DEMONSTRATIO MATHEMATICA
Vol. XXIV No 1-2 1991

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FIXED POINT FREE COMPONENTS IN LEXICOGRAPHIC SUMS WITH THE FIXED POINT PROPERTY

If in a lexicographic sum P of partial orders $P_t, t\in T$, the index set T and all components P_t have the fixed point property, then P has the fixed point property. This result was obtained in [4] for the case where P is a chain complete partial order. The general case, where P is not necessarily chain complete, was proven in [8, Theorem 7], and a different, very short proof for the general case was also given in [6, Theorem 2.2]. There are many examples of lexicographic sums, where the index set T and the sum P have fixed point property, but some of the components P_t do not. In [6, Theorem 3.3] the authors determined precisely which components P_t in a chain complete sum P may be fixed point free, while P still has the fixed point property. The goal of this paper is to remove the condition of chain completeness for P. The main result is the Characterization Theorem 3.2; it is an extension of Theorem 3.3 in [6] to the class of all partially ordered sets.

Section 1 contains notations and some basic facts about lexicographic sums. In section 2 we introduce contracting subsets of partially ordered sets and list some results from [6], where contracting sets have played an essential role. In section 3 we define a "local" fixed point property for intervals and prove the main theorem.

1. Preliminaries

If P is partially ordered set and $f:P\longrightarrow P$ is an order preserving map, then $\text{fix } f=\{p\in P \mid f(p)=p\}$ is the set of all fixed points of f. P has the fixed point property (fpp), if fix $f\neq\emptyset$ for each order preserving map $f:P\longrightarrow P$. If there is an order preserving map $g:P\longrightarrow P$ such that fix $g=\emptyset$, then g is called a fixed point free map, and we say that P is fixed point free. A partial order P is chain complete if for every non-empty chain C in P the supremum, sup C, and the infimum, inf C, exist in P.

Let T be a partially ordered set and let for each $t\in T$, P_t be a partially ordered set. We define the lexicographic sum $P=L\{P_t|t\in T\}$ with index set T and components $P_t, t\in T$, to be the set $\{(t,x)|t\in T,\ x\in P_t\}=\bigcup\{\{t\}\times P_t|t\in T\}$. The order on P is given by

$(s,x)\leq(t,y)$ if and only if either $s<t$, or $s=t$ and $x\leq y$.

When there is no confusion possible, we shall use only the elements of the components rather than the pairs, i.e., $x\in P_s\subset P$ rather than $(s,x)\in\{s\}\times P_s\subset P$.

For each lexicographic sum there is the order preserving canonical projection $\pi:P\longrightarrow T$ defined as $\pi(t,x)=t$. If we pick for each $t\in T$ an element $p_t\in P_t$, then the mapping $\phi:T\longrightarrow P$ defined by $\phi(t)=p_t$, satisfies $\pi\circ\phi=id_T$, so that π is a retraction, and T is a retract of P. For an order preserving map $f:P\longrightarrow P$ and for an index $t\in T$, let $T(f,t)=\{s\in T|\exists p\in P_t,\ f(p)\in P_s\}-\{t\}$. We define the associated choice function for f to be a map $\beta_f:T\longrightarrow T$ where $\beta_f(t)\in T(f,t)$ if $T(f,t)\neq\emptyset$ and $\beta_f(t)=t$ otherwise. In [6, Lemma 1.1] it was shown that β_f is order preserving, and that for all $t\in T$ we have $f(P_t)\subset P_t$ if and only if $\beta_f(t)=t$.

If the index set T and all components P_t in a lexicographic sum P have the fpp then the sum P has the fpp [6, Theorem 2.2], but it is not a necessary condition that all components have the fpp. The components P_s without the fpp are called the fixed point free components of P, and the subset S of T consisting of all those indices $s\in T$ such that P_s is fixed point free, is called the fixed point free part of T.

2. Contracting Sets and the Fixed Point Property for Lexicographic Sums

In a lexicographic sum P with the fixed point property, fixed point free components can occur only at certain positions in the index set T. The notion of contracting subsets characterizes these positions as we show in this section.

Let P be a partial order, and let $f:P\longrightarrow P$ be an order preserving map. If there is a fixed point $p\in \text{fix } f$ and an element $q\in P$ so that $f(q)=p$ and $p\neq q$, but p and q are comparable, i.e. $p\leq q$ or $q\leq p$, then we call f a contracting map in P and p a contracting fixed point of f. The map f is called downward (upward) contracting, if $p<q$ $(q<p)$, and two-sided contracting, if p is both upward and downward contracting. We call a subset Q in P contracting in P, if every order preserving map $f:P\longrightarrow P$ for which $\text{fix } f\subset Q$ holds, is a contracting map in P; any such map f will be called Q-contracting. Since the identity map on P is not a contracting map, P itself is not contracting in P. The empty set is contracting in P if and only if P has the fixed point property. More information on contracting sets may be found in [5]; singleton contracting sets are considered in [2].

Example I. Let P be a non-empty, finite chain and let $Q\neq\emptyset$ be any proper subset of P. Then Q is a contracting subset in P. On the other hand, if P is an infinite chain, even a complete chain, not every proper subset of P needs to be contracting. Let $P=N\cup\{\infty\}$, with $n<\infty$ for all $n\in N$, be the one-point completion of the natural numbers. The subset $Q_1=\{\infty\}$ is not contracting since the only fixed point of the extended successor function $f(n)=n+1$, for $n\in N$, and $f(\infty)=\infty$, is not contracting. The subset $Q_2=N$ is contracting in P since every order preserving map f on P with $\text{fix } f\subset N$ must satisfy $f(\infty)\in N$.

The following results, Lemma 2.1, 2.2 and Characterization Theorem 2.4 are results from [6]. We list them here without proof; the proofs are given in [6, Lemma 3.1, 3.2, and Theorem

3.3].

Lemma 2.1. Let $P=L\{P_t|t\in T\}$ be a lexicographic sum with fixed point free part $S\subset T$. If P has the fixed point property, then S is a contracting subset of T.

We say that a partial order P has the comparability property if for every order preserving mapping $f:P\rightarrow P$ there is $p\in P$ so that either $f(p)\leq p$ or $p\leq f(p)$. If P is chain-complete, then the comparability property and the fixed point property of P are equivalent (e.g. [4, Theorem 1]).

Lemma 2.2. Let $P=L\{P_t|t\in T\}$ be a lexicographic sum with fixed point free part $S\subset T$. Suppose that S is a contracting subset of T, and that T has the fixed point property. Then P has the comparability property.

In case S is empty, i.e. all P_t, $t\in T$, have the fixed point property, the conclusion in Lemma 2.2 can be strengthened to the fixed point property of P (case 1 in the proof of Lemma 3.2 in [6] does not occur if S is empty). In this special case we get as a consequence

Corollary 2.3. (Theorem 2.2 in [6] and also Theorem 7 in [8]). Let $P=L\{P_t|t\in T\}$ be a lexicographic sum, and let P_t have the fixed point property for all $t\in T$. The following statements are equivalent: (1) P has the fpp.
(2) T has the fpp.

From Lemma 2.1 and Lemma 2.2. we obtain immediately

Theorem 2.4. Let $P=L\{P_t|t\in T\}$ be a chain complete lexicographic sum with fixed point free part $S\subset T$. The following statements are equivalent:

(1) P has the fpp.
(2) T has the fpp, and S is a contracting set in T.

We give two simple examples for Theorem 2.4.

Example II. Let $T=\{0,a,b,1\}$ be the 4-element Boolean algebra. The contracting subsets of T are all proper subsets of T except the three subsets $\{0,1\}$, $\{0,a,1\}$, and $\{0,b,1\}$. Each of these three subsets admits at least one order preserving, non-contracting map whose fixed points are contained in the set. In the context of any lexicographic sum over the index set T, this means that up to any three of the four components P_t, $t \in T$, may be fixed point free, but never simultaneously P_0 and P_1. Note also that the subset $\{a,b\} \subset T$ is contracting, since any order preserving map f with $\emptyset \neq \text{fix } f \subset \{a,b\}$ has exactly one fixed point, and that is a contracting fixed point.

Example III. Let $T=N \cup \{\infty\}$ be as in Example I and let P_t, $t \in N$, be any partial orders with or without the fixed point property. If P_∞ has the fixed point property and all components are such that the lexicographic sum P is chain complete, then P has the fixed point property.

3. Intervals and the Fixed Point Property for Lexicographic Sums

In the proof of Lemma 2.2 [6, Lemma 3.2], the problem of finding a fixed point for an order preserving map f in a lexicographic sum is localized to a single component by a contracting fixed point on the index set T. This is a contracting fixed point for the associated choice function $\beta_f : T \longrightarrow T$. Chain completeness of the entire lexicographic sum then produces a fixed point for the map f. Our next goal is to eliminate from Theorem 2.4 the hypothesis that the lexicographic sum is chain complete. To this end we first introduce a fixed point property for intervals of partial orders.

For a partial order P and elements $p,q \in P$, let $[p,\rightarrow)=\{x \in P | p \leq x\}$ be the upper interval generated by p, let $(\leftarrow,q]=\{x \in P | x \leq q\}$ be the lower interval generated by q, and if $p \leq q$, let $[p,q]=\{x \in P | p \leq x \leq q\}$ be the proper interval generated by p and q. We say that P has the fixed point property for intervals, if all intervals of P (lower, upper and proper intervals) have the fixed point property. A partial order which either is chain complete or has the fixed point property, also has the fixed point property for intervals ([4, Theorem 1] and [6, Corollary 1.4], also [3] and [7]). Since every order preserving map on a proper interval $[p,q]$ of P can be extended to an order preserving map on $[p,\rightarrow)$ or on $(\leftarrow,q]$, the fixed point property for upper intervals (or lower intervals) alone, will imply the fixed point property for proper intervals. However, the fixed point property for all intervals of P does not imply the fixed point property for the partial order P. The four element crown is an example of a fixed point free partial order, where all intervals have the fixed point property.

In the context of lexicographic sums, we have that for $p,q\in P_t$, the interval $[p,q]$ is completely contained in P_t. Therefore we obtain the following lemma

Lemma 3.1. If $P=L\{P_t|t\in T\}$ is either chain complete or has the fixed point property, then for each $t\in T$, P_t has the fixed point property for proper intervals.

On the other hand, there are lexicographic sums which have the fixed point property, but also have components that do not satisfy the fixed point property for upper or lower intervals. A sum of this type that is chain complete is given in Example IV, and a sum of this type that is not chain complete is given in Example V.

Example IV. Let $P=L\{P_t|t\in T\}$ be a lexicographic sum where $T=3$, the three element chain, and $P_0=N$, the natural numbers, $P_1=1$, the one-element chain, and $P_2=N^d$, the dual natural numbers. P is chain complete, has the fixed point property, but P_0 does not have the fixed point property for upper intervals, and P_2 does not have the fixed point property for lower intervals.

Example V. Let $T_1=2$, $P_0=N$, and $P_1=\{a,b,c\}$, with $a<c$ and $b<c$ as the only order relations. $P=L\{P_t|t\in T_1\}$ is not chain complete, and P_0 does not have the fixed point property for upper intervals. However, P has the fixed point property since it has a largest element and since every chain in P has a least element.

Let $P=L\{P_t|t\in T\}$ be a lexicographic sum, and let $S\subset T$ be a contracting set in T. We say that the system $\{P_s | s \in S\}$ is a contracting fixed point system, if for every S-contracting map $\alpha: T \longrightarrow T$ there is a downward contracting fixed point $s \in \text{fix}\ \alpha$, such that P_s has the fixed point property for lower intervals, or there is an upward contracting fixed point $s \in \text{fix}\ \alpha$, such that P_s has the fixed point property for upper intervals, or there is a two sided contracting fixed point $s \in \text{fix}\ \alpha$, such that P_s has the fixed point property for proper intervals.

Note that in Example V, $S=\{0\} \subset T_1$ is contracting in T_1, and the system $\{P_s | s \in S\}$ consisting of P_0 only is a contracting fixed point system, since 0 is a downward contracting fixed point and since P_0 has the fixed point property for lower intervals.

Now we are ready to generalize Theorem 2.4 from the class of chain complete partial orders to the class of all partials orders. Observe first that the two conditions (1) and (2) in Theorem 2.4 are no longer equivalent for arbitrary partial orders, as the example below will show.

Example VI. Let $T_1=3$, $P_0=1$, $P_1=N$, and let $P_2=2$ be two element anti-chain. Then $\{1,2\}=S_1 \subset T_1$ is a contracting set in T_1. T_1 has the fixed point property, but $P=L\{P_t | t \in T_1\}$ does not. Note that P is not chain complete, so that Theorem 2.4 does not apply to the example. On the other hand, Lemma 2.1 will apply, if we represent P lexicographically as follows: Let T_2 be as in the second part of Example V, and let $S_2=\{\infty\} \subset T_2$. S_2 is not contracting in T_2, so that by Lemma 2.1, P does not have the fixed point property.

Theorem 3.2. Let $P=L\{P_t | t \in T\}$ be a lexicographic sum with fixed point free part $S \subset T$. The following statements are equivalent:

(1) P has the fpp.

(2) T has the fpp, S is contracting in T, and $\{P_s | s \in S\}$ is a contracting fixed point system.

Proof. (2)$\longrightarrow$(1): Let $f: P \longrightarrow P$ be order preserving and let $\beta_f: T \longrightarrow T$ be its associated choice function as defined in section 1. Since T has the fixed point property, fix $\beta_f \neq \emptyset$. If fix β_f is not contained in S, then there is $t\in T-S$ such that $\beta_f(t)=t$, i.e. $f(P_t)\subset P_t$, and thus f has a fixed point in P_t. If fix $\beta_f\subset S$, then β_f is S-contracting. Let $s\in \text{fix}\ \beta_f$ be such that s is downward contracting and P_s has the fixed point property for lower intervals. Then there is $t\in T$, $t>s$, and $p\in P_t$ so that $\beta_f(t)=s$ and $p>f(p)\in P_s$. By definition of β_f, $f(P_s)\subset P_s$, so that $f((\leftarrow,f(p)])\subset(\leftarrow,f(p)]$ holds, and f has a fixed point on $(\leftarrow,f(p)]$. If s is upward contracting, we can dualize our argument, and if s is two sided contracting, the argument is similar.

(1)⟶(2): If P has the fixed point property, then so does T since it is a retract of P. Furthermore, the fixed point free part $S\subset T$ of P is contracting because of Lemma 2.1. If S is empty, then $\{P_s|s\in S\}$ is a contracting fixed point system because T has the fixed point property. Suppose now that $S\neq\emptyset$ and that $\{P_s|s\in S\}$ is not a contracting fixed point system. Then there is an S-contracting map $\alpha:T\longrightarrow T$ so that for no $s\in \text{fix}\ \alpha\neq\emptyset$, P_s has the appropriate fixed point property for intervals (upper, lower, or proper intervals, respectively).

For each $s\in \text{fix}\ \alpha$ we choose a fixed point free partial order $Q_s\subset P_s$ in the following manner. If s is downward contracting let $u_s\in P_s$ be such that the lower interval $Q_s=(\leftarrow,u_s]$ is fixed point free. Similarly, we choose $p_s\in P_s$ such that $Q_s=[p_s,\rightarrow)$ and $Q_s=[p_s,u_s]$ are fixed point free when s is upward or two-sided contracting, respectively. Finally, if s is non-contracting we put $Q_s=P_s$. In addition, we choose a point $r_s\in Q_s$ for each non-contracting index $s\in \text{fix}\ \alpha$. Obviously, the lexicographic sum $Q = L\{Q_s|s\in \text{fix}\ \alpha\}$ is fixed point free since all components Q_s are fixed point free.

We establish the contraction that P is fixed point free by constructing a retraction $\phi:P\longrightarrow Q$. For each $s\in \text{fix}\ \alpha$ we let ϕ be the standard retraction of P_s onto the interval Q_s (for an explicit definition of the retraction see for example [1]), and for any $t\in T-\text{fix}\ \alpha$ we let ϕ be the constant map onto the element $r_{\alpha(t)}\in Q_{\alpha(t)}$ (We thank the referee for shortening part of this proof).

The General Characterization Theorem 3.2 provides us with a simple sufficient condition for the fixed point property of a lexicographic sum P over the index set T. Just observe that if every component P_s, $s \in S$, in the fixed point free part S of T has the interval fixed point property then the system $\{P_s | s \in S\}$ is a contracting fixed point system.

Corollary 3.3. Let $P=L\{P_t | t \in T\}$ be a lexicographic sum with fixed point free part $S \subset T$. If

(1) T has the fpp, and

(2) S is a contracting set, and

(3) P_s has the interval fixed point property for every $s \in S$,

then P has the fpp.

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Received July 20, 1989.