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ON A CERTAIN CLASS OF KÄHLERIAN MANIFOLDS

1. Introduction

A Kählerian manifold K^{2m} of even dimension $n = 2m$ is a Riemannian manifold admitting a parallel tensor field $F_j{}^i$ such that $F_i{}^r F_r{}^j = -\delta_i^j$, $F_{ji} = -F_{ij}$, where we put $F_{ji} = F_j{}^r g_{ri}$. In the present paper we shall consider the so-called Bochner curvature tensor (Bochner [2], Tachibana [6], Yano and Bochner [10]) defined by

$$\begin{aligned} B_{hijk} = R_{hijk} + \frac{1}{n+4} (&g_{ik}R_{hj} - g_{hk}R_{ij} + g_{hj}R_{ik} - g_{ij}R_{hk} + \\ &+ F_h{}^t R_{tj} F_{ik} - F_i{}^t R_{tj} F_{hk} + F_{hj} F_i{}^t R_{tk} - F_{ij} F_h{}^t R_{tk} + \\ &+ 2F_h{}^t R_{ti} F_{jk} + 2F_{hi} F_j{}^t R_{tk}) + \\ &- \frac{R}{(n+2)(n+4)} (g_{hj}g_{ik} - g_{ij}g_{hk} + F_{hj}F_{ik} - F_{ij}F_{hk} + 2F_{hi}F_{jk}), \end{aligned}$$

where R_{hijk}, R_{hj} and R are the curvature tensor, Ricci tensor and the scalar curvature, respectively (for the geometric meaning of the Bochner tensor see Blair [1], Yano [9]). Kählerian manifolds with parallel or vanishing Bochner curvature tensor were investigated by Chen and Yano [3], Matsumoto [4], Matsumoto and Tanno [5], Tachibana and Liu [7].

In this paper we consider a class of Kählerian manifolds in which Bochner curvature tensor satisfies the condition

$B_{hijk,lm} = B_{hijk,ml}$, where the comma indicates covariant differentiation with respect to the metric of the manifold. It is known that the class of Kählerian manifolds with parallel Bochner curvature tensor is contained in the union of a class of manifolds with vanishing Bochner curvature tensor and a class of symmetric manifolds. On the other hand, there exist Kählerian manifolds for which $B_{hijk,l} \neq 0$, and $R_{hijk,lm} = R_{hijk,ml}$. The following problem arises: is the class of Kählerian manifolds satisfying the condition $B_{hijk,lm} = B_{hijk,ml}$ essentially greater than the class of Kählerian manifolds for which $R_{hijk,lm} = R_{hijk,ml}$ or not? (It is obvious that the second class is contained in the first one).

In this paper the following theorem is proved: In Kählerian manifolds the condition $B_{hijk,lm} = B_{hijk,ml}$ implies $R_{hijk,lm} = R_{hijk,ml}$. This theorem gives the following answer to the above problem: the class of Kahlerian manifolds satisfying the condition $B_{hijk,lm} = B_{hijk,ml}$ is contained in the union of the class of manifolds with vanishing Bochner curvature tensor and the class of manifolds satisfying the condition $R_{hijk,lm} = R_{hijk,ml}$.

2. Preliminary results

We shall first recall the identities valid for Kählerian manifolds which are necessary for what follows (see Yano [9]):

$$(2.1)\quad \begin{cases} R_{kjt}{}^{h}F_{i}{}^{t} = R_{kji}{}^{t}F_{t}{}^{h}, \quad R_{kjht}F_{i}{}^{t} = R_{kjit}F_{h}{}^{t}, \\ R_{kji}{}^{h} + R_{kjs}{}^{t}F_{i}{}^{s}F_{t}{}^{h} = 0, \quad R_{kjih} = R_{kjst}F_{i}{}^{s}F_{h}{}^{t}, \\ R_{j}{}^{t}F_{t}{}^{h} = R_{t}{}^{h}F_{j}{}^{t}, \quad R_{jt}F_{i}{}^{t} + R_{it}F_{j}{}^{t} = 0, \\ R_{j}{}^{h} + R_{s}{}^{t}F_{j}{}^{s}F_{t}{}^{h} = 0, \quad R_{ji} = R_{ts}F_{j}{}^{t}F_{i}{}^{s}. \end{cases}$$

We put

$$(2.2)\quad \begin{cases} T_{iklm} = R_{ik,lm} - R_{ik,ml} = -(R_{tk}R^{t}{}_{ilm} + R_{it}R^{t}{}_{klm}), \\ T_{ij} = T_{hijk}g^{hk} = -(R_{rs}R^{r}{}_{ij}{}^{s} - R_{ir}R^{r}{}_{j}) = T_{ji}. \end{cases}$$

We see that

$$T_{iklm} = T_{kilm} = - T_{kiml} \tag{2.3}$$

and by a straightforward computation we find

$$\left\{\begin{array}{l} F_i{}^t T_{jklt} = F_l{}^t T_{jkit}, \\ F^{st} T_{tlhs} = F_h{}^t T_{tl}, \\ F_j{}^t T_{ti} = - F_i{}^t T_{tj}. \end{array}\right. \tag{2.4}$$

We put

$$W_{hijk} = F_h{}^t T_{tijk}, \quad W_{ij} = W_{hijk} g^{hk}. \tag{2.5}$$

From (2.1), (2.2), (2.3) and (2.4) we have

$$\left\{\begin{array}{l} W_{hijk} = - W_{ihjk} = W_{ihkj}, \\ F_m{}^s W_{klhs} = F_h{}^s W_{klms}, \quad F_m{}^s W_{skhl} = - T_{mkhl}, \\ W_{ij} = F_j{}^t T_{ti}, \quad W_{ij} = - W_{ji}, \quad F_m{}^t W_{tk} = T_{mk}. \end{array}\right. \tag{2.6}$$

The Bochner curvature tensor has the following properties (see Yano [9])

$$\left\{\begin{array}{l} B_{kjih} = - B_{jkih} = B_{jkhi} = B_{hijk}, \\ B_{kjih} + B_{jikh} + B_{ikjh} = 0, \\ B_{tji}{}^t = 0, \\ F_h{}^t B_{kjit} = F_i{}^t B_{kjht}, \\ F_i{}^t F_h{}^s B_{kjts} = B_{kjih}, \\ B_{kjts} F^{ts} = 0, \quad B_{tjis} F^{ts} = 0. \end{array}\right. \tag{2.7}$$

3. Main result

Theorem. **Let K^{2m} be a Kählerian manifold whose Boehner curvature tensor does not vanish identically. If $B_{hijk,lm} = B_{hijk,ml}$, then $R_{hijk,lm} = R_{hijk,ml}$.**

Proof. From (1.1) and (2.2) we have

$$
\begin{aligned}
(3.1)\quad B_{hijk,lm} - B_{hijk,ml} &= R_{hijk,lm} - R_{hijk,ml} + \\
&+ \frac{1}{n+4}\left(g_{hj}T_{iklm} - g_{ij}T_{hklm} + g_{ik}T_{hjlm} - g_{hk}T_{ijlm} + \right. \\
&+ F_{ik}F_h{}^tT_{tjlm} - F_{hk}F_i{}^tT_{tjlm} + F_{hj}F_i{}^tT_{tklm} - \\
&\left. - F_{ij}F_h{}^tT_{tklm} + 2F_{jk}F_h{}^tT_{tilm} + 2F_{hi}F_j{}^tT_{tklm}\right).
\end{aligned}
$$

Substituting this relation into the equation

$$B_{hijk,lm} - B_{hijk,ml} + B_{jklm,hi} - B_{jklm,ih} + B_{lmhi,jk} - B_{lmhi,kj} = 0$$

and using the well-known relation (see Walker [8])

$$R_{hijk,lm} - R_{hijk,ml} + R_{jklm,hi} - R_{jklm,ih} + R_{lmhi,jk} - R_{lmhi,kj} = 0,$$

we obtain

$$
\begin{aligned}
& g_{hj}T_{iklm} - g_{ij}T_{hklm} + g_{ik}T_{hjlm} - g_{hk}T_{ijlm} + F_{ik}F_h{}^tT_{tjlm} - \\
& - F_{hk}F_i{}^tT_{tjlm} + F_{hj}F_i{}^tT_{tklm} - F_{ij}F_h{}^tT_{tklm} + 2F_{jk}F_h{}^tT_{tilm} + \\
& + 2F_{hi}F_j{}^tT_{tklm} + g_{jl}T_{kmhi} - g_{kl}T_{jmhi} + g_{km}T_{jlhi} - g_{jm}T_{klhi} + \\
& + F_{km}F_j{}^tT_{tlhi} - F_{jm}F_k{}^tT_{tlhi} + F_{jl}F_k{}^tT_{tmhi} - F_{kl}F_j{}^tT_{tmhi} + \\
& + 2F_{lm}F_j{}^tT_{tkhi} + 2F_{jk}F_l{}^tT_{tmhi} + g_{lh}T_{mijk} - g_{mh}T_{lijk} + g_{mi}T_{lhjk} + \\
& + g_{li}T_{mhjk} + F_{mi}F_l{}^tT_{thjk} - F_{li}F_m{}^tT_{thjk} + F_{lh}F_m{}^tT_{tijk} - F_{mh}F_l{}^tT_{tijk} + \\
& + 2F_{hi}F_l{}^tT_{tijk} + 2F_{lm}F_h{}^tT_{tijk} = 0.
\end{aligned}
$$

The contraction of this equation with g^{ij} gives, in view of (2.2), (2.4), (2.5) and (2.6), the following result

$$(3.2)\quad -(n+4)\, T_{hklm} + T_{kmhl} - T_{klhm} + T_{lhmk} - T_{mhlk} +$$
$$+ g_{km}T_{lh} - g_{kl}T_{mh} + g_{mh}T_{lk} - g_{lh}T_{mk} +$$
$$+ F_{km}W_{lh} - F_{kl}W_{mh} + F_{mh}W_{kl} - F_{lh}W_{km} +$$
$$+ F_m{}^sW_{klhs} - F_l{}^sW_{kmhs} + F_l{}^sW_{mhks} - F_m{}^sW_{lhks} - 4F_k{}^sW_{lmhs} = 0.$$

Now transvecting (3.2) with $F_j{}^h$, using (2.5) and (2.6) and replacing j by h, we obtain

$$(3.3)\quad -(n+4)\, W_{lmhk} + W_{mhlk} - W_{mklh} + W_{klhm} - W_{hlkm} - 4W_{hklm} +$$
$$+ g_{km}W_{hl} - g_{hm}W_{kl} + g_{hl}W_{km} - g_{kl}W_{hm} +$$
$$+ F_{km}T_{hl} - F_{hm}T_{kl} + F_{hl}T_{km} - F_{kl}T_{hm} +$$
$$- F_h{}^sT_{mkls} + F_k{}^sT_{mhls} - F_k{}^sT_{hlms} + F_h{}^sT_{klms} = 0.$$

Symmetrizing the last equation in pairs of indices (h,k), (l,m) and applying (2.4) and (2.6) we find

$$-(n+8)\,(W_{hklm} + W_{lmhk}) + 2(W_{klhm} - W_{kmhl} - W_{lhmk} + W_{mhlk}) +$$
$$+ 2(F_m{}^sT_{klhs} - F_l{}^sT_{kmhs} + F_l{}^sT_{mhks} - F_m{}^sT_{lhks}) = 0.$$

Substituting this result into (3.3) we have

$$n(W_{lmhk} - W_{hklm}) + 2(g_{km}W_{lh} - g_{kl}W_{mh} + g_{lh}W_{mk} - g_{mh}W_{lk}) +$$
$$+ 2(F_{kl}T_{mh} - F_{km}T_{lh} + F_{lh}T_{mk} - F_{mh}T_{lk}) = 0.$$

Transvecting this equation with $F_i{}^h$, in view of (2.5) we obtain (after replacing i by h)

$$nF_h{}^sW_{lmks} = nT_{hklm} + 2(-g_{km}T_{hl} + g_{kl}T_{hm} + g_{lh}T_{mk} - g_{mh}T_{lk}) +$$
$$+ 2(F_{hl}W_{mk} - F_{hm}W_{lk} + F_{kl}W_{mh} - F_{km}W_{lh}).$$

Substituting this relation into (3.2) we have

$$(3.4)\quad -n(n+8)T_{hklm} + 2n\, T_{kmhl} - 2n\, T_{klhm} + 2n\, T_{lhmk} - 2n\, T_{mhlk} =$$
$$= (n+12)(g_{kl}T_{mh} - g_{km}T_{lh} + g_{hl}T_{mk} - g_{hm}T_{lk}) +$$
$$+ (n+4)(F_{kl}W_{mh} - F_{km}W_{lh} + F_{hl}W_{mk} - F_{hm}W_{lk}).$$

Symmetrizing this equation in pairs of indices (h,k), (l,m), we obtain

$$-n(n+8)(T_{hklm} + T_{lmhk}) - 4n\, T_{klhm} - 4n\, T_{mhlk} =$$
$$= (n+12((-2g_{km}T_{lh} + 2g_{hl}T_{mk}) + (n+4)(-2F_{km}W_{lh} + 2F_{hl}W_{mk}).$$

Symmetrizing this now in h,l, we have

$$n(T_{hklm} + T_{lmhk} + T_{klhm} + T_{mhlk}) = 4(g_{km}T_{lh} - g_{hl}T_{mk}).$$

Alternating now the indices l, m in the above relation, we find

$$n(2T_{hklm} + T_{klhm} + T_{mhlk} - T_{kmhl} - T_{lhmk}) =$$
$$= 4(g_{km}T_{lh} - g_{hl}T_{mk} - g_{kl}T_{mh} + g_{hm}T_{lk}).$$

Substituting this result into (3.4), we obtain

$$(3.5)\quad nT_{hklm} = g_{km}T_{hl} + g_{hm}T_{kl} - g_{hl}T_{km} - g_{kl}T_{hm} +$$
$$+ W_{km}F_{hl} + W_{hm}F_{kl} - W_{hl}F_{km} - W_{kl}F_{hm}.$$

From the equation $B_{hijk,lr} - B_{hijk,rl} = 0$ and Ricci identity, it follows that

$$B_{sijk}R^{s}{}_{hlr} + B_{hsjk}R^{s}{}_{ilr} + B_{hisk}R^{s}{}_{jlr} + B_{hijs}R^{s}{}_{klr} = 0.$$

Now transvecting this equation with $R^r{}_m$, we obtain

$$B^s{}_{ijk}R_{shlr}R^r{}_m - B^s{}_{hjk}R_{silr}R^r{}_m + B^s{}_{khi}R_{sjlr}R^r{}_m -$$

$$- B^s{}_{jhi}R_{sklr}R^r{}_m = 0.$$

Interchanging the indices m and l, using the identity $R_{shlr}R^r{}_m + R_{shmr}R^r{}_l = - T_{lmhs}$ and (3.5), we find

$$\begin{aligned}(3.6)\quad & B^s{}_{ijk}(g_{ms}T_{lh} + g_{ls}T_{mh} - g_{mh}T_{ls} - g_{lh}T_{ms}) + \\ & + B^s{}_{ijk}(W_{ms}F_{lh} + W_{ls}F_{mh} - W_{mh}F_{ls} - W_{lh}F_{ms}) + \\ & - B^s{}_{hjk}(g_{ms}T_{li} + g_{ls}T_{mi} - g_{mi}T_{ls} - g_{li}T_{ms}) - \\ & - B^s{}_{hjk}(W_{ms}F_{li} + W_{ls}F_{mi} - W_{mi}F_{ls} - W_{li}F_{ms}) + \\ & + B^s{}_{khi}(g_{ms}T_{lj} + g_{ls}T_{mj} - g_{mj}T_{ls} - g_{lj}T_{ms}) + \\ & + B^s{}_{khi}(W_{ms}F_{lj} + W_{ls}F_{mj} - W_{mj}F_{ls} - W_{lj}F_{ms}) + \\ & - B^s{}_{jhi}(g_{ms}T_{lh} + g_{ls}T_{mk} - g_{mk}T_{ls} - g_{lk}T_{ms}) - \\ & - B^s{}_{jhi}(W_{ms}F_{lk} + W_{ls}F_{mk} - W_{mk}F_{ls} - W_{lk}F_{ms}) = 0.\end{aligned}$$

Since $F_j{}^rW_{ms}B^s{}_{kir} - F_k{}^rW_{ms}B^s{}_{jir} = T_{rm}B^r{}_{ikj}$ in virtue of (2.7), by contracting (3.6) with g^{lh} and using (2.7), we obtain

$$\begin{aligned}(3.7)\quad & -(n+2)T_{ms}B^s{}_{ijk} + T_{js}B^s{}_{imk} - T_{ks}B^s{}_{imj} + g_{mj}T_{rs}B^s{}_{ki}{}^r - \\ & - g_{mk}T_{rs}B^s{}_{ji}{}^r + F_{mi}W_{sr}B^{sr}{}_{jk} + F_{mj}W_{sr}B^s{}_{ki}{}^r + \\ & + F_{mk}W_{rs}B^s{}_{ji}{}^r + F_{ms}W_{rj}B^s{}_{ki}{}^r - F_{ms}W_{rk}B^s{}_{ji}{}^r = 0.\end{aligned}$$

Summing (3.7) cyclically in m,j and k, we obtain

$$(3.8)\qquad -(n+4)(T_{ms}B^{s}{}_{ijk} + T_{js}B^{s}{}_{ikm} + T_{ks}B^{s}{}_{imj}) +$$
$$+ F_{mi}W_{sr}B^{sr}{}_{jk} + F_{ji}W_{sr}B^{sr}{}_{km} + F_{ki}W_{sr}B^{sr}{}_{mj} +$$
$$+ 2F_{jk}W_{sr}B^{s}{}_{mi}{}^{r} + 2F_{mj}W_{sr}B^{s}{}_{ki}{}^{r} + 2F_{km}W_{sr}B^{s}{}_{ji}{}^{r} = 0.$$

Since $W_{sr}B^{s}{}_{km}{}^{r} - W_{sr}B^{s}{}_{mk}{}^{r} = -W_{sr}B^{sr}{}_{km}$, the transvection of (3.8) with F^{ij} gives, in virtue of (2.7)

$$(3.9)\qquad W_{sr}B^{sr}{}_{km} = 0$$

and with help of the Bianchi identity

$$(3.10)\qquad W_{sr}B^{s}{}_{km}{}^{r} = 0.$$

Therefore, from (3.8) it follows

$$(3.11)\qquad T_{ms}B^{s}{}_{ijk} + T_{js}B^{s}{}_{ikm} + T_{ks}B^{s}{}_{imj} = 0.$$

Transvecting now (3.10) with $F_{h}{}^{m}$, we obtain

$$(3.12)\qquad T_{rs}B^{s}{}_{ij}{}^{r} = 0.$$

Substituting (3.9), (3.10) and (3.12) into (3.7), we find

$$(3.13)\qquad -(n+1)T_{ms}B^{s}{}_{ijk} + F_{ms}W_{rk}B^{s}{}_{ki}{}^{r} - F_{ms}W_{rk}B^{s}{}_{ji}{}^{r} = 0.$$

Transvecting (3.11) with $F_{h}{}^{i}$ and making use of (2.6) and (2.7),we have

$$(3.14)\qquad W_{ms}B^{s}{}_{hjk} + W_{js}B^{s}{}_{hkm} + W_{ks}B^{s}{}_{hmj} = 0.$$

Transvecting (3.13) with $F_h{}^m$ and using (2.6), we obtain

$$(n+1)W_{hs}B^s{}_{ijk} + W_{js}B^s{}_{ikh} + W_{ks}B^s{}_{ihj} = 0.$$

This implies, in view of (3.14),

$$W_{ms}B^s{}_{hjk} = 0 \tag{3.15}$$

and, by transvection with $F_k{}^m$,

$$T_{ms}B^s{}_{hjk} = 0. \tag{3.16}$$

Now (3.6) can be written as

$$T_{lh}B_{mijk} + T_{mh}B_{lijk} - T_{li}B_{mhjk} - T_{mi}B_{lhjk} +$$

$$+ T_{lj}B_{mkhi} + T_{mj}B_{lkhi} - T_{lk}B_{mjhi} - T_{mk}B_{ljhi} -$$

$$- W_{mh}F_{ls}B^s{}_{ijk} - W_{lh}F_{ms}B^s{}_{ijk} + W_{mi}F_{ls}B^s{}_{hjk} + W_{li}F_{ms}B^s{}_{hjk} -$$

$$- W_{mj}F_{ls}B^s{}_{khi} - W_{lj}F_{ms}B^s{}_{khi} + W_{mk}F_{ls}B^s{}_{jhi} + W_{lk}F_{ms}B^s{}_{jhi} = 0.$$

Transvecting the last equation with T^{lh} and using (3.15), (3.16) and the relation $W_{mh}F_{ls}T^{lh} = -W_m{}^rF_s{}^lT_{lr} = -W_m{}^rW_{rs}$, we obtain

$$T_{lh}T^{lh}B_{mijk} = 0.$$

Since the metric is positive definite, the manifold is analytic and $B_{mijk} \neq 0$, this implies $T_{lh} = 0$ and, consequently, $W_{lh} = 0$. Formula (3.5) implies now $T_{hklm} = 0$, and, in virtue of (3.1), we obtain $R_{hijk,lm} = R_{hijk,ml}$. This completes the proof.

C o r o l l a r y . Let K^{2m} be a Kählerian manifold whose Bochner curvature tensor does not vanish identically.

If $B_{hijk,l} = c_l B_{hijk}$, for some vector field c_l, then $R_{hijk,lm} = R_{hijk,ml}$.

P r o o f . Let V be an open subset of K^{2m} on which $B_{hijk} \neq 0$. We put $f(x) = B_{hijk}(x)B^{hijk}(x)$ for x V. Since $f_{,l} = 2c_l f$, c_l is a locally gradient and using the equation $B_{hijk,lm} = (c_{l,m} + c_l c_m)B_{hijk}$, we see that the condition $B_{hijk,lm} = B_{hijk,ml}$ is satisfied on V. Now applying our Theorem, we obtain $R_{hijk,lm} = R_{hijk,ml}$ on V. Since the manifold is analytic, we have $R_{hijk,lm} = R_{hijk,ml}$ on K^{2m}.

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