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MAXIMAL ANTICHAINS AND DISSECTING IDEALS

1. Introduction

Let (X, \leq) be a poset. A subset I of X is called an ideal of (X, \leq) if whenever $y \in I$ and $x \in X$ are such that $x \leq y$ then $x \in I$. A subset A of X is called an antichain if any two distinct elements of A are incomparable. Let $A(X)$ be the set of all antichains of (X, \leq) . An antichain A is called maximal if it is a maximal element of $A(X)$ with respect to set-theoretic inclusion. Let $MA(X)$ be the set of all maximal antichains of (X, \leq) .

It is well known (see, for example, [2], Ch. III, Thm. 3) that the set of all ideals of (X, \leq) forms a distributive lattice under set-theoretic inclusion, and that every finite distributive lattice can be represented in this way. For any subset Y of X let $\text{Max}(Y)$ and $\text{Min}(Y)$ be the set of maximal, respectively minimal, elements of Y . On the set of all antichains of (X, \leq) a partial order can be defined as follows. If $A, B \in A(X)$ then $A \leq B$ if for each $a \in A$ there exists $b \in B$ such that $a \leq b$. Furthermore, if X is finite then the mapping $Y \mapsto \text{Max}(Y)$ is an order-isomorphism between the set of ideals of (X, \leq) and the set $(A(X), \leq)$. Thus $(A(X), \leq)$ is a distributive lattice, and every finite distributive lattice arises in this way.

R.P. Dilworth [3] showed that if X is finite, then the set $MSA(X)$ of all antichains of (X, \leq) of maximum size forms a distributive lattice under the order induced from $(A(X), \leq)$, in fact, a sublattice of $(A(X), \leq)$. It was shown by K.M. Koh [4] that every finite distributive lattice can be represented in

this way.

In [1], the author showed that the set $MA(X)$ of all maximal antichains of the finite poset (X, \leq) forms a lattice under the order induced from $(A(X), \leq)$, and that furthermore every finite lattice can be represented in this way. In general, the lattice $(MA(X), \leq)$ is only a subposet of $(A(X), \leq)$ but not a sublattice. This suggests to consider the question of what can be said about the sublattice of $(A(X), \leq)$ which is generated by $MA(X)$. This question shall be dealt with in this paper.

2. The lattice of dissecting ideals

We call an ideal I of a poset (X, \leq) dissecting if whenever $x \in I$ and $y \in X \setminus I$ are such that x and y are incomparable then there exists $y' \in X$ with $y' < y$ such that x and y' are incomparable.

Theorem 2.1. Let (X, \leq) be a poset. Then the set of all dissecting ideals forms a complete sublattice of the lattice of all ideals of (X, \leq) .

Proof. First note that X and the empty set both are dissecting. Let S be a non-empty set of dissecting ideals of (X, \leq) and let J and M be the set-theoretic union respectively intersection of all members of S . Let $x \in J$ and $y \in X \setminus J$ such that x and y are incomparable. There exists $I \in S$ such that $x \in I$. As $I \subseteq J$, we also have $y \in X \setminus I$, and thus there exists $y' \in X$ with $y' < y$ such that x and y' are incomparable. Therefore J is dissecting. Let $x \in M$ and $y \in X \setminus M$. There exists $I \in S$ such that $y \in X \setminus I$, and as $M \subseteq I$ we have $x \in I$. Hence there exists $y' \in X$ with $y' < y$ such that x and y' are incomparable, and M is dissecting.

A first connection between maximal antichains and dissecting ideals is given as follows.

Proposition 2.2. Let (X, \leq) be a poset, let $A \in MA(X)$, and let $I = \{x \in X \mid \text{there exists } a \in A \text{ such that } x \leq a\}$. Then I is a dissecting ideal.

Proof. Clearly I is an ideal. Let $x \in I$ and $y \in X \setminus I$ such that

x and y are incomparable. By definition of I , there exists $a \in A$ such that $x \leq a$. By maximality of A , there exists $b \in A$ such that y and b are comparable, and as $y \in X \setminus I$, we must have $y < b$. If $a = b$ then $x \leq y$, which is a contradiction, thus we have $a \neq b$. If $b \leq x$ then $b \leq a$ which is a contradiction to $a \neq b$ and A being an antichain. If $x \leq b$ then again we get $x \leq y$, giving a contradiction. Thus x and b are incomparable, and as $y \in X \setminus I$ and $b \in I$ we have $b < y$. Therefore I is dissecting.

Next we see that the lattice of dissecting ideals is, in fact, generated by the ideals which are generated by the maximal antichains.

Theorem 2.3. Let (X, \leq) be a poset. Then the lattice of dissecting ideals of (X, \leq) is the complete sublattice of the lattice of all ideals of (X, \leq) which is generated by the empty set and the ideals $I(A) = \{x \in X \mid \text{there exists } a \in A \text{ such that } x \leq a\}$ for the maximal antichains A .

Proof. Let D be the set of all dissecting ideals and L the complete lattice generated by \emptyset and all ideals $I(A)$ for the maximal antichains A . By Theorem 2.1 and Proposition 2.2 it is clear that $L \subseteq D$. For $x \in X$ define $K(x)$ to be the intersection of all ideals $I(A)$ where A is a maximal antichain containing x . Obviously $K(x) \in L$. Let I be a non-empty dissecting ideal. Let $x \in I$ and $y \in X \setminus I$. If $x < y$ then clearly $y \notin K(x)$. Suppose x and y are incomparable. Then there exists $y' < y$ such that x and y' are incomparable, and there exists $A \in \text{MA}(X)$ with $\{x, y'\} \subseteq A$. It then follows that $y' \notin I(A)$, and thus $y' \notin K(x)$. Hence we have $K(x) \subseteq I$. As $x \in K(x)$, it follows that I is the union of all ideals $K(x)$ for $x \in I$, and therefore $I \in L$, which concludes the proof of the theorem.

Note that the lattice of dissecting ideals of (X, \leq) contains an atom (which is then unique) if and only if (X, \leq) has at least one minimal element (and then the atom is the set of all minimal elements). Thus the complete lattice generated by the ideals $I(A)$ for the maximal chains A is the set of all dissecting ideals whenever (X, \leq) has no minimal element, or otherwise it is the set of non-empty dissecting ideals.

3. Dissecting ideals in finite posets

If (X, \leq) is a finite poset then by Theorem 2.1 its dissecting ideals form a finite distributive lattice, which in turn is isomorphic to the lattice of all ideals of a finite poset (Y, \leq) . We shall now investigate in which way (Y, \leq) can be constructed from (X, \leq) .

Lemma 3.1. Let (X, \leq) be a finite poset and $x \in X$. Let $K(x)$ be defined as in the proof of Theorem 2.3, let $A(x) = \text{Max}(K(x))$, and let $B(x) = \{y \in X \mid x \text{ and } y \text{ are incomparable}\}$. Then $A(x) = \{x\} \cup \text{Min}(B(x)) \in \text{MA}(X)$.

Proof. Clearly $\{x\} \cup \text{Min}(B(x))$ is an antichain. Let $y \in X$. Then either y is comparable with x or $y \in B(x)$. But if $y \in B(x)$, as X is finite there exists $y' \in \text{Min}(B(x))$ with $y' \leq y$. Thus $\{x\} \cup \text{Min}(B(x)) \in \text{MA}(X)$, and $K(x) \subseteq I(\{x\} \cup \text{Min}(B(x)))$. But on the other hand, if $A \in \text{MA}(X)$ with $x \in A$ then $A \subseteq \{x\} \cup B(x)$ and thus $\{x\} \cup \text{Min}(B(x)) \subseteq I(A)$, hence $I(\{x\} \cup \text{Min}(B(x))) \subseteq K(x)$. Therefore $A(x) = \{x\} \cup \text{Min}(B(x))$.

Lemma 3.2. Let (X, \leq) be a finite poset, and let $x, y \in X$. Then $A(x) = A(y)$ if and only if $\{z \in X \mid z < x\} = \{z \in X \mid z < y\}$.

Proof. Without loss of generality, we can assume that $x \neq y$. Let $A(x) = A(y)$, and let $z \in X$ with $z < x$. If $y \leq z$ then $y < x$ and thus $y \in A(y) \setminus A(x)$, which is a contradiction. If y, z are incomparable then we get a contradiction from the fact that $x \in A(y)$, thus $x \in \text{Min}(B(y))$. By symmetry, we then get $\{z \in X \mid z < x\} = \{z \in X \mid z < y\}$.

Conversely, suppose that $\{z \in X \mid z < x\} = \{z \in X \mid z < y\}$, and let $z \in A(x)$. Note that x and y must be incomparable. First assume that $z = x$. By the equality above, there can be no $z' \in X$ incomparable with y such that $z' < x$. Thus $x \in \text{Min}(B(y)) \subseteq A(y)$. Now assume that $z \in \text{Min}(B(x))$. Clearly we can not have $z < y$. If $z > y$ then we get a contradiction to minimality of z in $B(x)$. Hence either $z = y \in A(y)$ or z and y are incomparable. If z, y are incomparable then suppose there exists $z' < z$ such that $z' \in B(y)$. We can not have $z' < x$ (and clearly nor can we have

$x \leq z'$), therefore $z' \in B(x)$ in contradiction to the minimality of z . Thus $z \in \text{Min}(B(y)) \subseteq A(y)$. We have shown that $A(x) \subseteq A(y)$, and equality holds by symmetry.

Lemma 3.3. Let (X, \leq) be a finite poset, and let I be a dissecting ideal. Then I is join-irreducible in the lattice of dissecting ideals if and only if there exists $x \in X$ such that $I = K(x)$.

Proof. Let $x \in X$ and let I, J be dissecting ideals such that $K(x) = I \vee J$. As $x \in K(x)$, it follows that $x \in I$ or $x \in J$. But from the proof of Theorem 2.3 it follows that $K(x) \subseteq I$ or $K(x) \subseteq J$, and thus $K(x) = I$ or $K(x) = J$. Therefore $K(x)$ is join-irreducible. Let I be a join-irreducible dissecting ideal. Again, the proof of Theorem 2.3 shows that I is the join of all ideals $K(x)$ for $x \in I$, and thus there exists $x \in I$ such that $I = K(x)$.

On a poset (X, \leq) we define an equivalence relation \sim by $x \sim y$ if and only if $\{z \in X \mid z < x\} = \{z \in X \mid z < y\}$. Let $E(X)$ be the set of equivalence classes of this relation. Note that each element of $E(X)$ is an antichain of (X, \leq) . For $C, D \in E(X)$ let $C \leq D$ if and only if $\{z \in X \mid z < c\} \subseteq \{z \in X \mid z < d\}$ whenever $c \in C$ and $d \in D$. It is easy to see that this relation is a partial order.

Lemma 3.4. Let (X, \leq) be a finite poset, let $x, y \in X$, and let $C(x)$ and $C(y)$ be the elements of $E(X)$ containing x respectively y . Then $A(x) \subseteq A(y)$ if and only if $C(x) \leq C(y)$.

Proof. Let $A(x) \subseteq A(y)$. Then there exists $y' \in A(y)$ such that $x \leq y'$. Let $z \in X$ with $z < x$. If $y \leq z$ then $y < x$ which is a contradiction to $A(x) \subseteq A(y)$. If y and z are incomparable, then we have $z \in B(y)$ but $z < y'$, which again is a contradiction. Thus $z < y$, and we have $C(x) \leq C(y)$. Conversely, suppose that $C(x) \leq C(y)$. Then $x \leq y$ (and hence also $A(x) \subseteq A(y)$) or $x \in \text{Min}(B(y))$. Let $x' \in A(x)$, and suppose there exists $y' \in A(y)$ such that $y' < x'$. Thus $y' < x$ but not $y' < y$, which is a contradiction. Therefore, as $A(y)$ is a maximal antichain, there exists $y'' \in A(y)$ such that $x' \leq y''$, and hence $A(x) \subseteq A(y)$.

We now can give the main result of this section.

Theorem 3.5. Let (X, \leq) be a finite poset. Then the lattice of dissecting ideals of (X, \leq) is isomorphic to the lattice of all ideals of $(E(X), \leq)$.

Proof. By the preceding Lemmas, the poset of join-irreducible dissecting ideals is isomorphic to $(E(X), \leq)$. The rest follows from [2], Ch. III, Thm. 3.

4. A representation theorem

We shall finally give a characterization of those finite lattices which are isomorphic to the lattice of dissecting ideals of a partially ordered set. In view of Theorem 3.5 this mainly reduces to the problem of characterizing the posets $(E(X), \leq)$.

Lemma 4.1. Let (X, \leq) be a finite poset with a unique minimal element. Then there exists a finite poset (Y, \leq) such that (X, \leq) is isomorphic to $(E(Y), \leq)$.

Proof. Let x_0 be the minimal element of X . Define $Y = X \times X$, and for $(x_1, x_2), (x'_1, x'_2) \in Y$ let $(x_1, x_2) \leq (x'_1, x'_2)$ if and only if $(x_1, x_2) = (x'_1, x'_2)$ or $x_1 < x'_1$ and $x_2 \leq x'_2$. It is clear that this is a partial order on Y . Now let $(x, y_1), (x, y_2) \in Y$. If $(z_1, z_2) \in Y$ is such that $(z_1, z_2) < (x, y_1)$ then $z_1 < x$ and $z_2 \leq x$. Thus we also have $(z_1, z_2) < (x, y_2)$. By symmetry, we get $(x, y_1) \sim (x, y_2)$. On the other hand, let $(x_1, y_1), (x_2, y_2) \in Y$ such that $x_1 \neq x_2$. If $x_1 < x_2$ then $(x_1, x_1) < (x_2, y_2)$ and $(x_1, x_2) < (x_2, y_2)$, but at least one of (x_1, x_1) and (x_1, x_2) is incomparable with (x_1, y_1) . If x_1 and x_2 are incomparable, then both are distinct from x_0 , and we have $(x_0, x_1) < (x_1, y_1)$ but we do not have $(x_0, x_1) < (x_2, y_2)$. Thus we have seen that $E(Y) = \{(x, y) \mid y \in X \mid x \in X\}$, and it is easy to see in a similar way that the mapping $x \mapsto \{(x, y) \mid y \in X\}$ gives the desired isomorphism.

Theorem 4.2. Let (L, \leq) be a finite distributive lattice with more than one element. Then (L, \leq) is isomorphic to the lattice of dissecting ideals of a finite poset if and only if (L, \leq) contains a unique atom.

Proof. It is clear that a finite distributive lattice

contains a unique atom if and only if the poset of its join-irreducible elements contains a unique minimal element. By Lemma 4.1 and Theorem 3.5 every finite distributive lattice with a unique atom is isomorphic to the lattice of dissecting ideals of a finite poset. The converse follows from Theorem 2.1 and the remark after Theorem 2.3.

As a sequence we get that every finite distributive lattice (L, \leq) is isomorphic to the lattice generated by the maximal antichains of some finite poset (X, \leq) (which is the lattice of non-empty dissecting ideals of (X, \leq)).

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