+$ be two WH p -factorisations. Then $k = \tilde{k}$ and there is $C \in \mathbb{C} \setminus \{0\}$ such that $g_- = C \tilde{g}_-$, $g_+ = C^{-1} \tilde{g}_+$.*

Proof. We start by proving the uniqueness of the index. If $k > \tilde{k}$, then $r^{k-\tilde{k}} g_+ \tilde{g}_+^{-1} = \tilde{g}_- g_-^{-1}$, which is equivalent to $r^{k-\tilde{k}-1} \frac{g_+}{x+i} \tilde{g}_+^{-1} = \frac{\tilde{g}_-}{x-i} g_-^{-1}$. Since $k - \tilde{k} - 1 \geq 0$, the left hand side belongs to \mathcal{H}_1^+ , the right hand side to \mathcal{H}_1^- ; thus both would be zero, which is impossible. We can see analogously that we cannot have $k < \tilde{k}$. It follows that

$g_- g_+ = \tilde{g}_- \tilde{g}_+$, which is equivalent to $g_- \tilde{g}_-^{-1} = g_+^{-1} \tilde{g}_+$. Again, since the left hand side belongs to $(x - i) \mathcal{H}_1^-$ and the right hand side to $(x + i) \mathcal{H}_1^+$, we conclude that both sides are equal to a non-zero constant (cf. (12)). \square

We now want to prove that T_g is Fredholm in H_p^+ if and only if g admits a WH p -factorisation, and invertible if and only if this factorisation is canonical. We will need the following lemmas.

Lemma 5.6. *Let $g_-^{-1} \in \mathcal{H}_p^-$, $r_0 \in \mathcal{R}_0^+$. Then $P^+(g_-^{-1} r_0) \in \mathcal{R}_0^+$.*

Proof. Let $r_0(x) = \frac{1}{(x+z_0)^n}$ with $z_0 \in \mathbb{C}^+$, $n \in \mathbb{N}$. We have

$$\begin{aligned} P^+(g_-^{-1} R_0) &= P^+ \frac{g_-^{-1} - \sum_{j=0}^{n-1} (g_-^{-1})_{(-z_0)}^{(j)} (x+z_0)^j}{(x+z_0)^n} + \frac{1}{(x+z_0)^n} \sum_{j=0}^{n-1} (g_-^{-1})_{(-z_0)}^{(j)} (x+z_0)^j \\ &= \frac{1}{(x+z_0)^n} \sum_{j=0}^{n-1} (g_-^{-1})_{(-z_0)}^{(j)} (x+z_0)^j \in \mathcal{R}_0^+. \end{aligned}$$

\square

Note that it follows from Lemma 5.6 that $g_+^{-1} P^+(g_-^{-1} r_0) \in H_p^+$ for all $r_0 \in \mathcal{R}_0^+$ and therefore $g_+^{-1} P^+ g_-^{-1} I_+ : \mathcal{R}_0^+ \rightarrow H_p^+$ is well defined.

Lemma 5.7. *Let $g_- \in \mathcal{H}_p^-$, $g_-^{-1} \in \mathcal{H}_p^-$, $r_0 \in \mathcal{R}_0^+$. Then*

$$g_- P^-(g_-^{-1} r_0) \in H_p^-.$$

Proof. We prove this by induction considering, without loss of generality, that $r_0(x) = \frac{1}{(x+z_0)^n}$ with $z_0 \in \mathbb{C}^+$, $n \in \mathbb{N}$. For $n = 1$ we have

$$g_- P^-(g_-^{-1} \frac{1}{x+z_0}) = g_- \frac{g_-^{-1} - g_-^{-1}(-z_0)}{x+z_0} = g_-^{-1}(-z_0) \frac{g_-(-z_0) - g_-}{x+z_0} \in H_p^-.$$

Now assume that $g_- P^-(g_-^{-1} \frac{1}{(x+z_0)^n}) \in H_p^-$ and let $F_- = P^-(g_-^{-1} \frac{1}{(x+z_0)^n})$. Then

$$\begin{aligned} g_- P^-\frac{g_-^{-1}}{(x+z_0)^{n+1}} &= g_- P^-\left[\left(P^-\frac{g_-^{-1}}{(x+z_0)^n} + P^+\frac{g_-^{-1}}{(x+z_0)^n}\right) \frac{1}{x+z_0}\right] \\ &= g_- P^-\left[\left(P^-\frac{g_-^{-1}}{(x+z_0)^n}\right) \frac{1}{x+z_0}\right] \\ &= g_- P^-\frac{F_-}{(x+z_0)} = g_- \frac{F_- - F_-(-z_0)}{(x+z_0)} \\ &= \frac{g_- F_- - g_-(-z_0)F_-(-z_0)}{(x+z_0)} + F_-(-z_0) \frac{g_-(-z_0) - g_-}{(x+z_0)} \in H_p^- \end{aligned}$$

since both terms on the right hand side belong to H_p^- . \square

Theorem 5.8. *Let $g \in L_\infty$. The operator T_g is invertible in H_p^+ if and only if g admits a canonical WH p -factorisation*

$$g = g_- g_+. \tag{40}$$

Proof. (i) First we prove that if g admits a factorisation (40) then T_g is invertible. Assume thus that (40) is a canonical WH p -factorisation.

Then T_g is injective because

$$f_+ \in \ker T_g \Leftrightarrow g f_+ = f_- \text{ with } f_- \in H_p^-$$

and $gf_+ = f_- \Leftrightarrow g_+f_+ = g_-^{-1}f_-$; since the left hand side of the last equality belongs to \mathcal{H}_1^+ while the right hand side belongs to \mathcal{H}_1^- , both are zero and we conclude that $f_+ = 0$. On the other hand T_g is surjective. To prove this, let $T_0 = g_+^{-1}P^+g_-^{-1}I_+ : \mathcal{R}_0^+ \rightarrow H_p^+$ and let $r_0 \in \mathcal{R}_0^+$. Then, using Lemmas 5.6 and 5.7,

$$\begin{aligned} T_g T_0 r_0^+ &= P^+ g P^+ g_+^{-1} P^+ (g_-^{-1} r_0^+) = P^+ g g_+^{-1} P^+ (g_-^{-1} r_0^+) \\ &= P^+ g_- P^+ (g_-^{-1} r_0^+) = r_0^+ - P^+ g_- P^- (g_-^{-1} r_0^+) = r_0^+. \end{aligned}$$

Now take any $\varphi_+ \in H_p^+$ and let (r_n^+) , with $r_n^+ \in \mathcal{R}_0^+$ for all $n \in \mathbb{N}$, be a sequence such that $r_n^+ \rightarrow \varphi_+$ in H_p^+ . Let moreover T be the continuous extension of T_0 to H_p^+ . We have $T_g T r_n^+ = T_g T_0 r_n^+ = r_n^+$ and it follows that $T_g T \varphi_+ = \varphi_+$, i.e., $\varphi_+ \in \text{Im } T_g$.

(ii) Conversely, assume that T_g is invertible in H_p^+ . By Theorem 4.3 this is equivalent to $T_{g^{-1}}$ being invertible in $H_{p'}^+$, since we must have $g \in \mathcal{GL}_\infty$ by Corollary 3.5. Let then $u_+ \in H_p^+$, $v_+ \in H_{p'}^+$ be the unique solutions of

$$T_g u_+ = \lambda_+^{-1}, \quad T_{g^{-1}} v_+ = \lambda_+^{-1}$$

where λ_+ is defined in (8). Then we have

$$g u_+ = \lambda_+^{-1} + u_-, \quad g^{-1} v_+ = \lambda_+^{-1} + v_- \quad \text{with } u_- \in H_p^-, \quad v_- \in H_{p'}^-. \quad (41)$$

Multiplying these two equations we get

$$\lambda_+ u_+ v_+ - \lambda_+^{-1} = u_- + v_- + \lambda_+ u_- v_-.$$

Since the left hand side is in \mathcal{H}_1^+ and the right hand side is in \mathcal{H}_1^- we conclude that both are zero, therefore

$$\lambda_+ u_+ v_+ = \lambda_+^{-1}, \quad u_- + v_- + \lambda_+ u_- v_- = 0. \quad (42)$$

The first equality in (42) implies that $(\lambda_+ u_+)(\lambda_+ v_+) = 1$; thus, defining $g_+ = \lambda_+ v_+$, we have that $g_+ \in \mathcal{H}_{p'}^+$, $g_+^{-1} = \lambda_+ u_+ \in \mathcal{H}_p^+$. From the second equality in (42) we have $(1 + \lambda_+ v_-)(1 + \lambda_+ u_-) = 1$. So, defining $g_- = 1 + \lambda_+ u_- = 1 + \lambda_- r^{-1} u_-$, we have $g_- \in \mathcal{H}_p^-$, $g_-^{-1} = 1 + \lambda_+ v_- \in \mathcal{H}_{p'}^-$. From (41), $g = g_- g_+$, so it is left to show that $g_+^{-1} P^+ g_-^{-1} I_+ : \mathcal{R}_0^+ \rightarrow H_p^+$ is bounded.

Let $r_0 \in \mathcal{R}_0^+$ and let $f_+ = (T_g)^{-1} r_0$. Then $P^+(gf_+) = r_0$, i.e., $gf_+ = r_0 + P^-(gf_+)$. Now

$$\begin{aligned} gf_+ = r_0 + P^-(gf_+) &\Leftrightarrow g_-^{-1} gf_+ = g_-^{-1} r_0 + g_-^{-1} P^-(gf_+) \\ &\Leftrightarrow g_+ f_+ - P^+(g_-^{-1} r_0) = P^-(g_-^{-1} r_0) + g_-^{-1} P^-(gf_+). \end{aligned}$$

The left hand side of the last equality belongs to \mathcal{H}_1^+ , while the right hand side belongs to \mathcal{H}_1^- , so both are equal to zero and we conclude that

$$g_+ f_+ = P^+(g_-^{-1} r_0) \in H_{p'}^+, \quad g_-^{-1} P^-(gf_+) = -P^-(g_-^{-1} r_0) \in H_{p'}^-. \quad (43)$$

Taking these relations into account, we get

$$\begin{aligned} g_+^{-1} P^+(g_-^{-1} r_0) &= g_+^{-1} P^+(g_-^{-1} T_g f_+) = g_+^{-1} P^+(g_-^{-1} P^+(gf_+)) \\ &= g_+^{-1} P^+ \left[g_-^{-1} (gf_+ - P^-(gf_+)) \right] \\ &= g_+^{-1} P^+ \left[\underbrace{g_+ f_+}_{\in H_{p'}^+} - \underbrace{g_-^{-1} P^-(gf_+)}_{\in H_{p'}^-} \right] = f_+ = (T_g)^{-1} r_0 \end{aligned}$$

and it follows that $g_+^{-1} P^+ g_-^{-1} I_+ : \mathcal{R}_0^+ \rightarrow H_p^+$ is bounded. \square

Remark 5.9. Remark that (39) was not needed to prove the injectivity of T_g and the relation $\mathcal{R}_0^+ \subset \text{Im } T_g$ in the first part of the proof above.

Theorem 5.10. Let $g \in L_\infty$. The operator T_g is Fredholm in H_p^+ if and only if g admits a WH p -factorisation $g = g_- r^k g_+$ and, in that case, its Fredholm index is $\text{Ind } T_g = -k$.

Proof. (i) Assume that T_g is Fredholm and let $-k$ be its Fredholm index. Then if $k \leq 0$ we have $r^{-k} \in H_\infty^+$ and $\text{Ind } T_{r^{-k}} = k$. Therefore $T_g T_{r^{-k}} = T_{g r^{-k}}$ is a Toeplitz operator which is Fredholm with index zero. By Corollary 3.5, $T_{g r^{-k}}$ is invertible. Therefore $g r^{-k}$ admits a canonical WH p -factorisation $g r^{-k} = g_- g_+$ and it follows that $g = g_- r^k g_+$ is a WH p -factorisation. If $k \geq 0$ we have $r^{-k} \in H_\infty^-$ and $\text{Ind } T_{r^{-k}} = k$, so $T_{r^{-k}} T_g = T_{g r^{-k}}$ is a Toeplitz operator which is Fredholm with index zero, therefore invertible, and we conclude analogously that g admits a canonical WH p -factorisation.

(ii) Conversely, let us now assume that $g = g_- r^k g_+$ is a WH p -factorisation. If $k \geq 0$ then $T_g = T_{g_- g_+} T_{r^k}$; if $k \leq 0$ then $T_g = T_{r^k} T_{g_- g_+}$. Since $T_{g_- g_+}$ is invertible by Theorem 5.8 and T_{r^k} is Fredholm with index $-k$, we conclude that T_g is Fredholm with index $-k$. \square

Corollary 5.11. *If $g \in L_\infty$ admits a WH p -factorisation $g = g_- r^k g_+$, then the invertibility of T_g corresponds to the integer k . For $k > 0$ the operator $g_+^{-1} P^+ g_-^{-1} T_{r^{-k}}$ is a left inverse for T_g and $\dim \ker T_g^* = k$; for $k < 0$ the operator $T_{r^{-k}} (g_-^{-1} P^+ g_+^{-1} I_+)$ is a right inverse for T_g and $\dim \ker T_g = |k|$.*

It is clear from Definition 5.4 that $g \in L_\infty$ admits a WH p -factorisation $g = g_- r^k g_+$ if and only if $g_0 = r^{-k} g$ admits a canonical WH p -factorisation. In that case T_{g_0} is invertible by Theorem 5.8. Therefore the factors g_\pm satisfy (37), (38) and the range of $T_{g_0} = T_{g_- g_+}$ is closed in H_p^+ , i.e.,

$$T_{r^{-k} g} = T_{g_- g_+} \text{ has closed range.} \tag{44}$$

We can see from the proof of Theorem 5.8 that the converse is also true and thus condition (39) in Definition 36 can be replaced by (44). In fact, from the first part of the proof of Theorem 5.8 we see that (37) and (38) imply that T_{g_0} is injective and $\mathcal{R}_0^+ \subset \text{Im } T_{g_0}$. Since \mathcal{R}_0^+ is dense in H_p^+ , by adding (44) to (37) and (38) we conclude that T_{g_0} is surjective, therefore invertible. Thus we have the following.

Corollary 5.12. *The factorisation $g = g_- r^k g_+$, with $k \in \mathbb{Z}$, is a WH p -factorisation for $g \in L_\infty$ if and only if g_\pm satisfy (37), (38) and (44).*

This means that we can replace (39) by another condition which is easy to verify for a large class of functions in L_∞ , including all piecewise continuous functions ([6, 10]), as we show in section 6.

We can now ask when does a function $g \in L_\infty$ admit a WH p -factorisation and, if it exists, how to obtain its factors and, in particular, its index. We address this question in the following section.

6 Piecewise continuous symbols

There are several classes of functions for which simple criteria for existence of a WH p -factorization can be established. (see for instance [3, 5, 13]). We will consider here only the class $PC(\mathbb{R}_\infty)$ of all piecewise continuous functions with finite limits at $\pm\infty$.

To each $g \in PC(\mathbb{R}_\infty)$ we associate

$$g_p(x, w) = \frac{g(x^-) + g(x^+)}{2} + \frac{g(x^-) - g(x^+)}{2} \coth \left(\left(\pi \left(\frac{i}{p} + w \right) \right) \right) \tag{45}$$

for $x \in \mathbb{R}_\infty$, $w \in \mathbb{R}_\pm\infty$, where \mathbb{R}_∞ and $\mathbb{R}_\pm\infty$ are the one-point and the two-point compactifications of the real line, respectively, and for $x = \infty$ we take $g(\infty^-) = g(+\infty)$, $g(\infty^+) = g(-\infty)$ ([6]).

If $g \in C(\mathbb{R}_\infty)$ then $g_p(x, w) = g(x)$ for all $x \in \mathbb{R}_\infty$ and $w \in \mathbb{R}_\pm\infty$. It is also easy to see that if $p = 2$ then

$$g_2(x, w) = \frac{g(x^+) + g(x^-)}{2} + \frac{g(x^+) - g(x^-)}{2} \text{th}(\pi w). \tag{46}$$

The image of g_p in the complex plane is a closed curve Γ obtained by adding to the image of $g(x)$, with $x \in \mathbb{R}_\infty$ such that g is continuous at x , certain arcs of a circle (or line segments, when $p = 2$) connecting the points

corresponding to $g(x^-)$ and $g(x^+)$ whenever these two values are different. If

$$\inf_{(x,w) \in \mathbb{R}_\infty \times \mathbb{R}_\pm \infty} |g_p(x,w)| > 0 \tag{47}$$

then we can associate with g_p an integer, designated by $ind\ g_p$, defined as the winding number of Γ around the origin. If $g \in C(\mathbb{R}_\infty)$ then condition (47) means that $g \in \mathcal{GC}(\mathbb{R}_\infty)$ and in that case $ind\ g_p = ind\ g$.

Definition 6.1. We say that $g \in PC(\mathbb{R}_\infty)$ is p -nonsingular, or p -regular, if and only if (47) holds.

Note that the product of two p -regular functions may be p -singular, as shown in an example presented below in this section. However, if two functions $g, h \in PC(\mathbb{R}_\infty)$ have no common points of discontinuity, then $(gh)_p(x,w) = g_p(x,w)h_p(x,w)$ and, if gh is p -regular, then $ind\ (gh)_p = ind\ g_p\ ind\ h_p$ ([6]).

For Toeplitz operators with piecewise continuous symbols we have the following.

Theorem 6.2 ([6, 17]). Let $g \in PC(\mathbb{R}_\infty)$. The operator T_g has closed range in H_p^+ if and only if g is p -regular. In this case T_g is Fredholm, the invertibility of T_g corresponds to $k = ind\ g_p$ and the Fredholm index of T_g is $Ind\ T_g = -k$.

Corollary 6.3. If $g \in PC(\mathbb{R}_\infty)$, then g admits a WH p -factorisation if and only if g is p -regular and, in this case, $g = g_- r^k g_+$ with $k = ind\ g_p$.

In the case $p = 2$ we see from the previous corollary that $g \in PC(\mathbb{R}_\infty)$ is 2-regular if and only if g is locally sectorial, i.e., for all $x \in \mathbb{R}_\infty$ we have

$$g(x^\pm) \neq 0 \tag{48}$$

$$\frac{g(x^+)}{g(x^-)} = e^{2\pi i \alpha} \quad \text{with } |\Re(\alpha)| < 1/2. \tag{49}$$

Condition (49) means precisely that the discontinuities are such that the line segment connecting the points $g(x^-)$ and $g(x^+)$ in the complex plane does not include the origin.

Now, having considered the Fredholm properties of T_{r^k} with $k \in \mathbb{Z}$, in Proposition 3.1, it is natural to consider also the case when the exponent k is not an integer, to illustrate the previous results.

Let $p = 2, \alpha \in \mathbb{R} \setminus \mathbb{Z}$ with $|\alpha| \leq 1/2$, and let $g_{(\alpha,\infty)}(x) = \left(\frac{x-i}{x+i}\right)_\infty^\alpha = e^{\alpha \log \frac{x-i}{x+i}}$ where $\log w = \log |w| + i \arg(w)$ with $\arg(w) \in [0, 2\pi[$. In this case $g_{(\alpha,\infty)}(x)$ is analytic in the whole complex plane except for the branch cuts $\pm i [0, \infty[$ in the imaginary axis; thus $\left(\frac{x-i}{x+i}\right)_\infty^\alpha$ is continuous on \mathbb{R} , but it has a discontinuity at ∞ : since $\lim_{x \rightarrow +\infty} \frac{x-i}{x+i} = 2\pi, \lim_{x \rightarrow -\infty} \frac{x-i}{x+i} = 0$, we have

$$\left(\frac{x-i}{x+i}\right)_\infty^\alpha (-\infty) = e^{i2\pi\alpha}, \quad \left(\frac{x-i}{x+i}\right)_\infty^\alpha (+\infty) = 1.$$

The image of $(g_{(\alpha,\infty)})_2(x,w)$ in the complex plane consists of the upper half of the unit circle and a line segment connecting the points 1 and -1 , when $\alpha = 1/2$; it consists of an arc of the unit circle in the upper half-plane connecting the points $e^{2i\pi\alpha}$ and 1, and a line segment from 1 to $e^{2i\pi\alpha}$, when $0 < \alpha < 1/2$. It follows from Corollary 6.3 that $g_{(\alpha,\infty)}$ admits a WH 2-factorisation if and only if $|\alpha| < 1/2$ and, in that case, the factorisation is canonical.

We see in particular that $g_{(1/4,\infty)}$ is 2-regular but $g_{(1/4,\infty)}^2 = g_{(1/2,\infty)}$ is 2-singular, which illustrates our previous assertion that the product of two 2-regular functions may be 2-singular if they have common discontinuities.

More generally, defining for $c \in \mathbb{R}, \alpha \in \mathbb{C} \setminus \mathbb{Z}$,

$$(z)_c^\alpha = e^{\alpha \log z}$$

where the branch cut connecting 0 to ∞ intersects the unit circle at the point $z_0 = \frac{c-i}{c+i}$, the function $\left(\frac{x-i}{x+i}\right)_c^\alpha$ is continuous for all points of \mathbb{R}_∞ except the point c . If the discontinuity points of $g \in PC(\mathbb{R}_\infty)$ are $c_1, c_2, \dots, c_n \in \mathbb{R}$

and (eventually) ∞ , then g can be represented in the form ([6])

$$g(x) = g_0(x) \left(\frac{x-i}{x+i} \right)_\infty^{\alpha_0} \prod_{j=1}^n \left(\frac{x-i}{x+i} \right)_{c_j}^{\alpha_j} \tag{50}$$

where $g_0 \in C(\mathbb{R}_\infty)$ and

$$\alpha_j = \frac{1}{2\pi i} \log \frac{g(c_j^-)}{g(c_j^+)}, \text{ for } j = 1, 2, \dots, n, \quad \alpha_0 = \frac{1}{2\pi i} \log \frac{g(+\infty)}{g(-\infty)}.$$

Using Corollary 47 we have the following:

Theorem 6.4 ([6, 17]). *Let $g \in PC(\mathbb{R}_\infty)$ be given by (50). The operator T_g is Fredholm if and only if $-1/p < \Re(\alpha_0) < 1 - 1/p$, $1/p - 1 < \Re(\alpha_j) < 1/p$ for all $j = 1, 2, \dots, n$; in that case its Fredholm index is $k = -ind g_0$.*

Although there are explicit formulae for the factors in a WH p -factorisation of $g \in PC(\mathbb{R}_\infty)$, once its index is known ([17]), the factors can in some cases be obtained by inspection. For example, if $-1/2 < \alpha < 1/2$ we can write

$$\left(\frac{x-i}{x+i} \right)_\infty^\alpha = (x-i)^\alpha \left(\frac{1}{x+i} \right)^\alpha \tag{51}$$

choosing appropriate branches such that $(x-i)^{\pm\alpha} \in \mathcal{H}_2^-$, $\left(\frac{1}{x+i}\right)^{\pm\alpha} \in \mathcal{H}_2^+$. Since the range of T_g , with g given by the left hand side of (51), is closed by Theorem 6.2, it follows from Corollary 5.12 that (51) is a canonical WH 2-factorisation of g .

7 The spectrum of the singular integral operator S_J in $L_2(J)$

Let S_J , where $J \subset \mathbb{R}$ is an interval, denote the singular integral operator

$$S_J : L_2(J) \rightarrow L_2(J), \quad S_J \varphi(x) = \frac{1}{\pi i} \int_J \frac{\varphi(t)}{t-x} dt, \quad x \in J. \tag{52}$$

In this section we study the spectrum of S_J in three cases, $J = \mathbb{R}$, $J = \mathbb{R}^+$, $J = [0, 1]$, using the results of the previous sections. The spectrum of S_J in these three cases was described in [17] using a slightly different approach. Furthermore, using the factorisation of the symbol of an associated Toeplitz operator, we obtain expressions for the resolvent operator $(S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+})^{-1}$ when $\lambda \notin \sigma(S_{\mathbb{R}^+})$. Here $I_{\mathbb{R}^+}$ denotes the identity operator in $L_2(\mathbb{R}^+)$.

First case: $J = \mathbb{R}$

Using (18) we can write $S_{\mathbb{R}} - \lambda I$ as a paired operator:

$$S_{\mathbb{R}} - \lambda I = P^+ - P^- - \lambda(P^+ + P^-) = (1 - \lambda)P^+ - (1 + \lambda)P^-.$$

Clearly $\lambda = \pm 1$ belong to the point spectrum of $S_{\mathbb{R}}$, since $S_{\mathbb{R}} - I = -2P^-$, $S_{\mathbb{R}} + I = 2P^+$ and P^\pm have infinite dimensional kernels. For $\lambda = \pm 1$ the operator $S_{\mathbb{R}} - \lambda I$ is invertible, with inverse $\frac{1}{1-\lambda^2}(S_{\mathbb{R}} + \lambda I)$. Thus $\sigma(S_{\mathbb{R}}) = \{1, -1\}$.

Second case: $J = \mathbb{R}^+$.

For $\varphi \in L_2(\mathbb{R}^+)$ we have

$$S_{\mathbb{R}^+} \varphi(x) = \frac{1}{\pi i} \int_{\mathbb{R}^+} \frac{\varphi(t)}{t-x} dt, \quad x \in \mathbb{R}^+.$$

Let χ_\pm denote the characteristic functions of \mathbb{R}^\pm , respectively. Identifying $L_2(\mathbb{R}^+)$ with a subspace of $L_2(\mathbb{R})$, $L_2(\mathbb{R}^+) = \chi_+ L_2(\mathbb{R})$, we have

$$S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+} = \chi_+ (S_{\mathbb{R}} - \lambda I)|_{\chi_+ L_2}. \tag{53}$$

We use now the Fourier transform, a natural tool to deal with convolution integrals,

$$\mathcal{F} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{ixt} dt = \frac{1}{\sqrt{2\pi}} \lim_{A \rightarrow \infty} \int_{-A}^A f(x) e^{ixt} dt \quad (54)$$

which is an isometric isomorphism from $\chi_+ L_2$ onto H_2^+ . We have

$$P^+ = \mathcal{F} \chi_+ \mathcal{F}^{-1} = \mathcal{F}^{-1} \chi_- \mathcal{F}, \quad P^- = \mathcal{F} \chi_- \mathcal{F}^{-1} = \mathcal{F}^{-1} \chi_+ \mathcal{F}. \quad (55)$$

Thus we can reduce the study of the invertibility of $S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+}$ to the study of the invertibility of the Toeplitz operator

$$T_{-(\text{sign} x + \lambda)} : H_2^+ \rightarrow H_2^+$$

where $\text{sign}(x) = \chi_+(x) - \chi_-(x)$, since, for all $f_+ \in H_2^+$,

$$\begin{aligned} T_{(\text{sign} x + \lambda)} f_+ &= P^+(\text{sign} x + \lambda) f_+ = \mathcal{F} \chi_+ \mathcal{F}^{-1} (\chi_+ - \chi_- + \lambda) \mathcal{F} f_+ \\ &= \mathcal{F} \chi_+ (\mathcal{F}^{-1} \chi_+ \mathcal{F} - \mathcal{F}^{-1} \chi_- \mathcal{F} + \lambda) \mathcal{F} f_+ \\ &= \mathcal{F} \chi_+ (P^- - P^+ + \lambda) \mathcal{F} f_+ = -\mathcal{F} \chi_+ (S_{\mathbb{R}} - \lambda I) \mathcal{F} f_+ \\ &= -\mathcal{F} \chi_+ (S_{\mathbb{R}} - \lambda I) \mathcal{F}^{-1} f_+ = -\mathcal{F} (S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+}) \mathcal{F}^{-1} f_+. \end{aligned}$$

The symbol $g_\lambda(x) = -\text{sign}(x) - \lambda$ is 2-regular if and only if $\lambda \notin [-1, 1]$; in this case, $\text{ind}(g_\lambda)_2 = 0$ and therefore, by Theorem 6.2, the operator T_{g_λ} is invertible. For $\lambda \in [-1, 1]$, the range of T_{g_λ} is not closed. We conclude therefore that $\sigma(S_{\mathbb{R}^+}) = [-1, 1]$. Since $g_\lambda(x)$ is real for all x , $T_{g_\lambda} = T_{g_\lambda}^*$ and by Coburn's Lemma $\ker T_{g_\lambda} = \ker T_{g_\lambda}^* = \{0\}$.

Third case: $J = [0, 1]$. Let $Y : L_2([0, 1]) \rightarrow \chi_+ L_2$ be the isometric isomorphism

$$Y \varphi(t) = e^{-t/2} \varphi(e^{-t}) \chi_+(t). \quad (56)$$

Identifying $L_2(\mathbb{R}^+)$ with $\chi_+ L_2$ as usual, we define

$$M = Y J Y^{-1} : L_2(\mathbb{R}^+) \rightarrow L_2(\mathbb{R}^+),$$

$$M f(x) = \frac{1}{\pi i} \int_{\mathbb{R}^+} \frac{e^{1/2(t-x)}}{1 - e^{t-x}} f(t) dt, \quad x > 0. \quad (57)$$

M is a convolution operator: $M f = \chi_+ E * f$ where

$$E(x) = \frac{e^{-x/2}}{1 - e^{-x}}, \quad x \in \mathbb{R}. \quad (58)$$

Applying the Fourier transform to M , and taking into account that ([6])

$$(\mathcal{F} E)(w) = \text{th}(\pi w), \quad w \in \mathbb{R}, \quad (59)$$

we have, for all $f_+ \in H_2^+$,

$$\mathcal{F} M \mathcal{F}^{-1} f_+ = \mathcal{F} (\chi_+ E * (\mathcal{F}^{-1} f_+)) = P^+ \text{th}(\pi x) f_+ = T_{\text{th}(\pi x)} f_+. \quad (60)$$

Thus the spectrum of S_J is the spectrum of the Toeplitz operator with symbol $\text{th}(\pi x)$ and, since the range of this symbol is $[-1, 1]$, we conclude as in the previous case that $\sigma(S_J) = [-1, 1]$.

Let us now apply the previous results and a WH p -factorisation of the symbol $g_\lambda(x) = -(\text{sign} x + \lambda)$ to obtain an expression for the resolvent $(S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+})^{-1}$ with $\lambda \in \mathbb{C} \setminus [-1, 1]$. Defining, for $z \neq 0$, $\log z = \log |z| + i \arg z$ with $\arg z \in]-\pi, \pi]$, let $z^\alpha = e^{\alpha \log z}$ for

$$\alpha = \frac{1}{2\pi i} \log \frac{\lambda - 1}{\lambda + 1}. \quad (61)$$

Note that $|\Re(\alpha)| < 1/2$ and

$$(-ix)^\alpha = \exp(\alpha \log |x| - i\alpha \frac{\pi}{2} \operatorname{sign} x) \in \mathcal{H}_2^+$$

$$(ix)^{-\alpha} = \exp(-\alpha \log |x| - i\alpha \frac{\pi}{2} \operatorname{sign} x) \in \mathcal{H}_2^-,$$

so that $(-ix)^\alpha (ix)^{-\alpha} = e^{-i\alpha\pi \operatorname{sign} x}$. Now, since $\frac{\lambda-1}{\lambda+1} = \exp(i2\pi\alpha)$, we have

$$\lambda + \operatorname{sign} x = \lambda + 1 = (\lambda + 1) \exp(i\alpha\pi)(-ix)^\alpha (ix)^{-\alpha}, \text{ for } x > 0,$$

$$\lambda + \operatorname{sign} x = (\lambda + 1) \exp(i2\alpha\pi) = (\lambda + 1) \exp(i\alpha\pi)(-ix)^\alpha (ix)^{-\alpha}, \text{ for } x < 0.$$

Defining

$$(g\lambda)_+ = (\lambda + 1) \exp(i\alpha\pi)(-ix)^\alpha$$

$$(g\lambda)_- = (ix)^{-\alpha},$$

we conclude that $g\lambda = -(g\lambda)_- (g\lambda)_+$ is a canonical WH 2-factorisation (cf. Corollary 5.12) and

$$(T_{-(\operatorname{sign} x + \lambda)})^{-1} = -(g\lambda)_+^{-1} P^+ (g\lambda)_-^{-1} I_{|H_2^+} = (g\lambda)_+^{-1} \mathcal{F} \chi_+ \mathcal{F}^{-1} (g\lambda)_-^{-1} I_{|H_2^+}.$$

Let us now consider the operator

$$(S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+})^{-1} = \mathcal{F}^{-1} (T_{-(\operatorname{sign} x + \lambda)})^{-1} \mathcal{F}_{|(\chi_+ L_2)}.$$

Assume that $-1/2 < \Re(\alpha) < 0$ (the case where $0 < \Re(\alpha) < 1/2$ can be studied analogously). Since ([11])

$$\left(\mathcal{F}^{-1} (g\lambda)_-^{-1} \right) (t) = \chi_-(t) \frac{(-t)^{-\alpha-1}}{\Gamma(-\alpha)}$$

and

$$(g\lambda)_+^{-1}(x) = \frac{e^{-i\alpha\pi}}{\lambda + 1} (-ix)(-ix)^{-\alpha-1}$$

with

$$\left(\mathcal{F}^{-1} (-ix)^{-\alpha-1} \right) (t) = \frac{\chi_+(t) t^\alpha}{\Gamma(1 + \alpha)},$$

we have, for $\varphi_+ \in L_2(\mathbb{R}^+)$,

$$\begin{aligned} & \left(\mathcal{F}^{-1} (g\lambda)_+^{-1} \mathcal{F}(\chi_+ \mathcal{F}^{-1} (g\lambda)_-^{-1} \mathcal{F}\varphi_+) \right) (t) = \\ & = \frac{e^{-i\alpha\pi}}{(\lambda + 1)\Gamma(-\alpha)\Gamma(1 + \alpha)} \chi_+ \frac{d}{dt} \int_0^t (t-u)^\alpha \int_u^\infty (s-u)^{-\alpha-1} f(s) ds du. \end{aligned}$$

Taking into account that

$$\Gamma(-\alpha)\Gamma(1 + \alpha) = \frac{\pi}{\sin(\alpha\pi)} = \frac{-2i\pi}{e^{i\alpha\pi} - e^{-i\alpha\pi}}$$

we have, for $t > 0$,

$$(S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+})^{-1} f(t) = \frac{1}{i\pi} \frac{1}{(\lambda^2 - 1)} \frac{d}{dt} \left[\int_0^t (t-x)^\alpha \left(\int_x^\infty \frac{f(s)}{(s-x)^{\alpha+1}} ds \right) dx \right].$$

Since

$$\frac{d}{dt} \left[\int_0^t (t-x)^\alpha \left(\int_x^\infty \frac{f(s)}{(s-x)^{\alpha+1}} ds \right) dx \right]$$

$$\begin{aligned}
&= \frac{d}{dt} \left[\int_0^t f(s) \left(\int_0^s \frac{(t-x)^\alpha}{(s-x)^{\alpha+1}} dx \right) ds + \int_t^\infty f(s) \left(\int_0^t \frac{(t-x)^\alpha}{(s-x)^{\alpha+1}} dx \right) ds \right] \\
&= \frac{d}{dt} \left[\int_0^t f(s) \left(\int_0^{s/t} \frac{1}{(1-y)y^{\alpha+1}} dy \right) ds - \int_t^\infty f(s) \left(\int_{s/t}^\infty \frac{1}{(1-y)y^{\alpha+1}} dy \right) ds \right] \\
&= f(t) \int_0^\infty \frac{1}{(1-y)y^{\alpha+1}} dy - t^\alpha \int_0^\infty \frac{f(s)}{(t-s)s^\alpha} ds \\
&= i\pi\lambda f(t) - t^\alpha \int_0^\infty \frac{f(s)}{(t-s)s^\alpha} ds,
\end{aligned}$$

where we employed the change of variables

$$x = t - \frac{(t-s)}{1-y},$$

and the relation

$$\int_{\mathbb{R}^+} \frac{y^{-\alpha-1}}{\pi(1-y)} dy = -\cotg(\alpha\pi) = i\lambda$$

where the integral is understood as a principal value.

Hence,

$$(S_{\mathbb{R}^+} - \lambda I_{\mathbb{R}^+})^{-1} f(t) = \frac{1}{(\lambda^2 - 1)} \left[\lambda f(t) - \frac{t^\alpha}{i\pi} \int_{\mathbb{R}^+} \frac{f(s)}{(t-s)s^\alpha} ds \right] \quad (t > 0).$$

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