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UNIFORM ESTIMATES WITHOUT BOUNDS FOR THE MESH-SIZE FOR FINITE-DIFFERENCE SCHEMES, APPROXIMATING HALF-LINEAR ELLIPTIC SYSTEMS

1. Introduction

Using finite-difference methods for elliptic differential equations of second order, involving first derivatives of the unknown function, we get difficulties. They are caused by the finite-difference approximation of the first derivatives. Using central divided differences, we obtain a bound for the mesh-size h (possibly small), but approximation of order two. Using one-sided divided differences, we do not get a bound for h, but only approximation of first order.

Now we propose a finite-difference scheme for half-linear elliptic systems with first derivatives and mixed second derivatives which does not need a bound for h but has an approximation of order two. It is a generalization of the method of Samarskij [3], used for ordinary differential equations (introduction of an artificial Reynolds-number), to the case of partial differential equations with mixed second derivatives. Analogous results have been attained for parabolic type equations.

Other generalizations of the method of Samarskij, especially for degenerated parabolic equations, strongly oscillating coefficients and with regard to an optimal choice of the Reynolds-number, have been carried out by Stoyan [4], [5].

The general treatment of elliptic systems with finite-difference methods in the given paper is essentially taken from Malec [1], [2].

2. The problem

Let $l > 0$ be a real number and $G = \{x \in R^n : -l < x_j < l, j=1(1)n\}$ with the closure $\bar{G}$ and the boundary Γ. By $\partial_i u$ and $\partial_{ij} u = \partial_i(\partial_j u)$ we denote the partial derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ of a function $u(x)$, respectively. Further, let $U_0 = C(\bar{G}) \cap C^2(G)$ and $\bar{U}_0$ be the set of vector-functions $\bar{u} = (u^1,\dots,u^p)$ with $u^m \in U_0$, $m = 1(1)p$. In the following, vectors of R^p, vectors whose p components are functions and classes of such vectors are marked by a bar. Inequalities between vectors hold component-wise. By maps $f^m: G \times R^p \to R$ and the given functions a^m_{ij}, $b^m_j \in C(G)$, $\sup_G |a^m_{ij}| < \infty$, $\sup_G |b^m_j| < \infty$, the operator $\bar{L} = (L^1,\dots,L^p)$ such that

$$L^m \bar{u} \equiv \sum_{i,j=1}^{n} a^m_{ij}(x)\partial_{ij}u^m + \sum_{j=1}^{n} b^m_j(x)\partial_j u^m + f^m(x,\bar{u}), \quad m=1(1)p,$$

will be defined. Then we seek an approximation $\bar{w}$ of a solution $\bar{u} \in \bar{U}_0$ of the problem

$$\text{(1)} \qquad \bar{L}\bar{u} = \bar{0}, \quad x \in G,$$

$$\text{(2)} \qquad \bar{u}(x) = \bar{g}(x), \quad x \in \Gamma,$$

with a given vector $\bar{g} = (g^1,\dots,g^p)$ such that $g^m \in C(\Gamma)$, $m=1(1)p$.

Now we need the following assumptions:

(A_1) $\bar{y} \in R^p$, $\bar{\eta} \in R^p$ with $\eta^j = 0$ for $j \neq m \Longrightarrow$

$$\operatorname{sgn}\eta^m\left[f^m(x,\bar{y}+\bar{\eta}) - f^m(x,\bar{y})\right] \leq -K|\eta^m|, \quad m=1(1)p.$$

(A_2) $\bar{y} \in R^p$, $\bar{\eta} \in R^p$ with $\eta^m = 0 \Longrightarrow$

$$\left|f^m(x,\bar{y}+\bar{\eta})-f^m(x,\bar{y})\right| \leqslant M \sum_{j=1}^{p} \left|\eta^j\right|, \quad m=1(1)p; \; K > (p-1)M.$$

(A_3)
$$a^m_{ij}(x) = a^m_{ji}(x), \quad x \in G.$$

(A_4)
$$\exists \bar{c}_j \geqslant \bar{0}, \; \bar{c}_j \in R^p \Longrightarrow a^m_{jj}(x) \geqslant c^m_j + \sum_{\substack{i=1\\ i\neq j}}^{n} \left|a^m_{ij}(x)\right|,$$

$$x \in G; \; j=1(1)n; \; m=1(1)p.$$

(A_5)
$$b^m_j \not\equiv 0 \Longrightarrow c^m_j > 0.$$

(A_6) The problem (1), (2) has one and only one solution.

3. Discretization

Let $N \geqslant 2$ be an integer and $h > 0$ a real number such that $hN = 1$ holds and denote

$$\bar{X} = \left\{x : x_j = m_j h; \; m_j = -N(1)N; \; j = 1(1)n\right\},$$
$$X = \left\{x : x_j = m_j h; \; m_j = (-N+1)(1)(N-1); \; j=1(1)n\right\}, \; Y=\bar{X}\setminus X.$$

Let W be the class of functions $w : \bar{X} \longrightarrow R$ defined and bounded on X and $\bar{W}$ the set of vector-functions $\bar{w} = (w^1,\dots,w^p)$ with $w^m \in W$. With $e_j = (\delta_{1j},\dots,\delta_{nj})$ we define the divided differences for $w^m \in W$:

$$D^+_j w^m(x) = \left(w^m(x+e_jh) - w^m(x)\right)\cdot h^{-1}, \quad D^-_j w^m(x) = D^+_j w^m(x-e_jh),$$

$$D_j w^m(x) = \begin{cases} D^+_j w^m(x), & \text{if} \quad b^m_j(x) \geqslant 0,\\ D^-_j w^m(x), & \text{if} \quad b^m_j(x) < 0,\end{cases}$$

$$D^-_{ij} w^m = \frac{1}{2}\,(D^-_i D^+_j + D^+_i D^-_j)w^m, \quad D^+_{ij} w^m = \frac{1}{2}\,(D^+_i D^+_j + D^-_i D^-_j)w^m,$$

$$D_{ij}w^m(x) = \begin{cases} D^+_{ij}w^m(x), & \text{if } i \neq j \text{ and } a^m_{ij}(x) \geq 0, \\ D^-_{ij}w^m(x), & \text{if } i = j \text{ or } a^m_{ij}(x) \leq 0. \end{cases}$$

Further, let

$$\alpha^m_j(x) = \begin{cases} 0, & \text{if } b^m_j \equiv 0, \\ \dfrac{\left|b^m_j(x)\right|^2}{16c^m_j\left(a^m_{jj}(x) + \frac{h}{2}\left|b^m_j(x)\right|\right)}, & \text{if } b^m_j \not\equiv 0, \end{cases}$$

$$R^m_{ij}(x) = \begin{cases} \dfrac{h\left|b^m_j(x)\right|}{2a^m_{jj}(x)}, & \text{if } i = j, \\ h^2 \max\left[\alpha_i(x), \alpha_j(x)\right], & \text{if } i \neq j, \end{cases}$$

$$\nu^m_{ij} = \left(1 + R^m_{ij}\right)^{-1},$$

$$L^m_D\bar{w} \equiv \sum_{i,j=1}^{n} a^m_{ij}(x)\nu^m_{ij}(x)D_{ij}w^m(x) + \sum_{j=1}^{n} b^m_j(x)D_jw^m(x) + f^m(x,\bar{w}).$$

The following finite-difference problem will be investigated

(3) $$\bar{L}_D\bar{w} = \bar{0}, \quad x \in X,$$

(4) $$\bar{w}(x) = \bar{g}(x), \quad x \in Y.$$

4. Approximation

In the following, let $\bar{u} \in \bar{U}_0$ be the solution of (1), (2), and $\bar{v} \in \bar{U}_0$ an arbitrary function. Further, let $U_1 = U_0 \cap C^3(G)$, $U_2 = U_1 \cap C^4(G)$ and U^*_0 be the class of functions $v \in U_0$ with

$\sup_G |\partial_{ij}v| < \infty$, where every $\partial_{ij}v$ can be continuously extended on $\bar{G}$, and

$$U_1^* = \left\{ v \in U_1 \; : \; \sup_G \; |\partial_{ijk}v| < \infty \; ; \; i,j,k = 1(1)n \right\},$$

$$U_2^* = \left\{ v \in U_2 \; : \; \sup_G \; |\partial_{ijkl}v| < \infty \; ; \; i,j,k,l = 1(1)n \right\}.$$

The point-wise approximation error will be defined by

$$\bar{\zeta}(x,\bar{v}) = \left| \bar{L}_D \bar{v} - \bar{L} \, \bar{v} \right|, \quad x \in X,$$

and the uniform approximation error by

$$\zeta(\bar{v}) = \sup \left\{ \zeta^m(x,\bar{v}) \; : \; x \in X, \; m = 1(1)p \right\}.$$

Obviously, $\zeta(\bar{v})$ exists for every $\bar{v} \in \bar{U}_k^*$ with $k = 0,1,2$.

T h e o r e m 1. For $h \to 0$ it holds

(i) $\bar{\zeta}(x,\bar{v}) \to \bar{0}$, if $\bar{v} \in \bar{U}_0$,

(ii) $\bar{\zeta}(x,\bar{v}) = O(h^k)$, if $\bar{v} \in \bar{U}_k$, $k = 1,2$,

(iii) $\zeta(\bar{v}) \to 0$, if $\bar{v} \in \bar{U}_0$,

(iv) $\zeta(\bar{v}) = O(h^k)$, if $\bar{v} \in \bar{U}_k$, $k = 1,2$.

P r o o f . At first a partition of $\bar{\zeta}(x,\bar{v})$ into three parts takes place

$$\bar{\zeta} = \bar{\zeta}_1 + \bar{\zeta}_2 + \bar{\zeta}_3 \quad \text{with} \quad \bar{\zeta}_1 = \sum_{j=1}^{n} \bar{\zeta}_{1j} \quad \text{and}$$

$$\zeta_{1j}^m = b_j^m D_j v^m - b_j^m \partial_j v^m + a_{jj}^m (\nu_{jj}^m - 1) D_{jj} v^m,$$

$$\zeta_2^m = \sum_{i,j=1}^{n} a_{ij}^m (D_{ij} v^m - \partial_{ij} v^m),$$

$$\zeta_3^m = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij}^m (\nu_{ij}^m - 1) D_{ij} v^m, \quad m = 1(1)p.$$

Using Taylor's formula with the remainder in an arbitrary form and regarding the case $\bar{v} \in \bar{U}_2$, we obtain $\zeta_2^m = O(h^2)$ and $\zeta_3^m = O(h^2)$ because of $\nu_{ij}^m - 1 = -R_{ij}^m(1 + R_{ij}^m)^{-1} = O(h^2)$, $i \neq j$. If b_j^{m+} and b_j^{m-} are the positive and negative part of b_j^m, respectively, we get

$$\zeta_{1j}^m = b_j^{m+}D_j^+v^m + b_j^{m-}D_j^-v^m - b_j^m\partial_j v^m + a_{jj}^m(\nu_{jj}^m - 1)D_{jj}v^m =$$

$$= \frac{h^2}{4}\sum_{j=1}^{n} \frac{|b_j^m|^2}{a_{jj}^m(1+R_{jj}^m)}\partial_{jj}v^m + O(h^2) = O(h^2).$$

Thus, (ii) is proved for $k = 2$. Assertion (iv) for $k = 2$ follows immediately from the properties of $\bar{U}_2^*$. Analogously, the case $k = 1$ is to treat. For proving (i) and (iii), we use Taylor's formula with the remainder in integral form and, for this integral, the integral main value theorem. Then (i) follows from the continuity of $\partial_{ij}v^m$ on G and (iii) from the properties of $\bar{U}_0^*$.

5. Estimates

At first we need some lemmas.

Lemma 1. With the assumptions (A_4), (A_5) we have

$$\nu_{jj}^m a_{jj}^m - \sum_{\substack{i=1\\ i\neq j}}^{n} \nu_{ij}^m |a_{ij}^m| \geq 0, \quad x \in X, \; j = 1(1)n.$$

Proof:

$$\nu_{jj}^m a_{jj}^m - \sum_{\substack{i=1\\ i\neq j}}^{n} \nu_{ij}^m |a_{ij}^m| = a_{jj}^m(1+R_{jj}^m)^{-1} - (a_{jj}^m - c_j^m)(1+h^2\alpha_j^m)^{-1} =$$

$$= \left[h^2 a_{jj}^m \frac{|b_j^m|^2}{16c_j^m(a_{jj}^m + \frac{h}{2}|b_j^m|)} - \frac{h}{2}|b_j^m| + \frac{h}{2}\frac{|b_j^m|c_j^m}{a_{jj}^m}\right] \times$$

$$\times\left(1+R_{jj}^{m}\right)^{-1}\left(1+h^{2}\alpha_{j}^{m}\right)^{-1} \geqslant \left[\frac{h\left|b_{j}^{m}\right|}{4}\left(\frac{a_{jj}^{m}}{c_{j}^{m}\left(a_{jj}^{m}+\frac{h}{2}\left|b_{j}^{m}\right|\right)}\right)^{\frac{1}{2}} -\right.$$

$$\left. - \left(\frac{c_{j}^{m}\left(a_{jj}^{m}+\frac{h}{2}\left|b_{j}^{m}\right|\right)}{a_{jj}^{m}}\right)^{\frac{1}{2}}\right]^{2} \left(1+R_{jj}^{m}\right)^{-1}\left(1+h^{2}\alpha_{j}^{m}\right)^{-1} \geqslant 0.$$

Similarly as in [2], we can estimate the expression

$$Q_{D}^{m}z = \sum_{i,j=1}^{n} \nu_{ij}^{m}a_{ij}^{m}D_{ij}z + \sum_{j=1}^{n} b_{j}^{m}D_{j}z \quad \text{for} \quad z \in W.$$

L e m m a 2. Let the assumptions (A_3) - (A_5) be fulfilled. For $z(x_0) = \max\{z(x):\ x \in \bar{X}\}$, $x_0 \in X$, we get

$$Q_{D}^{m}z(x) \leqslant \frac{2}{h^{2}}\left[z(x_0)-z(x)\right] \sum_{j=1}^{n}\left(\nu_{jj}^{m}a_{jj}^{m} + \frac{h}{2}\left|b_{j}^{m}\right|\right).$$

P r o o f :

$$Q_{D}^{m}z(x) = \sum_{j=1}^{n}\left(b_{j}^{m+}D_{j}^{+}z + b_{j}^{m-}D_{j}^{-}z\right) + \sum_{j=1}^{n}\nu_{jj}^{m}a_{jj}^{m}D_{jj}z + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\nu_{ij}^{m}a_{ij}^{m}D_{ij}z =$$

$$= \frac{1}{h^{2}}\sum_{j=1}^{n}\left[z(x+e_{j}h)-z(x_0)\right]\left(\nu_{jj}^{m}a_{jj}^{m}+hb_{j}^{m+} - \sum_{\substack{i=1\\ i\neq j}}^{n}\nu_{ij}^{m}\left|a_{ij}^{m}\right|\right) +$$

$$+ \frac{1}{h^{2}}\sum_{j=1}^{n}\left[z(x-e_{j}h)-z(x_0)\right]\left(\nu_{jj}^{m}a_{jj}^{m}-hb_{j}^{m-} - \sum_{\substack{i=1\\ i\neq j}}^{n}\nu_{ij}^{m}\left|a_{ij}^{m}\right|\right) +$$

$$+ \frac{1}{h^{2}}\left[z(x)-z(x_0)\right]\sum_{\substack{i,j=1\\ i\neq j}}^{n}\nu_{ij}^{m}\left|a_{ij}^{m}\right| -$$

$$- \frac{1}{h^2} \left[z(x)-z(x_0)\right] \sum_{j=1}^{n} \left(2\vartheta_{jj}^m a_{jj}^m + h\left|b_j^m\right|\right) +$$

$$+ \frac{1}{2h^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \vartheta_{ij}^m \left|a_{ij}^m\right| \left[z(x+e_i h + \operatorname{sgn}(a_{ij}^m)e_j h) +\right.$$

$$\left.+ z(x-e_i h - \operatorname{sgn}(a_{ij}^m)e_j h) - 2z(x_0)\right].$$

Lemma 1 and the inequalities $b_j^{m+} \geq 0$, $b_j^{m-} \leq 0$ imply the assertion. Now the estimation theorem can be proved.

T h e o r e m 2. If

(i) $\overline{w}, \overline{y} \in \overline{W}$,

(ii) the assumptions (A_1) - (A_5) hold,

(iii) $\left|L_D^m \overline{w} - L_D^m \overline{y}\right| \leq \delta = \text{const}$, $x \in X$,
$\left|w^m(x) - y^m(x)\right| \leq \delta\left[K - (p-1)M\right]^{-1}$, $x \in Y$, $m = 1(1)p$,

then

(5) $$\left|w^m(x) - y^m(x)\right| \leq \delta\left[K - (p-1)M\right]^{-1}, \quad x \in \overline{X},\ m = 1(1)p.$$

P r o o f . Suppose that the inequality (5) is not fulfilled for certain $m \in \{1,\dots,p\}$ and certain $x \in \overline{X}$. Let

$$\left|w^j(x_0) - y^j(x_0)\right| = \max\left\{\left|w^m(x) - y^m(x)\right| : x \in \overline{X},\ m = 1(1)p\right\}.$$

Then we have

(6) $$\left|w^j(x_0) - y^j(x_0)\right| > \delta\left[K - (p-1)M\right]^{-1}.$$

For $x_0 \in Y$ the inequality (6) contradicts to assumption (iii). Therefore, let $x_0 \in X$ and $\left|w^j(x_0)-y^j(x_0)\right| = w^j(x_0)-y^j(x_0) = z(x_0)$ (i.e. $z = w^j - y^j$). Application of Lemma 2 provides

$$-\delta \leqslant L_D^j\overline{w} - L_D^j\overline{y} \leqslant Q_D^j z(x_0) + f^j(x_0,\overline{w}) - f^j(x_0,\overline{y}) \leqslant$$

$$\leqslant -Kz(x_0) + M \sum_{\substack{m=1 \\ m\neq j}}^{p} \left|w^m(x_0) - y^m(x_0)\right| \leqslant z(x_0)\left[M(p-1)-K\right].$$

Taking into account (6), we get a contradiction. The case $\left|w^j-y^j\right| = y^j-w^j$ is analogous. Hence, the assertion is proved.

6. Uniqueness, stability and convergence

Theorem 3. By the assumptions (A_1) - (A_5), the problem (3), (4) has at most one solution $\overline{w} \in \overline{W}$, and is stable in relation to this solution.

Proof. Suppose that $\overline{y} \in \overline{W}$ is a second solution of (3), (4). Then Theorem 2 holds with $\delta = 0$ and from (5) uniqueness follows immediately. Let $\varepsilon > 0$ be an arbitrary real number and let us put $\delta(\varepsilon) = \varepsilon\left[K - (p-1)M\right]$. An arbitrary function $\overline{y} \in \overline{W}$ now is assumed to be a solution of the inequalities

$$\left|L_D^m\overline{y}\right| \leqslant \delta(\varepsilon),\ x \in X,\quad \left|g^m(x)-y^m(x)\right| \leqslant \delta(\varepsilon)\left[K-(p-1)M\right]^{-1},\quad x \in Y.$$

Then, by Theorem 2, the inequality $\left|w^m-y^m\right| \leqslant \varepsilon$, $m = 1(1)p$, holds, i.e. the problem (3), (4) is stable in relation to the solution $\overline{w} \in \overline{W}$.

A function $\overline{w} \in \overline{W}$, defined on the mesh $\overline{X} = \overline{X}(N)$, will be denoted by $\overline{w}_N$.

Theorem 4. If

(i) for every N there exists at least one solution $\overline{w}_N \in \overline{W}$ of (3), (4),

(ii) the assumptions (A_1) - (A_6) hold,

(iii) $\overline{u} \in \overline{U}_k^*$ is the solution of (1), (2), $k = 0,1,2$,

then

$$(7)\qquad \lim_{N\to\infty} \max\left\{\left|w_N^m(x) - u^m(x)\right| : x \in \overline{X}(N),\ m = 1(1)p\right\} = 0,$$

(8) the speed of convergence equals to the speed of approximation.

P r o o f . Because of $\bar{L}_D\bar{w} = \bar{0}$ we have $\left|\bar{L}_D\bar{w}-\bar{L}_D\bar{u}\right| = \bar{\zeta}(x,\bar{u})$ and $\left|L_D^m\bar{w}-L_D^m\bar{u}\right| \leq \zeta(\bar{u})$, as well as $\bar{u}(x)-\bar{w}(x) = \bar{0}$ for $x \in Y(N)$. Applying Theorem 2 with $\delta = \zeta(\bar{u})$, we obtain the estimate

$$\left|w_N^m(x)-u^m(x)\right| \leq \zeta(\bar{u})\left[K - (p-1)M\right]^{-1} \quad \text{for} \quad x \in X(N) \text{ and } m=1(1)p.$$

By this and Theorem 1, the assertions (7), (8) are proved.

7. Existence and iteration method

By $\bar{Z}$ we denote the set of mesh-functions $\bar{y}$ defined on $\bar{X}$ with $\bar{y}(x) = \bar{g}(x)$ for $x \in Y$, and with the norm

$$\|\bar{y}\| = \max\left\{|y^m(x)| : x \in \bar{X},\ m=1(1)p\right\}.$$

Let $\bar{B} : \bar{Z} \to \bar{Z}$ be a map such that $\bar{y} = \bar{B}(\bar{z})$ is determined for $\bar{y},\bar{z} \in \bar{Z}$ by the $m(2N+1)^n$ equalities

$$y^m(x) = z^m(x) + \lambda L_D^m\bar{z}, \quad x \in X,$$

$$y^m(x) = g^m(x), \quad x \in Y.$$

By $\lambda > 0$ we denote a real parameter. Let us extend the assumption (A_1) to the following one

(A_1') $\bar{y} \in R^p$, $\bar{\eta} \in R^p$ with $\eta^j = 0$ for $j \neq m \Longrightarrow$

$$-K_1\left|\eta^m\right| \leq \operatorname{sgn} \eta^m\left[f^m(x,\bar{y}+\bar{\eta})-f^m(x,\bar{y})\right] \leq -K\left|\eta^m\right|, \ m=1(1)p.$$

L e m m a 3. If

(i) the assumptions (A_1'), (A_2) - (A_5) hold,

(ii) $\lambda\left[K_1 + \frac{2}{h^2}\sum_{j=1}^{n} \sup_{m,x} (\nu_{jj}^m a_{jj}^m + \frac{h}{2}\left|b_j^m\right|)\right] \leq 1,$

then the map $\bar{B}$ is contracting in $\bar{Z}$ with the contraction parameter $c = 1 - \lambda\left[K - (p-1)M\right] \in \langle 0,1)$.

P r o o f . Let $\bar{q}, \bar{r}, \bar{y}, \bar{w} \in \bar{Z}$ and $\bar{q} = \bar{B}(\bar{y})$, $\bar{r} = \bar{B}(\bar{w})$. Then it follows from Lemma 2, for $m \in \{1,\dots,p\}$ and $x \in X$,

$$q^m(x) - r^m(x) \leqslant$$

$$\leqslant \left(y^m(x)-w^m(x)\right) \left[1 + \frac{f^m(x,\bar{y})-f^m(x,y^1,\dots,y^{m-1},w^m,y^{m+1},\dots,y^p)}{y^m(x) - w^m(x)} - \right.$$

$$\left. - \frac{2\lambda}{h^2} \sum_{j\neq 1}^{n} (\nu^m_{jj}a^m_{jj} + \frac{h}{2}\left|b^m_j\right|)\right] + \|\bar{y}-\bar{w}\| \frac{2\lambda}{h^2} \sum_{j=1}^{n} (\nu^m_{jj}a^m_{jj} + \frac{h}{2}\left|b^m_j\right|) +$$

$$+ \lambda M \sum_{\substack{j=1\\ j\neq m}}^{p} \left|y^j(x) - w^j(x)\right|$$

in the case $y^m(x) \neq w^m(x)$. The case $y^m(x) = w^m(x)$ is simple. By the assumptions (A_1'), (ii), the expression in the square bracket is nonnegative. By (A_1'), we get the estimate

$$q^m(x) - r^m(x) \leqslant [1 - \lambda(K - (p-1)M)] \|\bar{y}-\bar{w}\|$$

and an analogous one, if q^m and r^m are exchanged. Thus, the inequality $\|\bar{q}-\bar{r}\| \leqslant c\|\bar{y}-\bar{w}\|$ is proved. The inequality $0 \leqslant c < 1$ follows from the assumption (ii) because of $1 - \lambda K + \lambda(p-1)M \geqslant \geqslant 1 - \lambda K_1 \geqslant 0$ and $K > (p-1)M$.

From Lemma 3 it follows immediately the theorem.

T h e o r e m 5. Let the assumptions of Lemma 3 be fulfilled. Then for the problem (3), (4) one and only one solution $\bar{w} \in \bar{W}$ exists which can be computed with the convergent iteration method

$$\bar{y}^{(k+1)} = \bar{B}(\bar{y}^{(k)}),\ \bar{y}^{(k+1)}(x) = \bar{g}(x), \quad x \in Y, \quad k = 0,1,2,\dots,$$

where $\bar{y}^{(0)} \in \bar{Z}$ is arbitrary.

The proof is based on the fixed point theorem of Banach.

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