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FREE P-BILATTICES

Introduction

In a number of papers [Gi1], [Gi2], [Gi3], M.L. Ginsberg introduced algebras called bilattices having two lattice structures and one additional basic unary operation acting on both lattices in very regular way. Bilattices originated as an algebraization of some non-classical logics that appeared recently in investigation on artificial intelligence. The structure of P-bilattices (bilattices satisfying some additional identities) was described in [RT].

The purpose of this paper is to characterize free P-bilattices.

In section 1 we collect some useful facts about bilattices. Free P-bilattices are investigated in section 2. In section 3 the theorem about free P-bilattices is specified for distributive bilattices. Finally, in section 4 we discuss cardinality of free distributive bilattices on n generators and in section 5 we give two examples of free distributive bilattices.

1. Preliminaries

A bilattice is an algebra $\underline{B} = (B,\wedge,\vee,0_1,1_1,\circ,+,0_2,1_2,')$ such that

(B1) $\underline{B}_1 = (B,\wedge,\vee,0_1,1_1)$ and $\underline{B}_2 = (B,\circ,+,0_2,1_2)$ are bounded lattices,

(B2) $':B \longrightarrow B$ is an unary operation satisfying the following identities:

(i) $x''=x$,

(ii) $(x\vee y)'=x'\wedge y'$,

$(x\wedge y)'=x'\vee y'$,

(iii) $(x+y)'=x'+y'$,

$(x\circ y)'=x'\circ y'$.

Let $\underline{L} = (L,\wedge,\vee,0,1)$ be a bounded lattice and let $B(L)=L\times L$. On the set $B(L)$ we define four binary operations $\wedge$, $\vee$, $\circ$ and $+$ in the following way:

(1) $(B(L),\wedge,\vee) := (L,\wedge,\vee)\times(L,\wedge,\vee)^d$,

where $(L,\wedge,\vee)^d$ is the dual of $(L,\wedge,\vee)$,

(2) $(B(L),\circ,+) := (L,\wedge,\vee)\times(L,\wedge,\vee)$,

and one unary and four nullary operations as follows:

$(a,b)' := (b,a)$,

$0_1:=(0,1)$,

$1_1:=(1,0)$,

$0_2:=(0,0)$,

$1_2:=(1,1)$.

It was shown in [RT] that the algebra $\underline{B}(L) = (B(L),\wedge,\vee,0_1,1_1,\circ,+,0_2,1_2,')$ is a bilattice and it was called a product bilattice associated with the lattice $\underline{L}$.

For more information about bilattices see [Gi] and [T] and for basic facts concerning universal algebra and lattice theory (specially free lattices) [G] and [CD].

2. Free Padmanabhan bilattices

A bilattice satisfying the following identities:

(2.1) $((x\wedge y)\circ z)\wedge(y\circ z) = (x\wedge y)\circ z$,

(2.2) $((x\circ y)\wedge z)\circ(y\wedge z) = (x\circ y)\wedge z$,

(2.3) $((x\wedge y)+z)\wedge(y+z) = (x\wedge y)+z$,

(2.4) $((x+y)\wedge z)+(y\wedge z) = (x+y)\wedge z$

is called a Padmanabhan bilattice or briefly a P-bilattice. The following representation of P-bilattices was presented in [RT].

Theorem 2.1. [RT] An algebra $\underline{B}=(B,\wedge,\vee,0_1,1_1,\circ,+,0_2,1_2,')$ of type $(2,2,0,0,2,2,0,0,1)$ is a P-bilattice if and only if there is a bounded lattice $\underline{L}=(L,\wedge,\vee,0,1)$, such that $\underline{B}$ is isomorphic to the product bilattice $\underline{B}(L)$ associated to the lattice $\underline{L}$. ∎

In this section we give a characterization of free Padmanabhan bilattices on n generators. A free bounded lattice on n generators is denoted by $\underline{F}_L(n)$ and a free bounded distributive lattice on n generators is denoted by $\underline{F}_D(n)$.

Theorem 2.2. Let $\underline{F}_L(2n)$ be a free bounded lattice on 2n generators. A product bilattice $\underline{B}(F_L(2n))$ associated with the lattice $\underline{F}_L(2n)$ is a free P-bilattice on n generators.

Proof. Let $G=\{g_1,g_2,\dots,g_n,g_{n+1},\dots,g_{2n}\}$ be a set of free generators of the lattice $\underline{F}_L(2n)$.
Let S be the following n-elements subset of the set $B(F_L(2n))$:

$$S = \{(g_1,g_{n+1}),(g_2,g_{n+2}),\dots,(g_i,g_{n+i}),\dots,(g_n,g_{2n})\}.$$

The proof is made in two steps.

Step A. Note that elements (1,0), (0,1), (0,0), (1,1) are in $B(F_L(2n))$, and

$$(g_i,g_{i+n})\vee(0,0) = (g_i,0),$$

$$(g_i,g_{i+n})\wedge(0,0) = (0,g_{i+n}),$$

$$(g_i,0)' = (0,g_i),$$

$$(0,g_{i+n})' = (g_{i+n},0),$$

$$(g_i,g_{i+n})\vee(1,1) = (1,g_{i+n}),$$

$$(g_i,g_{i+n})\wedge(1,1) = (g_i,1),$$

$$(1,g_{i+n})' = (g_{i+n},1),$$

$$(g_i,1)' = (1,g_i)$$

for $1\le i\le n$. Moreover $(g_k,0)+(0,g_l) = (g_k,g_l)$ for $1\le k,l\le 2n$.

In this way we received all elements of the set $G\times G$ from elements of the set S.

Let (a,b) be in the set $F_L(2n)\times F_L(2n)$. Then elements a,b are in $F_L(2n)$. Since G is the set of free generators of the lattice $\underline{F}_L(2n)$, the elements a, b can be expressed by elements from the set G. It follows that the element (a,b) can be built up from elements in the set $G \times G$.

Step B. Let $\underline{B}$ be a P-bilattice and let $f: S \longrightarrow B$ be a mapping. By Theorem 2.1, there is a bounded lattice $\underline{L}=(L,\wedge,\vee,0,1)$ such that $\underline{B} \cong \underline{B}(L)$. Put $f((g_i,g_{i+n})) = (k_i,k_{i+1})$, where (k_i,k_{i+1}) is in $B(L)$. Let $f_1: G \longrightarrow L$ be a mapping such that

$$f_1(g_i) = k_i,$$
$$f_1(g_{i+n}) = k_{i+1}.$$

Since $\underline{F}_L$ is a free lattice on the set G, it follows that the mapping $f_1: G \longrightarrow L$ can be extended to a homomorphism $h_1: \underline{F}_L \longrightarrow \underline{L}$.

Let $h: B(F_L) \longrightarrow B(L)$ be the following mapping

$$h((x_1,x_2)) := (h_1(x_1),h_1(x_2)).$$

The mapping h is an extension of f, because

$$h((g_i,g_{i+n}))=(h_1(g_i),h_1(g_{i+n}))=(f_1(g_i),f_1(g_{i+n}))=(k_i,k_{i+1}).$$

Moreover h is a bilattice homomorphism, since

$$h((x_1,x_2)') = h((x_2,x_1)) = (h_1(x_2),h_1(x_1)) = (h_1(x_1),h_1(x_2))' = (h((x_1,x_2)))',$$

$$h((x_1,x_2)\wedge(y_1,y_2)) = h((x_1\wedge y_1,x_2\vee y_2)) = (h_1(x_1\wedge y_1),h_1(x_2\vee y_2)) = (h_1(x_1)\wedge h_1(y_1),h_1(x_2)\vee h_1(y_2)) =$$
$$= (h_1(x_1),h_1(x_2))\wedge(h_1(y_1),h_1(y_2)) = h((x_1,x_2))\wedge h((y_1,y_2)),$$

and similarly

$$h((x_1,x_2)\vee(y_1,y_2)) = h((x_1,x_2))\vee h((y_1,y_2)),$$
$$h((x_1,x_2)\circ(y_1,y_2)) = h((x_1,x_2))\circ h((y_1,y_2)),$$
$$h((x_1,x_2)+(y_1,y_2)) = h((x_1,x_2))+h((y_1,y_2)).$$

Let the greatest and least elements of $\underline{F}_L$ be denoted by 1_F and 0_F respectively. Then we have that

$$h((0_F,1_F)) = (h_1(0_F),h_1(1_F)) = (0,1),$$
$$h((1_F,0_F)) = (h_1(1_F),h_1(0_F)) = (1,0),$$
$$h((1_F,1_F)) = (h_1(1_F),h_1(1_F)) = (1,1),$$
$$h((0_F,0_F)) = (h_1(0_F),h_1(0_F)) = (0,0). \blacksquare$$

A free Padmanabhan bilattice on n generators will be denoted by $\underline{BF}_L(n)$.

For n=0 and n=1 $\underline{F}_L(n)$ is finite and it is easy to see that $|BF_L(0)| = |F_L(0)|^2$ and $|BF_L(1)| = |F_L(2)|^2$. Since $\underline{F}_L(n)$ for n≥3 is infinite (see [CD]) it is clear that the bilattice $\underline{BF}_L(2)$ is infinite.

3. A free distributive bilattices

A bilattice in which each basic binary operation distributes over each other is called distributive bilattice.

The structure Theorem 2.1. may be formulated in the case of distributive bilattice as follows:

Theorem 3.1. [T]. An algebra $\underline{B} = (B,\wedge,\vee,0_1,1_1,\circ,+,0_2,1_2,')$ of type (2,2,0,0,2,2,0,0,1) is a distributive bilattice if and only if there is a bounded distributive lattice $\underline{L} = (L,\wedge,\vee,0,1)$ such that $\underline{B}$ is isomorphic to the product bilattice $\underline{B}(L)$ associated with the lattice $\underline{L}$.∎

The following result is an easy corollary of Theorem 3.1. and Theorem 2.1.

Theorem 3.2. Let $\underline{F}_D(2n)$ be a free bounded distributive lattice on 2n generators. A product bilattice $\underline{B}(F_D(2n))$ associated with the lattice $\underline{F}_D(2n)$ is a free distributive bilattice on n generators.∎

A free distributive bilattice on n generators will be denoted by $\underline{BF}_D(n)$.

Since free distributive lattices on finite set of generators are finite [G], it follows easily that each $\underline{BF}_D(n)$ is finite and $|BF_D(n)| = |F_D(2n)|^2$.

By the result of K. Yamamoto [BD] the free distributive lattice on an even number of generators has even cardinality and $|BF_D(n)| = |F_d(2n)|^2$ is even, too.

By the recent result obtained by A. Kisielewicz [K] for free distributive lattices we know that the number of elements of the free bounded distributive lattice on n generators is equal to

$$|BF_D(n)| = \left(\sum_{k=1}^{2^{2^n}} \prod_{j=1}^{2^n-1} \prod_{i=1}^{j} \left(1-b_i^k b_j^k \prod_{m=0}^{\log_2 i} (1-b_m^i+b_m^i b_m^j)\right)\right)^2,$$

for any $n\geq 1$, where $b_i^k = [k/2^i]-2[k/2^{i+1}]$.

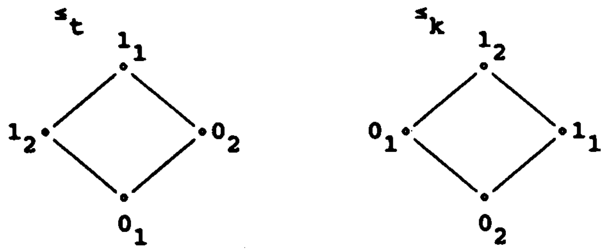
The exact values of $|BF_D(n)|$ can be computed for $n\leq 3$. For example:

$$|BF_D(0)| = 4,$$
$$|BF_D(1)| = 36,$$
$$|BF_D(2)| = 28224,$$
$$|BF_D(3)| = (7828354)^2 \approx 6{,}13\cdot 10^{13}.$$

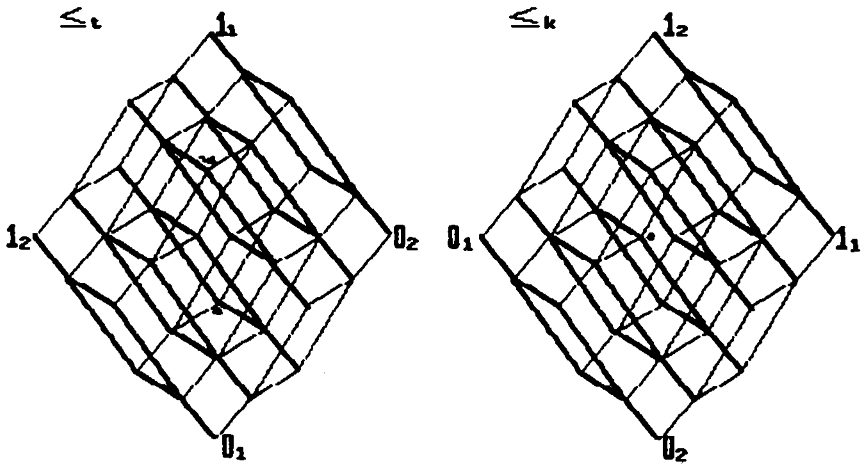
For n=4, $|BF_D(4)|$ is known to be greater than 2^{140}.

4. Examples of free distributive lattices

A. $\underline{FB}_D(0)$



B. $\underline{FB}_D(1)$



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