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OVER AN ORTHOGONAL SYSTEM OF QUASIGROUPS

This paper deals with orthogonal systems of quasigroups and identities over them. V.J.Dementjeva in [3] dealt with closure conditions in k-nets, which she reduced to an investigation of universal identities. We start from 3-basic quasigroups and show a connection between universal identities and identities over an orthogonal system of 3-basic quasigroups.

1. 3-basic quasigroups

The quadruple $(Q_1, Q_2, Q_3; A_{12}^3)$, where $\text{card } Q_1 = \text{card } Q_2 = \text{card } Q_3 \neq 0$ and A_{12}^3 is a map of $Q_1 \times Q_2$ onto Q_3 , is called a *3-basic quasigroup* if, in the equation $A_{12}^3(a_1, a_2) = a_3$, any two of the elements $a_1 \in Q_1$, $a_2 \in Q_2$, $a_3 \in Q_3$ uniquely determine the remaining one. Similarly as in the classical case we define *parastrophic operations*

$$(1) \quad A_{ij}^m(a_i, a_j) = a_m \Leftrightarrow A_{12}^3(a_1, a_2) = a_3$$

for all permutations (i, j, m) of the set $\{1, 2, 3\}$.

If $Q_1 = Q_2 = Q_3 = Q$ we get usual *quasigroup*. For example we have two equivalent denotations for a quasigroup $(Q; \cdot)$ or $((Q; A_{12}^3))$.

Key words: 3-basic quasigroup, k-net, orthogonal system of 3-basic quasigroups, isotopy, universal identity, position of a term.

A M S: 20 N05, 51A20

In detail:

$$\begin{aligned} x \cdot y = z &\Leftrightarrow A_{12}^3(x, y) = z, & y^S(\cdot)x = z &\Leftrightarrow A_{21}^3(y, x) = z, \\ x \setminus z = y &\Leftrightarrow A_{13}^2(x, z) = y, & z^S(\setminus)x = y &\Leftrightarrow A_{31}^2(z, x) = y, \\ z / y = x &\Leftrightarrow A_{32}^1(z, y) = x, & y^S(/)z = x &\Leftrightarrow A_{23}^1(y, z) = x. \end{aligned}$$

It is clear that $A_{ij}^m(x, y) = A_{ji}^m(y, x)$ for all (i, j, m) .

2. k-nets and orthogonal systems of quasigroups

A system \mathcal{N} of objects of two sorts, called points and lines, with an incidence relation between them is a k -net iff

- (i) the set of all lines is partitioned into k disjoint classes $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_k$, $k \geq 3$,
- (ii) two lines from different classes are incident to exactly one point,
- (iii) in every line class there is exactly one line incident to a given point,
- (iv) there are at least two points incident to every line.

A k -net is of order n if every line class \mathcal{L}_i has exactly n lines. In a k -net of order n there are exactly n^2 points and kn lines. If Q_i , $1 \leq i \leq k$, are sets such that $\text{card } Q_i = n$, then there exist bijections between Q_i and \mathcal{L}_i and maps $A_{ij}^m: Q_i \times Q_j \rightarrow Q_m$ such that every quadruple $(Q_i, Q_j, Q_m; A_{ij}^m)$ forms a 3-basic quasigroup for all $i \neq j \neq m \neq i$ and $i, j, m \in \{1, 2, \dots, k\}$. The geometric interpretation of the operations A_{ij}^m is the following: $A_{ij}^m(x, y)$ is the unique line of \mathcal{L}_m incident to the unique point incident to the line x of \mathcal{L}_i and the line y of \mathcal{L}_j . The set

$$\mathcal{A} = \{(Q_i, Q_j, Q_m; A_{ij}^m); i \neq j \neq m \neq i; i, j, m \in \{1, \dots, k\}\}$$

forms a so-called *orthogonal system of 3-basic quasigroups* considering that every system

$$(2) \quad A_{ij}^p(x, y) = a, \quad A_{ij}^q(x, y) = b, \quad p \neq q,$$

has a unique solution for all $a \in Q_p$, $b \in Q_q$. This solution is

$x=A_{pq}^i(a,b)$, $y=A_{pq}^j(a,b)$. The system \mathcal{A} is also called a coordinate system of the k -net.

If $Q_1=Q_2=\dots=Q_k=Q$ and if we take the operations A_{12}^m , $m \geq 3$, only then we get an orthogonal system of quasigroups Σ in the meaning of V.D.Belousov ([1]). If we take all A_{ij}^m on Q , we obtain a covering system $\bar{\Sigma}$ ([1]).

Immediately we see that

$$(3) \quad A_{ij}^m(x, A_{ip}^j(x, y)) = A_{ji}^m(A_{ip}^j(x, y), x) = A_{ip}^m(x, y)$$

for all $x \in Q_i$, $y \in Q_p$.

3. Isotopies

Let a line class \mathcal{L}_i of the k -net \mathcal{N} be labelled by elements of the set Q_i . If we take any set Q'_i , with $\text{card } Q'_i = \text{card } Q_i$ and a bijective map $\alpha_i: Q_i \rightarrow Q'_i$, we obtain a new labelling of \mathcal{L}_i by elements of Q'_i , $i=1, 2, \dots, k$, and new quasigroup operations B_{ij}^m defined on $Q'_i \times Q'_j$ such that from the incidence it follows that

$$A_{ij}^m(x, y) = z \iff B_{ij}^m(\alpha_i x, \alpha_j y) = \alpha_m z \quad \text{and}$$

$$(4) \quad A_{ij}^m(x, y) = \alpha_m^{-1} B_{ij}^m(\alpha_i x, \alpha_j y) \quad \text{for all } x \in Q_i, y \in Q_j.$$

The system

$$\mathcal{B} = \{(Q'_i, Q'_j, Q'_m; B_{ij}^m); i \neq j \neq m \neq i; i, j, m \in \{1, \dots, k\}\}$$

is a new coordinate system of \mathcal{N} . The k -tuple $(\alpha_1, \dots, \alpha_k) := \bar{\alpha}$ of bijective maps is called an isotopy of the coordinate systems \mathcal{A} and \mathcal{B} . Thus we have

Lemma 1. Isotopic orthogonal systems of 3-basic quasigroups are associated with the same k -net \mathcal{N} and correspond to changes in the labelling of lines in the line classes of \mathcal{N} .

Every change of numbering of the line classes of \mathcal{N} corresponds to a permutation ρ of the numbers $(1, 2, \dots, k)$.

An isotopy $\bar{\alpha} = (1, 1, \alpha_3, \dots, \alpha_k)$ is a torsion by [1] -it changes the labellings of the lines in the line classes $\mathcal{L}_3, \mathcal{L}_4, \dots, \mathcal{L}_k$, whereas an isotopy $\bar{\alpha} = (\alpha_1, \alpha_2, 1, \dots, 1)$ is a conjugation changing the labellings of the lines in $\mathcal{L}_1, \mathcal{L}_2$, i.e. changing the coordinates of the points.

4. Identities over \mathcal{A}

It is clear that an arbitrary term over a system \mathcal{A} of orthogonal 3-basic quasigroups need not make sense. We must form terms carefully, respecting the places belonging to the various Q_i 's.

In what follows let x_i, y_i, z_i, \dots be variables over Q_i , $i=1, 2, \dots, k$ and let t^i, t_n^i be terms over \mathcal{A} , $i=1, \dots, k$; $n=1, 2, \dots$. It is easy to show that only terms of the following forms are meaningful:

$$A_{ij}^p(x_i, x_j), A_{pq}^m(x_p, A_{sr}^q(y_s, y_r)), A_{pq}^m(A_{ij}^p(x_i, x_j), y_q),$$

$$A_{pq}^m(A_{ij}^p(x_i, x_j), A_{sr}^q(y_s, y_r)), \text{ and so on.}$$

If $A_{ij}^p(x_i, x_j) = x_p$ for all triples (i, j, p) , then

$$A_{pq}^m(A_{ij}^p(x_i, x_j), A_{sr}^q(x_s, x_r)) = A_{pq}^m(x_p, x_q) = x_m.$$

Now an identity w over \mathcal{A} consists of the following subterms:

$$(5) \quad \begin{cases} A_{pq}^m(t_1^p, t_1^q), A_{pq}^m(t_1^p, A_{sr}^q(t_2^s, t_2^r)), A_{pq}^m(A_{ij}^p(t_1^i, t_1^j), t_2^q), \\ A_{pq}^m(A_{ij}^p(t_1^i, t_1^j), A_{sr}^q(t_2^s, t_2^r)), \dots, \end{cases}$$

where the length of the subterms t_n^i , $1 \leq i \leq k$, $n=1, 2$, can be ≥ 1 (written $l(t_n^i) \geq 1$). Note $l(t_n^i) = 1 \Leftrightarrow t_n^i = x_i$ for some $x_i \in Q_i$ and

$$l(t_n^i) \geq 2 \Leftrightarrow t_n^i = A_{gh}^i(t_{n+1}^g, t_{n+1}^h).$$

It is clear that for all triples (i, j, m) , $i * j * m = i$, the following trivial identities are true:

$$(6) \quad \begin{cases} A_{ij}^m(x, A_{im}^j(x, y)) = y, \\ A_{ij}^p(x, A_{im}^j(x, y)) = A_{im}^p(x, y), \\ A_{pj}^q(A_{im}^p(x, y), A_{im}^j(x, y)) = A_{im}^q(x, y) \text{ for all } x \in Q_i, y \in Q_m. \end{cases}$$

Similarly for subterms $t^i \in Q_i$ and $t^m \in Q_m$:

$$(7) \quad \begin{cases} A_{ij}^m(t^i, A_{im}^j(t^i, t^m)) = t^m, \\ A_{ij}^p(t^i, A_{im}^j(t^i, t^m)) = A_{im}^p(t^i, t^m), \\ A_{pj}^q(A_{im}^p(t^i, t^m), A_{im}^j(t^i, t^m)) = A_{im}^q(t^i, t^m). \end{cases}$$

If an identity w contains at least one the left terms from (7), then it can be replaced by the corresponding right term from (7) and we obtain a new identity w' equivalent to w , written $w \Leftrightarrow w'$, with $l(w') \leq l(w) - 1$. In this case the identity w is called *reducible*.

Lemma 2. Let w be a non-reducible and non-trivial identity over \mathcal{A} . Then w contains three different variables at least.

Proof. It is easy to verify that every identity w containing $x_i \neq x_j$ only necessarily has among its subterms some term from the left sides of (7) and thus w is reducible. After reductions we obtain a trivial identity w' , $w' \Leftrightarrow w$.

Lemma 3. The minimal length of any non-reducible and non-trivial identity w over \mathcal{A} is six.

Proof. By Lemma 2 every non-reducible and non-trivial identity w contains three different variables at least and every variable occurs two times at least, because any identity has no isolated variables.

Let Σ be an orthogonal system of usual quasigroups. An identity w over Σ is said to be *universal* if the validity of w over every Σ' isotopic to Σ follows from the validity of w over Σ . If we transfer this definition to an orthogonal system of 3-basic quasigroups in the natural way, then we can conclude:

Theorem 1. Any identity w over an orthogonal system of 3-basic quasigroups is universal.

Proof. Let t^m be some subterm of w of the type

$$t^m = A_{pq}^m(A_{ij}^p(t_1^i, t_1^j), A_{sr}^q(t_2^s, t_2^r)).$$

Then under an isotopy $\bar{\alpha}$ we have

$$\alpha_m t^m = A_{pq}^m(A_{ij}^p(\alpha_i t_1^i, \alpha_j t_1^j), A_{sr}^q(\alpha_s t_2^s, \alpha_r t_2^r)), \text{ where if}$$

$t_1^i = A_{gh}^i(t_3^g, t_3^h)$, then under $\bar{\alpha}$ we get $\alpha_i t_1^i = A_{gh}^i(\alpha_g t_3^g, \alpha_h t_3^h)$ and

so on. In the end we obtain for some n subterm $t_n^i = A_{df}^i(x_d, x_f)$ for which $\alpha_i t_n^i = A_{df}^i(\alpha_d x_d, \alpha_f x_f)$. Thus from the identity w we obtain under $\bar{\alpha}$ the identity $\bar{\alpha}(w)$ which is of the same form as w , where instead of individual variables x_i, y_j, \dots there occur variables $\alpha_i x_i, \alpha_j y_j, \dots$. Thus w is universal.

Now let \mathcal{A} be an orthogonal system of 3-basic quasigroups $(Q_i, Q_j, Q_m; A_{ij}^m)$, $i, j, m \in \{1, \dots, k\}$ and let \mathcal{B} be an orthogonal system of usual quasigroups (Q, B_{ij}^m) with $\text{card } Q = \text{card } Q_i$, $i \in \{1, \dots, k\}$. We shall investigate the connection between corresponding identities over \mathcal{A} and over \mathcal{B} .

Theorem 2. Let w_A be an identity over \mathcal{A} and let w_B arise from w_A by substitutions between subterms

$$(8) \quad \begin{cases} t_{1A} = A_{ij}^m(x, y) := B_{ij}^m(x, y) = t_{1B}, \\ t_{3A} = A_{ij}^m(t_{1A}, t_{2A}) := B_{ij}^m(t_{1B}, t_{2B}) = t_{3B}, \end{cases}$$

where at least one of t_{1A}, t_{2A} has length > 1 . Then w_B is a universal identity over \mathcal{B} .

Proof. The identity w_B as well as w_A consists of subterms of the types (5) only. Then similarly as in Theorem 1, from w_B under any isotopy $\bar{\beta} = (\beta_1, \dots, \beta_k)$ we obtain the identity $\bar{\beta}(w_B)$ which is of the same form as w_B . Thus w_B is a universal identity.

Corollary. Conversely, let w_B be an identity over \mathcal{B} . Then w_B is universal if the corresponding expression w_A arising from w_B by (8), is an identity over \mathcal{A} , i.e. w_A has a meaning over \mathcal{A} .

5. Positions of terms

The question as to when an identity w over an orthogonal system \mathcal{B} of usual quasigroups is universal can be investigated independently of multi-basic systems. For this we shall define positions of subterms.

Let t_1, t_2, t_3 be terms over \mathcal{B} and let

$$B_{ij}^m(t_1, t_2) = t_3.$$

Then we say that terms t_1 and t_2 are in *inner positions* i and j with respect to B_{ij}^m , written:

$$\widehat{\text{pos}}_{ij}^m(t_1) = i, \quad \widehat{\text{pos}}_{ij}^m(t_2) = j,$$

and the term t_3 is in an *outer position* m with respect to B_{ij}^m , written: $\widetilde{\text{pos}}_{ij}^m(t_3) = \widetilde{\text{pos}}(B_{ij}^m(t_1, t_2)) = m$.

Illustration: Let

$$B_{13}^2(B_{24}^1(x, y), z) = B_{31}^2(y, B_{23}^4(x, z)).$$

Then we have

$$\widehat{\text{pos}}_{24}^1(x) = 2 = \widehat{\text{pos}}_{23}^4(x), \quad \widehat{\text{pos}}_{24}^1(y) = 4 \neq \widehat{\text{pos}}_{31}^2(y) = 3,$$

$$\widehat{\text{pos}}_{13}^2(z) = 3 = \widehat{\text{pos}}_{23}^4(z),$$

$$\widehat{\text{pos}}_{13}^2(B_{24}^1(x, y)) = 1 = \widetilde{\text{pos}}(B_{24}^1(x, y)),$$

$$\widehat{\text{pos}}_{31}^2(B_{23}^4(x, z)) = 1 \neq \widetilde{\text{pos}}(B_{23}^4(x, z)) = 4,$$

$$\widetilde{\text{pos}}(B_{13}^2(B_{24}^1(x, y), z)) = 2 = \widetilde{\text{pos}}(B_{31}^2(y, B_{23}^4(x, z))).$$

Position condition (P): Every subterm of a given identity occurs always in the same position, independently of whether this position is inner or outer.

Theorem 3. If an identity w over \mathcal{B} satisfies (P), the w is universal.

Proof. If w satisfies (P), then w consists of subterms of the types (5) and w makes sense over an orthogonal system of 3-basic quasigroups \mathcal{A} with $\text{card } Q_i = \text{card } Q$ for all $i \in \{1, \dots, k\}$. By Theorem 2 w is universal.

Theorem 4. Let w be an identity over \mathcal{B} satisfying (P). Then $\delta(w)$ is universal for all permutations δ of $\{1, \dots, k\}$, where $\delta(w)$ denotes the identity arising from w by replacement of B_{ij}^m by $B_{\delta i, \delta j}^{\delta m}$ for all $i * j * m * i$; $i, j, m \in \{1, \dots, k\}$.

Proof. From the assumption that w satisfies (P) it follows that every subterm of w occurs in one position only. Then also every subterm of $\delta(w)$ occurs in one position only, because δ is a permutation of $\{1, \dots, k\}$. So $\delta(w)$ satisfies (P), and therefore $\delta(w)$ is universal for all permutations δ .

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Received October 26, 1989.