THE DETERMINATION OF SUBALGEBRAS OF A GIVEN ALGEBRA
BY THE METHOD OF CLASSIFICATION

Introduction
In this study we present a method of determining all sub-
algebras of a given algebra. This method is based on the notion
of closure and its properties. Its practical application is
based on a calculating method also described here.

The method of classification itself is given in chapter
2 and followed by an example of determining subalgebras of
an abstract algebra.

The study concludes with a chapter devoted to ways of
calculating in practical use of the method—this is a de-
velopment of chapter 1 for algebras. Some formulas useful
in such calculations are given. After this chapter there is
one more example.

1. Composition of families of sets
Let \( X \) be an arbitrary set. We consider the set \( \mathcal{B}(\mathcal{B}(X)) \);
its elements are families of subsets of the set \( X \) (if \( A \) is
a set, then \( \mathcal{B}(A) \) denotes the family of its subsets).

In this set we define a binary operation called composition
and denoted by \( \cdot \). The definition is as follows.

For \( \alpha, \mathcal{B} \subseteq \mathcal{B}(X) \)

\[
\alpha \cdot \mathcal{B} = \left\{ A \cup B : A \in \alpha \land B \in \mathcal{B} \right\}.
\]  

(1.1)

The composition has the following properties. For arbitrary
\( \alpha, \mathcal{B}, \mathcal{C}, \alpha_1, \alpha_2 \ldots \subseteq \mathcal{B}(X); A, B \subseteq X; a \in X; \)
\((1.2)\) \[ \alpha \cdot \mathcal{Z} = \mathcal{Z} \cdot \alpha \quad \text{(commutativity)} \]

\((1.3)\) \[ \alpha \cdot \phi = \phi \]

(the empty family is the zero of composition)

\((1.4)\) \[ \alpha \cdot \{ \phi \} = \alpha \]

(the family consisting of the empty set is the unit of composition)

\((1.5)\) \[ \alpha \cdot (\mathcal{Z} \cdot \zeta) = (\alpha \cdot \mathcal{Z}) \cdot \zeta \quad \text{(associativity)} \]

\((1.6)\) \[ \alpha \subset \alpha \cdot \alpha \]

\((1.7)\) \[ \{ A \} \cdot \{ B \} = \{ A \cup B \} \]

(the composition of one-element families is a one-element family again)

\((1.8)\) \[ \alpha \subset \mathcal{Z} \Rightarrow \alpha \cdot \zeta \subset \mathcal{Z} \cdot \zeta \quad \text{(monotonicity)} \]

\((1.9)\) \[ (\alpha \cup \mathcal{Z}) \cdot \zeta = (\alpha \cdot \zeta) \cup (\mathcal{Z} \cdot \zeta) \]

(separability with respect to addition)

\((1.10)\) \[ \alpha \cdot \bigcup_{i=1}^{\infty} \alpha_{i} = \bigcup_{i=1}^{\infty} \alpha \cdot \alpha_{i} \]

\((1.11)\) \[ \bigwedge_{Y \in X} \{ A \} \cdot \alpha \Rightarrow A \subset Y \]

\((1.12)\) \[ \mathcal{B}(A) \cdot \mathcal{B}(B) = \mathcal{B}(A \cup B) \]

\((1.13)\) \[ \mathcal{B}(A \cup a) = \{ a \} \cdot \mathcal{B}(A) \cup \mathcal{B}(A) \cdot \{ a \} \]

\text{Remark.} We use the following simplification in denotation of subsets of the set \( X \). We omit brackets and commas, so, for instance, if \( X = \{ a, b, c, d \} \) then "bc" denotes the set \{b, c\}.

\text{Proof.} \[ \mathcal{B}(A \cup a) = \mathcal{B}(A) \cdot \mathcal{B}(\{ a \}) = \mathcal{B}(A) \cdot \{ a, \phi \} = \mathcal{B}(A) \cdot \{ a \} \cup \mathcal{B}(A) \cdot \{ \phi \} = \{ a \} \cdot \mathcal{B}(A) \cup \mathcal{B}(A). \]
Remark. In case \( a \not\in A \) the last formula gives the partition of \( \mathcal{B}(A \cup a) \) into the family of subsets of \( A \cup a \) containing \( a \) \( \{a\} \mathcal{B}(A) \) and the family of subsets not containing \( a \) \( \mathcal{B}(a) \).

The properties listed above give the conclusion that the algebra \( \left( \mathcal{B}(\mathcal{B}(x)), \cup, \cap, \cdot, \{\phi\} \right) \) is a net. This is a special net, it is commutative in contradiction to other well-known nets.

2. The method of classification

The following facts provide the theoretical base for the method of classification:

i) All subalgebras of any algebra form a multiplicative family of subsets of this algebra.

ii) There is a one-one correspondence between multiplicative families of subsets of a given set and closure operations in this set. The closure corresponding to a given family is such that a subset belongs to this family if and only if it is closed.

The idea of the method of classification is the following. In an algebra we introduce the closure operation corresponding to the multiplicative family of all subalgebras and we look for subalgebras as closed subsets.

Let \((A, \Omega)\) be an algebra, \( D \) - closure operation in \( A \) corresponding to the family of subalgebras \( D : \mathcal{B}(A) \rightarrow \mathcal{B}(A) \).

For every \( X \subseteq A \), \( D(X) \) is the subalgebra generated by \( X \), and \( D(\mathcal{B}(A)) \) is the family of all subalgebras of \((A, \Omega)\).

In \( \mathcal{B}(A) \) we introduce a relation of equivalence. For \( X, Y \subseteq A \) we define

\[
X \sim Y \iff D(X) = D(Y)
\]

(it is clearly an equivalence relation).

Let \( \|X\| \) denote the equivalence class of the set \( X \) (for \( X \subseteq A \)). The relation \( \sim \) divides \( \mathcal{B}(A) \) into classes of equivalence with the following properties:

1) Net - the notion introduced in [1].
i) There is exactly one subalgebra in each class and it is the greatest element of this class.

ii) If $A_1, A_2, \ldots$ - all subalgebras of $A$, then $\|A_1\|, \|A_2\|, \ldots$ - all and different equivalence classes.

iii) Let $B \subseteq A$. For $X \subseteq A$

\[(2.1) \quad B \subseteq X \subseteq D(B) \Rightarrow D(X) = D(B).\]

Conclusion. If $B \subseteq X \subseteq D(B)$ then $X \in \|D(B)\|$. Hence

\[\{X : B \subseteq X \subseteq D(B)\} \subseteq \|D(B)\|\]

iv) Let $B, C \subseteq A$.

\[(2.2) \quad \text{If } C \subseteq D(B) \text{ then } \{B\} \cdot \mathcal{B}(C) \subseteq \|D(B)\|.\]

Having introduced the necessary properties and formulas we can come to the essence of the matter. The method of classification is the following.

1. We divide the family $\mathcal{B}(A)$ of all subsets of given algebra into a number of disjoint components (we make a partition of $\mathcal{B}(A)$) using formula (1.13). In this way we get components of a special form:

\[\{B\} \cdot \mathcal{B}(C) \quad \text{where } B, C \subseteq A.\]

The partition has to be performed in such a way that every component of the form $\{B\} \cdot \mathcal{B}(C)$ satisfy the condition

\[C \subseteq D(B).\]

This may be achieved by means of refining the partition.

Each step of the calculation is an application of the formula $\mathcal{B}(a \cup A) = \{a\} \cdot \mathcal{B}(A) \cup \mathcal{B}(A)$. We may say it is a kind of taking an element out of brackets. In components of the form $\{B\} \cdot \mathcal{B}(C)$ the set $B$ consists of just those elements, which have been taken out of brackets. So if we want to get the condition $C \subseteq D(B)$ true for all components having at the same time as little number of components as possible - we must try
to have $D(B)$-s as large as possible. In other words, we must take out of brackets those elements of $A$ which generate the greatest subalgebras.

When we obtain the partition satisfying the above condition we make the second step.

2. We classify components with respect to their classes of equivalence. Since every component $\mathcal{B}(C)$ satisfies the condition $C \subseteq D(B)$, it is a subset of one class of equivalence, by virtue of property iv). In other words, all subsets of the algebra belonging to such a component generate the same subalgebra $D(B)$. So each component gives us one subalgebra.

All components form a partition of the family $\mathcal{B}(A)$. Hence classifying them we consider all subsets of $A$ - and in consequence we obtain all subalgebras.

It is a characteristic feature of the method of classification, that we consider in fact all subsets of the algebra, but not one by one: they are joined into components.

Example 1. Let $(A,\cdot,\ast,f)$ be the following algebra. $A = \{a, b, c, d\}$, $f$ - function, $\cdot$ - partial binary operation:

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|-----|-----|-----|-----|
| $a$ | $b$ | $a$ | $a$ | $d$ |

We divide the family $\mathcal{B}(abod)$ into components using formula (1.13). That is, as we mentioned before, a kind of taking an element out of brackets. To make the calculations short, we start with elements which generate large subalgebras.

$$\mathcal{B}(abod) = \{c\} \cdot \mathcal{B}(ab) \cup \mathcal{B}(bd) =$$

$$= \{cd\} \cdot \mathcal{B}(ab) \cup \{c\} \mathcal{B}(ab) \cup \mathcal{B}(abd).$$
We can see that \( D(c) = abc \) and \( D(cd) = abcd \), so the components \( \{cd\} \cdot J(ab) \) and \( \{c\} \cdot J(ab) \) give the subalgebras \( abcd \) and \( abc \) respectively.

\[
\mathcal{B}(abd) = \{a\} \cdot \mathcal{B}(bd) \cup \mathcal{B}(bd) = \{ad\} \cdot \mathcal{B}(b) \cup \{a\} \cdot \mathcal{B}(b) \cup \mathcal{B}(bd)
\]

\( D(ad) = abd, D(a) = ab \)

\( \mathcal{B}(bd) = \{\emptyset, b, d, bd\} \)

\( \emptyset, d \) are subalgebras. \( D(b) = ab, D(bd) = abd \).

We have obtained all subalgebras:

\[
\begin{array}{c}
\text{abcd} \\
\text{abc} \quad \text{abd} \\
\text{ab} \quad \text{d} \\
\text{\emptyset}
\end{array}
\]

(arrows denote inclusion, so they show the structure of the complete lattice of subalgebras).

3. The ways of calculating in the method of classification

The method of classification may seem hard to use, because the number of all subsets to classify in an algebra is incomparably higher than the number of its elements. The aim of this chapter is to give some formulas useful in calculations. These formulas describe connections between the closure \( D \) in an algebra \( A \) and composition in the family \( \mathcal{B}(\mathcal{B}(A)) \). They make possible effective determining of all subalgebras of a given algebra in a rather short way.

All formulas given here are derived from one fundamental formula (2.1) and are, roughly speaking, its progressive transformation and generalization. After finishing this chapter we will also give an example of determining subgroups of a group.
The determination of subalgebras by the method of classification with application of formulas given here.

Now we can state the formulas mentioned above.

Remark. If $X, Y \subseteq A$ we will write "$X \rightarrow Y$" instead of "$X$ generates $Y$", i.e.

$$X \rightarrow Y \iff Y = D(X).$$

Let $X, Y \subseteq A$; $a, b \in A$; $\alpha \subseteq B(A)$.

I. If $a \rightarrow b$ then:

(3.1) \[ D(ab \cup X) = D(a \cup X). \]

Proof. $b \in D(a)$ and $a \in D(a)$, so: we have $a \subseteq ab \subseteq D(a)$ and $D(ab) = D(a)$ by virtue of (2.1).

$D(ab \cup X) = D(D(ab) \cup D(X)) = D(D(a) \cup D(X)) = D(a \cup X)$ (we have used the following formula, valid for an arbitrary closure: if $D$ — closure in $A$ and $X, Y \subseteq A$ then $D(D(X) \cup D(Y)) = D(X \cup Y)$).

(3.2) \[ D(\{ab\} \cdot \alpha) = D(\{a\} \cdot \alpha). \]

Proof. $D(\{ab\} \cdot \alpha) = \{D(ab \cup X) : X \in \alpha\} = \{D(a \cup X) : X \in \alpha\} = D(\{a\} \cdot \alpha)$

II. If $X \rightarrow a$ then:

(3.3) \[ D(a \cup X \cup Y) = D(X \cup Y). \]

Proof.

$a \subseteq D(X)$

$D(a) \subseteq D(X)$

$X \subseteq X \cup a \subseteq D(X) \cup D(a) \subseteq D(X)$

$X \cup Y \subseteq X \cup a \cup Y \subseteq D(X \cup a) \cup D(Y) = D(X) \cup D(Y) \subseteq D(X \cup Y)$.
By virtue of (2.1) we obtain
\[ D(X \cup Y \cup a) = D(X \cup Y) \]

(3.4) \[ D({X} \cup a \cdot \alpha) = D({X} \cdot \alpha). \]

Proof. \[ D({X} \cup a \cdot \alpha) = \{ D(X \cup a \cup Y) : Y \in \alpha \} = \{ D(X \cup Y) : Y \in \alpha \} = D({X} \cdot \alpha). \]

Proof. \[ D({X} \cdot \beta(a \cup Y)) = D({X} \cdot \beta(a \cup Y)) = D({X} \cdot \beta(Y)) \]

III. If \( a \rightarrow b \) and \( b \rightarrow a \) then:

(3.6) \[ D(a \cup X) = D(b \cup X) \]

(3.7) \[ D({a} \cdot \alpha) = D({b} \cdot \alpha). \]

Proof. \[ D({a} \cdot \alpha) = D({ab} \cdot \alpha) = D({b} \cdot \alpha) \] by virtue of (3.2). (3.6) arises from (3.7) by substituting \( \alpha = \{X\}. \)

(3.8) \[ D(\beta(ab \cup X)) = D(\beta(a \cup X)) \]

(3.9) \[ D({X} \cdot \beta(ab \cup Y)) = D({X} \cdot \beta(a \cup Y)) \]

(proofs in IV).

IV. If \( a \cup X \rightarrow b \) and \( b \cup X \rightarrow a \) then

(3.10) \[ D({X} \cdot \beta(ab \cup Y)) = D({X} \cdot \beta(a \cup Y)). \]

Remark. Formulas (3.9) and (3.10) are identical, but their assumptions are different. We mention them separately to save up the rule of division into points I, II, III, IV - each of them has its own assumptions common for all formulas listed in it. But the proof is one for both:
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\[ D(\{x\} \cdot B(ab \cup Y)) = D(\{x\} \cdot B(ab) \cdot B(Y)) = \]
\[ = D(\{x\} \cdot \Phi, a, b, ab \cdot B(Y)) = D(\{x, x \cup a, x \cup b, x \cup ab\} \cdot B(Y)) = \]
\[ = D(\{x\} \cdot B(Y)) \cup D(\{x \cup a\} \cdot B(Y)) \cup D(\{x \cup b\} \cdot B(Y)) \cup D(\{x \cup ab\} \cdot B(Y)). \]

But \(x \cup a \rightarrow b\) and \(x \cup b \rightarrow a\) so we have by virtue of (3.4):

\[ D(\{x \cup a\} \cdot B(Y)) = D(\{x \cup ab\} \cdot B(Y)) = D(\{x \cup b\} \cdot B(Y)), \]
\[ D(\{x\} \cdot B(ab \cup Y)) = D(\{x\} \cdot B(Y)) \cup D(\{x \cup a\} \cdot B(Y)) = \]
\[ = D(\{x\} \cdot B(Y) \cup \{x\} \cdot \{a\} \cdot B(Y)) = D(\{x\} \cdot (B(Y) \cup \{a\} \cdot B(Y))) = \]
\[ = D(\{x\} \cdot B(a \cup Y)). \]

The formula (3.8) arises from (3.9) by substituting \(X = \Phi\).

The most important of these formulas (in the sense of their utility in the method of classification) are (3.5),(3.8), (3.9),(3.10), because they show how to determine closures of families of the form \(\{x\} \cdot B(Y)\) where \(X, Y \subseteq A\), and we know that all the components we get in the method of classification are just of this form.

To finish this chapter we shall give one more example. In this example we shall apply the formulas derived above.

Example 2.

We consider the group \(G\) of symmetries of a square.

The table of the group operation is following:

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We determine the family of all subgroups, i.e. the family $D(\mathcal{B}(G))$:

$$D(\mathcal{B}(G)) = D(\mathcal{B}(a)bcd_{gh}) = D(a,bcd_{gh}) =$$

$$= D(\mathcal{B}(abdfgh)) = D(\{a\} \cdot \mathcal{B}(bdfgh) \cup D(\mathcal{B}(bdfgh)).$$

$$D(\{a\} \cdot \mathcal{B}(bdfgh)) = D(\{a\} \cdot \mathcal{B}(dfgh)) = D(\{a\} \cdot \mathcal{B}(d)) =$$

$$= D(\{a, ad\}) = \{abc, abcd_{gh}\}.$$ 

$$D(\mathcal{B}(bd_{fgh})) = D(\{b\} \cdot \mathcal{B}(d_{fgh}) \cup D(\mathcal{B}(d_{fgh}) =$$

$$= D(\{b\} \cdot \mathcal{B}(dg)) \cup D(\{d\} \cdot \mathcal{B}(fg)) \cup D(\mathcal{B}(gh)) =$$

$$= D(\{b\} \cdot \mathcal{B}(dg)) \cup D(\{d\} \cdot \mathcal{B}(fg)) \cup D(\{f\} \cdot \mathcal{B}(gh)) \cup D(\mathcal{B}(gh)).$$

$$D(\{b\} \cdot \mathcal{B}(dg)) = D(\{b, bd, bg, bdg\}) = \{be, be_{df}, be_{gh}, G\}$$

$$D(\{a\} \cdot \mathcal{B}(fg)) = D(\{d, df, dg, dfg\}) = \{de, bed_{f}, G\}$$

$$D(\{f\} \cdot \mathcal{B}(gh)) = D(\{f\} \cdot \mathcal{B}(g)) = D(\{f, fg\}) = \{fe, G\}$$

$$D(\mathcal{B}(gh)) = D(\{\phi, g, h, gh\}) = \{e, ge, he, ebgh\}.$$ 

We have obtained all subgroups:

$$D(\mathcal{B}(G)) = \{eabcd_{fgh}, eabc, bedf, begh, be, de, fe, ge, he, e\}.$$
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