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SHARP ESTIMATION OF THE COEFFICIENTS OF QUASI-STARLIKE FUNCTIONS CLOSE TO IDENTITY

The present note deals with the sharp estimation of all coefficients of quasi-starlike functions under the additional assumption that the functions are close to identity. The obtained result is based on and coincides with the corresponding result obtained by L.Siewierski [4], [5], and subsequently by M.Schiffer and O.Tammi [3] in the class S_M of bounded univalent functions close to identity.

Let $\mathcal{G}^M$ denote the class of quasi-starlike functions [1], [2] of the form

$$g(z) = a_1 z^2 + a_2 z^2 \ldots, \qquad a_1 = \frac{1}{M}, \quad |z| < 1, \tag{1}$$

satisfying the equation

$$F(g(z)) = \frac{1}{M} F(z), \qquad |z| < 1, \tag{2}$$

where $F \in S^*$ is an arbitrary starlike function, and M a fixed real number, $1 < M < +\infty$.

Assume further, similarly as in [1], [2] and [6], that $\tilde{S}^*_M$ denotes the class of normalized quasi-starlike functions of the form

$$G(z) = z + A_2 z^2 + \ldots, \qquad |z| < 1, \tag{3}$$

where

$$G(z) = Mg(z), \quad g \in \mathcal{G}^M. \tag{4}$$

It follows easily from (2) that the functions in the classes $\mathcal{G}^M$ and $\tilde{S}^*_M$ are univalent and bounded. Namely:

$$|g(z)| < 1 \qquad \text{for } |z| < 1, \tag{5}$$

and consequently by (4)

$$|G(z)| < M \qquad \text{for } |z| < 1. \tag{5'}$$

We shall prove the following theorem.

Theorem. For an arbitrary natural number $n \geq 2$ there exists a number $M_n > 1$ such that for every function $g \in \mathcal{G}^M$, $1 < M < M_n$, the following sharp estimation holds

$$|a_n| \leq \frac{1}{M} \cdot \frac{2}{(n-1)}\left(1 - \frac{1}{M^{n-1}}\right). \tag{6}$$

The equality sign in (6) holds only for the function g(z) defined by the equation:

$$\frac{g(z)}{\left[1-(e^{i\varphi}g(z))^{n-1}\right]^{\frac{2}{n-1}}} = \frac{1}{M}\,\frac{z}{\left[1-(e^{i\varphi}z)^{n-1}\right]^{\frac{2}{n-1}}}. \tag{7}$$

Proof: Take an arbitrary function $f \in S_M$, where M is sufficiently close to identity.

Let

$$f(z) = z+b_2z^2+\dots, \quad |z| < 1.$$

Using the estimation given in [4] and [5], we obtain

$$|b_n| \leq \frac{2}{n-1}\left(1 - \frac{1}{M^{n-1}}\right) \quad \text{for} \quad 1<M<M_n, \text{ where } n \geq 2$$

The above inequality and the inclusion $\tilde{S}^*_M \subset S_M$ imply in view of (3) the following inequality

$$|A_n| \leqslant \frac{2}{n-1}\left(1-\frac{1}{M^{n-1}}\right), \quad 1 < M < M_n, \ n \geqslant 2.$$

In view of (1) and (4) we further obtain

$$|a_n| \leqslant \frac{1}{M}\cdot\frac{2}{(n-1)}\left(1-\frac{1}{M^{n-1}}\right) \quad \text{for} \quad 1<M<M_n, \ n\geqslant 2. \tag{9}$$

To show that the estimation (9) is sharp, we first check that the function g(z) defined by (7) belongs to the class G^M. Next by a simple computation we find that the modulus of the n-th coefficient of this function is equal to the right-hand side of (9). Hence the theorem has been proved.

R e m a r k 1. From equation (7) defining g(z) and from [4] and [5] it follows that the extremal function which realizes the equality sign of (8) in the class S_M of bounded univalent functions close to identity is the function (4). Consequently, the function (4) is quasi-starlike, and moreover it is the unique function realizing the maximum of the modulus of the n-th coefficient.

R e m a r k 2. Similarly as in [4], [5] and [3] there remains an interesting open problem of determinig or estimating the numbers in the sequence $\{M_n\}$.

From the estimations of a_2, a_3 and a_4 obtained for quasi-starlike functions it only follows that we respectively have: $M_2 = \infty$, $M_3 = 3$, $M_4 = A$, where $\frac{1}{A}$ is the only root of the equation $(2-3x^2)^3 = 3(1-x^3)^2$ in the interval (0,1).

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