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SOME COMMON FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS

1. Introduction

There are a multitude of fixed point theorems for multi-valued mappings defined on a complete metric space. The work of Nadler [6], Reich ([7], [8]), Wong [9] and Kaulkud - Pai [4] are worth mentioning. In most of these results one uses Hausdorff metric and the usual distance function between points and sets.

Recently Fisher [1] proved an interesting multi-valued fixed point theorem using a different distance function and employing an entirely new technique of proof.

In this paper we wish to present a common fixed point theorem for two multi-valued mappings. Our result generalizes the main theorem of Fisher [2].

2. Preliminaries and Notations

Let (X,d) be a metric space, and $B(X)$ the set of all non-empty, bounded subsets of X. The function $\delta(A,B)$ with A,B in $B(X)$ is defined by

$$\delta(A,B) = \sup\{d(a,b) : a \in A,\ b \in B\}.$$

If A consists of a single point a we write

$$\delta(A,B) = \delta(a,B),$$

and if B also consists of a single point b we write

$$\delta(A,B) = \delta(a,b) = d(a,b).$$

It follows easily from the definition that

$$\delta(A,B) = \delta(B,A) \geq 0,$$

and

$$\delta(A,B) \leq \delta(A,C) + \delta(C,B)$$

for all A,B,C in B(X).

We say that a multi-valued function F has a fixed point z if z is in Fz (Nadler [6]).

The following result was obtained by Fisher [1].

T h e o r e m A. Let F be a mapping of a complete metric space (X,d) into B(X) satisfying the inequality

$$\delta(Fx,Fy) \leq c.\max\{\delta(x,Fx),\ \delta(y,Fy),\ \delta(x,Fy),\ \delta(y,Fx),\ d(x,y)\}$$

for all x,y in X, where $0 \leq c < 1$. If F also maps B(X) into itself, then F has a unique fixed point z in X, and further $Fz = \{z\}$.

It is noted in Fisher [1] that Theorem A cannot be extended to a pair of multi-valued mappings F and G. However, in the special case when X is bounded, and F,G commute the result does in fact hold (Fisher [3]). In our result the mappings are not necessarily commuting. Also the metric space under consideration need not be bounded.

3. Results

Throught this section (X,d) stands for a complete metric space. Let R_+ be the set of non-negative reals. We denote by Φ the collection of all functions $\varphi: (R_+)^3 \to R_+$ satisfying the following conditions:

(i) φ is non-decreasing in each coordinate;

(ii) $\varphi(a,a,a) \leq a$ for all $a \in R_+$.

Our first result generalizes a recent result due to Fisher [2]. In doing so we are motivated by the functional inequality considered by Kubiaczyk [5].

T h e o r e m 1. Let F and G be mappings of (X,d) into $B(X)$ satisfying the inequality

$$(+) \qquad \delta(Fx,Gy) \leqslant k.\ \varphi(\delta(x,Fx),\ \delta(y,Gy),\ d(x,y))$$

for all x,y in X, where $0 \leqslant k < 1$ and $\varphi \in \Phi$. Then F and G have a common fixed point z. Further, $Fz = Gz = \{z\}$ and z is the unique common fixed point of F and G.

P r o o f . Let x be an arbitrary point in X and choose a point x_1 in $X_1 = Gx$. Then choose a point x_2 in $X_2 = Fx_1$ and so on. In general, choose a point x_{2n-1} in $X_{2n-1} = Gx_{2n-2}$ and a point

$$x_{2n} \text{ in } X_{2n} = Fx_{2n-1} \quad \text{for} \quad n=1,2,3,\dots .$$

Then

$$\delta(X_{2n},X_{2n-1}) = \delta(Fx_{2n-1},Gx_{2n-2})$$

$$\leqslant k\cdot\varphi(\delta(x_{2n-1},Fx_{2n-1}),\ \delta(x_{2n-2},Gx_{2n-2}),\ d(x_{2n-1},x_{2n-2})) \leqslant$$

$$\leqslant k\cdot\varphi(\delta(X_{2n-1},X_{2n}),\ \delta(X_{2n-2},X_{2n-1}),\ \delta(X_{2n-1},X_{2n-2})) \leqslant$$

$$\leqslant k\ \delta(X_{2n-2},X_{2n-1}), \quad \text{if} \quad \delta(X_{2n-1},X_{2n}) \leqslant \delta(X_{2n-2},X_{2n-1}).$$

If $\delta(X_{2n-1},X_{2n}) > \delta(X_{2n-1},X_{2n-2})$ then by (+), (i) and (ii) we obtain a contradiction.

Analogously we can prove that $\delta(X_{2n+1},X_{2n}) \leqslant k\ \delta(X_{2n},X_{2n-1})$. Thus by induction it follows that

$$\delta(X_n,X_{n+r}) \leqslant \delta(X_n,X_{n+1}) + \dots + \delta(X_{n+r-1},X_{n+r}) \leqslant$$

$$\leqslant (k^n + k^{n+1} + \dots + k^{n+r-1})\ \delta(x,Gx) \leqslant \left(\frac{k^n}{1-k}\right)\delta(x,Gx).$$

Now since $k < 1$, for an arbitrary $\varepsilon > 0$ we have

$$d(x_m,x_n) \leqslant \delta(X_m,X_n) < \varepsilon$$

for m,n greater than some N. It follows then that (x_n) is a Cauchy sequence in the complete metric space (X,d) and so has a limit z in X. Further

$$\delta(z,X_n) \leq d(z,x_m) + \delta(x_m,X_n) \leq d(z,x_m) + \delta(X_m,X_n) \leq d(z,x_m) + \varepsilon$$

for $m,n > N$. As m tends to infinity we get

$$\delta(z,X_n) < \varepsilon$$

for $n > N$. Now consider

$$\delta(Fz,X_{2n+1}) \leq \delta(Fz,Gx_{2n}) \leq k.\varphi(\delta(z,Fz),\delta(x_{2n},Gx_{2n}),d(z,x_{2n})) \leq$$

$$\leq k.\varphi(\delta(z,Fz),\delta(X_{2n},X_{2n+1}),\delta(z,X_{2n})) \leq k.\varphi(\delta(z,Fz),\varepsilon,\varepsilon)$$

for $2n > N$. On letting n tend to infinity we have

$$\delta(Fz,z) \leq k.\max\{\delta(z,Fz),\varepsilon\}$$

This yields $\delta(Fz,z) = 0$, which implies that $Fz = \{z\}$.

Similarly, we can prove that $Gz = \{z\}$. So $\{z\}$ is a common fixed point of F and G.

Now suppose that F and G have a second common fixed point w so that w is in Fw and Gw. Then

$$\delta(Fw,Gw) \leq k.\varphi(\delta(w,Fw),\delta(w,Gw),d(w,w)) \leq$$

$$\leq k.\varphi(\delta(Gw,Fw),\delta(Fw,Gw),0).$$

So $\delta(Fw,Gw) = 0$. Hence $Fw = Gw = \{w\}$. Now

$$d(z,w) = \delta(Fz,Gw) \leq k.\varphi(\delta(z,Fz),\delta(w,Gw),d(z,w)) \leq$$

$$\leq k.\varphi(0,0,d(z,w)).$$

As $k < 1$, we get $d(z,w) = 0$. So z is the unique common fixed point of F and G. This completes the proof.

C o r o l l a r y 1. Let S and T be mapping of X into itself satisfying the inequality

$$d(Sx,Ty) \leqslant k.\varphi(d(x,Sx),\ d(y,Ty),d(x,y))$$

for all x,y in X, where $0 \leqslant k < 1$ and $\varphi \in \Phi$. Then S and T have a unique common fixed point.

P r o o f . Define mappings F and G of X into $B(X)$ by putting

$$Fx = \{Sx\}, \quad Gx = \{Tx\}$$

for all x in X. It follows that F and G satisfy the conditions of Theorem 1. So there exists z in X with $Fz = Gz = \{z\}$. Hence z is a common fixed point of S and T.

Now suppose that w is a second fixed point of T. Then

$$d(z,w) = d(Sz,Tw) \leqslant k.\varphi(d(z,Sz),d(w,Tw),d(z,w)) = k.\varphi(0,0,d(z,w)).$$

So $k < 1$ implies that $z = w$. Thus z is the unique fixed point of T. Similarly, z is the unique fixed point of S. This completes the proof.

The result of the following corollary was given by Fisher [2].

C o r o l l a r y 2. Let F and G be mappings of (X,d) into $B(X)$ satisfying either of the following conditions

(A) $$\delta(Fx,Gy) \leqslant k.\max\{\delta(x,Fx),\ \delta(y,Gy),\ d(x,y)\}$$

where $0 \leqslant k < 1$;

(B) $$\delta(Fx,Gy) \leqslant \alpha\delta(x,Fx) + \beta\delta(y,Gy) + \gamma d(x,y)$$

for some $\alpha, \beta, \gamma \geqslant 0$ such that $\alpha+\beta+\gamma < 1$.

Then there exists a unique common fixed point of F and G.

P r o o f . If (A) holds, put $\varphi(a,b,c) = \max\{a,b,c\}$ in Theorem 1. In case of (B), let $k = \alpha+\beta+\gamma$. Then clearly (B) implies (A). So the desired result follows from the previous part.

R e m a r k . The results of this paper are not valid for $k = 1$. Let $X = \{1,2\}$. Let d be the metric of X defined by $d(1,2) = 1$. Let $F = G$ be the mappings on X such that $F(1) = 2$, $F(2) = 1$ and $\varphi(x,y,z) = \frac{1}{3}(x + y + z)$.

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