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ON APPROXIMATE SOLVING OF THE FOURIER PROBLEMS

1. Introduction

The Fourier problems for linear equations of parabolic type are reducible to the following Volterra integral equation

$$f(x,t) + \int_0^t \int_M N(x,t;y,s)\, f(y,x)\, d\mu(y)\, ds = g(x,t), \tag{1}$$

where M is sufficiently regular and compact manifold in R^n, $n \geq 2$ (see [1] p.90), μ - the measure function on M, N - the kernel of integral equation (1) that is continuous and unbounded function on $W \times W$, $W = M \times [0,T]$.

It is known that, after some finite number of iterations of equation (1), we shall get an integral equation with a bounded kernel (see [2] or [3]), and applying the method of successive approximations we can express the solution of this equation in the form of Neumann functional series. Unfortunately, this method is numerical unapplicable, because of construction of the iterated kernels and complexity of functional series.

Usually, Fourier problems are solved with the help of the method of finite element, the method of finite differences or variations methods and iterative methods.

In this paper we present an approximate method for solving the integral equation (1), method based on a partition of the unity and an approximation of continuous function.

This method can be applied for the integral equation (1) with the continuous and bounded or continuous and unbounded kernel.

2. Volterra weak-singular operators

Let $W = M \times [0,T]$ denote the compact manifold in the space-time $R^n \times R$, where M is compact manifold in R^n and $[0,T]$ and interval in R.

By $C(W)$ we denote the space of continuous bounded functions defined in W with the following norm

$$\|f\| = \sup \left\{ |f(x,t)| \ : \ (x,t) \in W \right\}. \tag{2}$$

Definition 1. The function N is called a Volterra weak-singular kernel, if it has the following properties:

1^0 it is defined and continuous for two arbitrary points x,y of the manifold M and for $t > s$, $t,s \in [0,T]$,

2^0 there exists on W the uniform limit with regard to (x,t)

$$n(x,t) = \lim_{k \to \infty} \int_0^t \int_M w_k(t-s) \ |N(x,t;y,s)| \ d\mu(y)ds, \tag{3}$$

where

$$w_k(t-s) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t-s < \frac{1}{2k} \\ 2k(t-s) - 1 & \text{for} \quad \frac{1}{2k} \leqslant t-s < \frac{1}{k} \\ 1 & \text{for} \quad \frac{1}{k} \leqslant t-s \leqslant T. \end{cases} \tag{4}$$

For every Volterra weak-singular kernel we define a sequence of continuous kernels as follows

$$N_k(x,t;y,s) = w_k(t-s) \ N(x,t;y,s). \tag{5}$$

Setting

$$n_k(x,t) = \int_0^t \int_M |N_k(x,t;y,s)| \ d\mu(y)ds, \tag{6}$$

it is easy to see that above introduced function is non-negative and continuous on W. Because $\{n_k\}$ is a non-decreasing sequence of the continuous functions which converges uniformly to the function $n(x,t)$ on W, it follows that $n(x,t)$ is a continuous function too.

In the space C(W) let us consider an operator N defined by the relation

$$(7)\qquad (\mathcal{N}f)(x,t) = \int_0^t \int_M N(x,t;y,s)\, f(y,s)\, d\mu(y)ds,$$

the kernel N being continuous on $W \times W$.

We denote by $\mathcal{A}$ the algebra of all linear and bounded operators defined in the space C(W). In this algebra we consider the set $\mathcal{K}$ of all compact and the set $\mathcal{K}'$ of all integral operators with continuous kernels. It is known that

$$(8)\qquad \mathcal{K}' \subset \mathcal{K} \subset \mathcal{A}$$

We shall now consider a class of integral operators with continuous and unbounded kernels and we shall state some properties of them.

T h e o r e m 1. For every Volterra weak-singular kernel N there exists a sequence $\mathcal{N}_k$ of the operators of the class $\mathcal{K}'$ convergent to an operator $\mathcal{N}$ of the class $\mathcal{K}$ and moreover

$$(9)\qquad |\mathcal{N}| = \max \left\{ n(x,t) : (x,t) \in W \right\}$$

holds.

P r o o f . From (4) and (5) we get for $p > q$

$$(10)\quad \left|N_p(x,t;y,s) - N_q(x,t;y,s)\right| = \left|N_p(x,t;y,s)\right| - \left|N_q(x,t;y,s)\right|.$$

Because the norm of the Volterra operator $\mathcal{N}_k$ with a continuous kernel is of the form

$$(11)\qquad |\mathcal{N}_k| = \max \left\{ \int_0^t \int_M \left|N_k(x,t;y,s)\right| d\mu(y)ds : (x,t) \in W \right\},$$

so from the above formula and (6) we obtain for $p > q$

$$(12) \qquad |\mathcal{N}_p - \mathcal{N}_q| = \max\left\{n_p(x,t) - n_q(x,t) : (x,t) \in W\right\}.$$

Hence, for $p > q$ and $p \leq q$

$$(13) \qquad |\mathcal{N}_p - \mathcal{N}_q| = \max\left\{\left|n_p(x,t) - n_q(x,t)\right| : (x,t) \in W\right\}$$

holds.

Next, from (3) we have

$$(14) \qquad |\mathcal{N}_p - \mathcal{N}_q| = \|n_p - n_q\| \longrightarrow 0 \quad \text{as} \quad p,q \longrightarrow \infty .$$

In this way we have proved that the sequence $\{\mathcal{N}_k\}$ is convergent in the algebra $\mathcal{A}$. Hence, the limit $\mathcal{N}$ belongs to the class $\mathcal{K}$ of compact operators. We get therefore

$$(15) \qquad |\mathcal{N}_k| = \max\left\{n_k(x,t) : (x,t) \in W\right\} = \|n_k\|$$

and

$$(16) \qquad |\mathcal{N}| = \lim_{k\to\infty} |\mathcal{N}_k| = \lim_{k\to\infty} \|n_k\| = \|n\|.$$

Thus, the proof of the theorem is completed.

R e m a r k 1. A Volterra weak-singular kernel N defines a Volterra weak-singular operator $\mathcal{N}$ by the following formula

$$(17) \qquad (\mathcal{N}f)(x,t) = \lim_{k\to\infty} \int_0^t \int_M N_k(x,t;y,s)\, f(y,s)\, d\mu(y)ds.$$

Now we shall state some properties of the defined above weak-singular operators of Volterra type.

C o r o l l a r y 1. The Volterra weak-singular kernel N defines the operator $\mathcal{N}$ in the unique manner.

C o r o l l a r y 2. If N and L are Volterra weak-singular kernels, then $\alpha N + \beta L$ is such a kernel too and it defines in the unique manner the operator $\alpha \mathcal{N} + \beta \mathcal{L}$ for $\alpha, \beta \in R$.

R e m a r k 2. The Volterra weak-singular operator $\mathcal{N}$ defined by formula (7), where N is Volterra weak-singular kernel (see Definition 1), is compact in $C(W)$.

3. Fundamental definitions and lemmas

For $f \in C(W)$ and positive δ, γ we define a modulus of continuity as

$$(18)\quad \omega(f;\delta,\gamma) = \sup\Big\{ |f(x,t)-f(y,s)| : |x-y| < \delta, |t-s| < \gamma; (x,t),(y,s) \in W \Big\}$$

and a modulus of oscillation of the operator $\mathcal{N}$ compact in $C(W)$ as

$$(19)\qquad \omega(\mathcal{N};\delta,\gamma) = \sup\Big\{ \omega(\mathcal{N}f;\delta,\gamma) : f \in C(W), \|f\| < 1 \Big\}.$$

L e m m a 1. Let $W = M \times [0,T]$ be a compact manifold and $\mathcal{N}$ an operator compact in $C(W)$.

Then $\omega(\mathcal{N};\delta,\gamma)$ is a non-negative and non-decreasing function on $[0,\infty)\times[0,\infty)$ which has the following properties:

1° $\omega(\mathcal{N}f;\delta,\gamma) \leq \|f\|\,\omega(\mathcal{N};\delta,\gamma)$ for all $f \in C(W)$, $\delta, \gamma \geq 0$,

2° $\omega(\mathcal{N};\delta,\gamma) \to 0$ as $\delta, \gamma \to 0_+$.

P r o o f . From the definition (19) it follows that $\omega(\mathcal{N};\cdot,\cdot)$ is a non-negative and non-decreasing function. The estimation 1° is clear for $f = 0$ and for $f \neq 0$, namely it follows from the equality

$$(20)\quad \omega(\mathcal{N}f;\delta,\gamma) = \omega(\|f\|\cdot\|f\|^{-1}\cdot\mathcal{N}f;\delta,\gamma) = \|f\|\,\omega(\|f\|^{-1}\cdot\mathcal{N}f;\delta,\gamma)$$

and from the definition (19).

Because $\mathcal{N}$ is an operator compact in $C(W)$, so the Arzela-Ascoli theorem implies the proof of 2°.

D e f i n i t i o n 2. A set of the functions $Z = (w_i)$, $i = 1,2,\dots,k$, is called a partition of the unity on the compact manifold V, if

1° $w_i \in C(W)$, $w_i(x) \geq 0$ for $i = 1,2,\dots,k$,

2° $\sum_{i=1}^{k} w_i(x) = 1$ for every $x \in V$.

D e f i n i t i o n 3. The diameter of a partition of the unity $Z = (w_i)$, $i = 1,2,\dots,k$, on the manifold V is defined as follows

$$\varrho(Z) = \sup\left\{d(\operatorname{supp} w_i) : w_i \in Z\right\}. \tag{21}$$

It is known that for every $\delta > 0$ there exists, on the manifold V, a partition of the unity Z with the diameter $\varrho(Z) \leq \delta$.

Now we shall prove the lemma on approximation of a continuous function in $C(W)$.

L e m m a 2. Let $Z = (u_j)$, $j = 1,2,\dots,m$, be a partition of the unity on the compact manifold M with the diameter $\varrho(Z) \leq \delta$, and $\mathcal{Z} = (v_k)$, $k = 1,2,\dots,n$ - a partition of the unity on interval $[0,T]$ with the diameter $\varrho(\mathcal{Z}) \leq \gamma$ for positive δ, γ.

Then the operator $T_{\delta,\gamma} : C(W) \longrightarrow C(W)$, $W = M \times [0,T]$, defined for an arbitrary $x_j \in \operatorname{supp} u_j$, $t_k \in \operatorname{supp} v_k$ and for $f \in C(W)$ by

$$T_{\delta,\gamma} f(x,t) = \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_j,t_k)\, u_j(x)\, v_k(t) \tag{22}$$

has the following properties

1° $\|T_{\delta,\gamma} f\| \leq \|f\|$,

2° $|T_{\delta,\gamma}| < 1$

3° $\|T_{\delta,\gamma} f - f\| \leq \omega(f;\delta,\gamma)$.

P r o o f . From the definition (22) we get 1° and 2°. It is easy to see that

$$(23)\quad \left| f(x,t)-T_{\delta,\gamma}\, f(x,t)\right| \leq \sum_{j=1}^{m}\sum_{k=1}^{n} \left| f(x,t)-f(x_j,t_k)\right| u_j(x)v_k(t).$$

In virtue of the definition of the modulus of continuity, the following inequality

$$(24)\quad \left| f(x,t)-f(x_j,t_k)\right| \; u_j(x)v_k(t) \leq \omega(f;\delta,\gamma)\; u_j(x)v_k(t)$$

for $(x,t) \in W$, $\varrho(Z) \leq \delta$ and $\varrho(\mathcal{Z}) \leq \gamma$ holds. The proof of 3^{0} follows from (24).

R e m a r k 3. The approximation of a continuous function is the basis of a certain method for solving integral equations of the form (1).

4. The approximate method

Rewrite the equation (1) in the abbreviated form

$$(25)\qquad f + \mathcal{N}f = g$$

where $\mathcal{N}$ is a Volterra weak-singular operator defined by (7).

We shall give the approximate method for solving equation (25) with the operator $\mathcal{N}$ compact in $C(W)$.

Consider the equation

$$(26)\qquad \bar{f} + \mathcal{N}_{\delta,\gamma}\,\bar{f} = g$$

in which the operator $\mathcal{N}_{\delta,\gamma}$ is defined as follows

$$(27)\qquad \mathcal{N}_{\delta,\gamma} = \mathcal{N}T_{\delta,\gamma},$$

where $T_{\delta,\gamma}$ is of the form (22).

We treat the solution of (26) as an approximate solution of (25). The equation (26) may be written in the following form

$$(28)\qquad \bar{f}(x,t) + \sum_{j=1}^{m}\sum_{k=1}^{n} \bar{f}(x_j,t_k)(\mathcal{N}u_jv_k)(x,t) = g(x,t),$$

where

$$(29)\quad (\mathcal{N}u_j v_k)(x,t) =$$

$$= \int\limits_{\operatorname{supp} u_j \subset [0,t]} \quad \int\limits_{\operatorname{supp} v_k \subset M} N(x,t;y,s)u_j(y)v_k(s)d\mu(y)ds.$$

Setting in (28): $x = x_p$ $(p=1,2,\dots,m)$, $t = t_q$ $(q=1,2,\dots,n)$, we obtain a system of $m\cdot n$ linear algebraic equations of the form

$$(30)\quad \bar{f}(x_p,t_q) + \sum_{j=1}^{m} \sum_{k=1}^{n} \bar{f}(x_j,t_k)(\mathcal{N}u_j v_k)(x_p,t_q) = g(x_p,t_q).$$

The approximate solution $\bar{f}(x,t)$ of the equation (25) is given by (28), where values $\bar{f}(x_j,t_k)$, $j=1,2,\dots,m$, $k = 1,2,\dots,n$, form the solution of the system (30).

Now we formulate the following theorem.

T h e o r e m 2. Let $Z = (u_j)$, $j = 1,2,\dots,m$, be a partition of the unity on the compact manifold M with the diameter $\varrho(Z) \leqslant \delta$ and let $\mathcal{Z} = (v_k)$, $k=1,2,\dots,n$ be a partition of the unity of interval $[0,T]$ with the diameter $\varrho(\mathcal{Z}) \leqslant \partial$.

If $(x_j,t_k) \in \operatorname{supp} u_j \times \operatorname{supp} v_k$ and

$$(31)\qquad \omega(\mathcal{N};\delta,\gamma)\ \left|R(\mathcal{N})\mathcal{N}\right| < 1$$

for the operator $\mathcal{N}$, compact in $C(W)$, then the system (30) has a unique solution and an estimation of the error of the approximate solution of the equation (25) is given by the following inequality

$$(32)\quad \|f-\bar{f}\| \quad \frac{\left|R(\mathcal{N})\mathcal{N}\right|}{1-\omega(\mathcal{N};\delta,\gamma)\ \left|R(\mathcal{N})\right|}\ \left[\omega(g;\delta,\gamma) + \|f\|\,\omega(\mathcal{N};\delta,\gamma)\right],$$

where $f = R(\mathcal{N})g$ is the solution of the equation (25) and $R(\mathcal{N})$ - the resolvent of the operator $\mathcal{N}$.

P r o o f . Suppose that the system of linear equations (30) has a solution for a given $g \in C(W)$. Then the approximate solution $\bar{f}$ of the equation (25) is given by (28) and $\bar{f} \in C(W)$. From (26) and (27) we get

$$\bar{f} - \mathcal{N}\bar{f} = g + \mathcal{N}(T_{\delta,\gamma}\bar{f} - \bar{f}),$$

hence

$$\bar{f} = R(\mathcal{N})g + R(\mathcal{N})\mathcal{N}(T_{\delta,\gamma}\bar{f} - \bar{f}).$$

In virtue of the Lemma 2 we obtain

$$\|\bar{f}\| \leqslant \|R(\mathcal{N})g\| + |R(\mathcal{N})\mathcal{N}|\omega(\bar{f};\delta,\gamma). \tag{33}$$

From

$$\bar{f}(x,t) = g(x,t) + \mathcal{N}T_{\delta,\gamma}\bar{f}(x,t)$$

it follows that

$$\omega(\bar{f};\delta,\gamma) \leqslant \omega(g;\delta,\gamma) + \|\bar{f}\|\,\omega(\mathcal{N};\delta,\gamma)$$

therefore, from (33) we have

$$\omega(\bar{f};\delta,\gamma) \leqslant \omega(g;\delta,\gamma) + \Big[\|R(\mathcal{N})g\| + |R(\mathcal{N})\mathcal{N}|\omega(\bar{f};\delta,\gamma)\Big]\omega(\mathcal{N};\delta,\gamma)$$

and hence

$$\omega(\bar{f};\delta,\gamma) \leqslant \frac{\omega(g;\delta,\gamma) + \|R(\mathcal{N})g\|\omega(\mathcal{N};\delta,\gamma)}{1 - \omega(\mathcal{N};\delta,\gamma)\,|R(\mathcal{N})\mathcal{N}|}. \tag{34}$$

If $g = 0$, then $\omega(g;\delta,\gamma) = 0$. So from (31) and (34) we get $\omega(\bar{f};\delta,\gamma) = 0$. In virtue of (33), we obtain $\bar{f} = 0$.

In this way we have proved that the system of linear equations (30) has only a trivial solution for $g = 0$, so for every $g \in C(W)$ it has a unique solution.

The estimation (32) follows from Lemma 2 and (34). Subtracting the functions $\bar{f}$ and f we get

$$\bar{f} - f = R(\mathcal{N})\mathcal{N}(T_{\delta,\gamma}\bar{f} - f)$$

and consequently (32) too. The proof is completed.

Remark 4. The smaller the diameter of the partition of the unity is the more precise solution of equation (25) is obtained.

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