Valeriu Popa

SOME FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS SATISFYING AN IMPLICIT RELATION

1. Introduction

Let $S$ and $T$ be two self mappings of a metric space $(X, d)$ Sessa [2] defines $S$ and $T$ to be weakly commuting if $d(STx, TSx) < d(Tx, Sx)$ for all $x$ in $X$. Jungck [1] defines $S$ and $T$ to be compatible if $\lim_{n \to \infty} (STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x$ for some $x$ in $X$. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but neither implications is reversible [3, Ex. 1] and [1, Ex. 2.2].

Lemma 1 [4]. Let $f$ and $g$ be two self mappings of the set $X = \{x, y\}$ with any metric $d$. If the range of $g$ contains the range of $f$, then the following statements are equivalent:

1) $f$ and $g$ commute,
2) $f$ and $g$ weakly commute,
3) $f$ and $g$ are compatible.

By Lemma 1, we suppose that $X$ contains at least three points.

Lemma 2 [1]. Let $f$ and $g$ be compatible self mappings on a metric space $(X, d)$. If $f(t) = g(t)$, then $fg(t) = gf(t)$.

The purpose of this paper is to prove some fixed point theorems for compatible mappings satisfying an implicit relation.

2. Implicit relations

Let $F$ be the set of all real continuous functions $F(t_1, \ldots, t_6) : R_+^6 \to R$ satisfying the following conditions:
$F_1$: $F$ is non-increasing in variables $t_5$ and $t_6$.

$F_2$: there exists $h \in (0,1)$ such that for every $u, v \geq 0$ with

$(F_a)$: $F(u, v, u, u + v, 0) \leq 0$ or

$(F_b)$: $F(u, v, u, v, 0, u + v) \leq 0$

we have $u \leq h \cdot v$.

$F_3$: $F(u, u, 0, 0, u, u) > 0, \forall u > 0$.

**EXAMPLE 1.** $F(t_1, \ldots, t_6) = t_1 - k \cdot \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, where $k \in (0, 1)$. $F_1$: Obviously.

$(F_a)$: Let be $u > 0$ and $F(u, v, v, u, u + v, 0) = u - k \cdot \max\{v, v, u, \frac{1}{2}(u + v)\} \leq 0$. If $u \geq v$, then $u \leq k \cdot u < u$, a contradiction. Thus $u < v$ and $u \leq v = h \cdot v$, where $h = k \in (0, 1)$.

$(F_b)$: Let be $u > 0$ and $F(u, v, u, v, 0, u + v) \leq 0$, then $u \leq h \cdot v$. If $u = 0$, then $u \leq h \cdot v$.

$F_3$: $F(u, u, 0, 0, u, u) = u - k \cdot u = (1 - k)u > 0, \forall u > 0$.

**EXAMPLE 2.** $F(t_1, \ldots, t_6) = \frac{t_1^2}{1 - c_1} \cdot \max\{t_2^2, t_3^2, t_4^2\} - c_2 \cdot \max\{t_3t_5, t_4t_6\} - c_3t_5t_6$, where $c_1 > 0$, $c_2, c_3 \geq 0$, $c_1 + 2c_2 < 1$, and $c_1 + c_3 < 1$.

$F_1$: Obviously.

$(F_a)$: Let be $u > 0$ and $F(u, v, u, u + v, 0) = u^2 - c_1 \cdot \max\{u^2, v^2\} - c_2 v(u + v) \leq 0$. If $u \geq v$, then $u^2(1 - c_1 - 2c_2) \leq 0$ which implies $c_1 + 2c_2 \geq 1$, a contradiction. Thus $u < v$ and $u \leq \sqrt{(c_1 + 2c_2)v} = hv$, where $h = \sqrt{c_1 + 2c_2} < 1$.

$(F_b)$: Let be $u > 0$ and $F(u, v, u, v, 0, u + v) = u^2 - u(av + bu + cv) \leq 0$. Then $u \leq \frac{a + c + d}{1 - b} v$, where $h_2 = \frac{a + c + d}{1 - b} < 1$. Therefore, $u \leq h, v$ where $h = \max\{h_1, h_2\}$.

If $u = 0$ then $u \leq hv$.

$F_3$: $F(u, u, 0, 0, u, u) = u^2(1 - (a + d)) > 0, \forall u > 0$.

**EXAMPLE 3.** $F(t_1, \ldots, t_6) = \frac{t_1^3}{1 - a} - at_2t_2 - bt_1t_3 + ct_4 - dt_5t_6$, where $a > 0$, $b, c, d \geq 0$, $a + b < 1$ and $a + c + d < 1$.

$F_1$: Obviously.
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\((F_a)\): Let be \(u > 0\) and \(F(u, v, v, u, u + v, 0) = u^3 - au^2v - bu^2v \leq 0\),
then \(u \leq (a + b)v = hv\), where \(h = a + b < 1\).

\((F_b)\): Let be \(u > 0\) and \(F(u, v, u, v, 0, u + v) \leq 0\), then \(u \leq hv\). If \(u = 0\),
then \(u \leq hv\).

\(F_3\): \(F(u, u, 0, 0, u, u) = u^3(1 - (a + d + c)) > 0, \forall u > 0\).

Example 5. \(F(t_1, \ldots, t_6) = t_3^3 - c \cdot \frac{t_4^2t_5^2 + t_6^2}{t_2 + t_3 + t_4 + 1}\), where \(c \in (0, 1)\).

\(F_1\): Obviously.

\((F_a)\): Let be \(u > 0\) and \(F(u, v, v, u + v, 0) = u^3 - \frac{av^2}{u+2v+1} \leq 0\) which
implies \(u \leq \frac{cv^2}{2v+u+1}\). But \(\frac{cv^2}{2v+u+1} \leq cv\) is equivalent to \(u + v + 1 > 0\) an
evident relation. Thus \(u \leq cv = hv\), where \(h = c < 1\).

\((F_b)\): Let be \(u > 0\) and \(F(u, v, v, 0, u + v) \leq 0\), then \(u \leq hv\). If \(u = 0\)
then \(u \leq hv\).

\(F_3\): \(F(u, u, 0, 0, u, u)u^3 - \frac{cuv}{v+1} = u^3 \cdot \frac{(1-c)u+1}{u+1} > 0, \forall u > 0\).

Remark. There exists a function \(F \in \mathcal{F}\) which is increasing in variables \(t_3\) or \(t_4\).

Example 6. \(F(t_1, t_2, \ldots, t_6) = t_1^2 - at_2^2 - \frac{bt_3t_5}{t_2 + t_3 + t_4 + 1}\), where \(a > 0, b \geq 0\) and
\(a + b < 1\).

\(F_1\): Obviously.

\((F_a)\): Let be \(u > 0\) and \(F(u, v, u, u + v, 0) = u^2 - av^2 \leq 0\) which implies
\(u \leq a^\frac{1}{2}v = vh\), where \(h = a^\frac{1}{2} < 1\).

\((F_b)\): Let \(u > 0\) be and \(F(u, v, u, 0, u + v) \leq 0\), then \(u \leq hv\). If \(u = 0\),
then \(u \leq hv\).

\(F_3\): \(F(u, u, 0, 0, u, u) = u^2(1 - a - b) > 0, \forall u > 0\).

3. Common fixed point theorems

Theorem 1. Let \((X, d)\) be a metric space and \(S, T, I, J : (X, d) \to (X, d)\) four mappings satisfying the inequality

\[(1) \quad F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \leq 0\]

for all \(x, y\) in \(X\), where \(F\) satisfies property \((F_3)\). Then \(S, T, I, J\) have at
most one common fixed point.

Proof. Suppose that \(S, T, I, J\) have two common fixed point \(z\) and \(z'\) with
\(z \neq z'\). Then by \((1)\) we have

\[F(d(Sz, Tz'), d(Iz, Jz'), d(Iz, Sz), d(Jz', Tz'), d(Iz, Tz'), d(Jz', Sz)) = F(d(z, z'), d(z, z'), 0, 0, d(z, z'), d(z, z')) \leq 0,\]
a contradiction to \((F_3)\).
THEOREM 2. Let $S, T, I, J$ be mappings from a complete metric space $(X, d)$ into itself satisfying the conditions:

(a) $S(X) \subset J(X)$ and $T(X) \subset I(X)$,
(b) one of $S, T, I, J$ is continuous,
(c) $S$ and $I$ as well as $T$ and $J$ are compatible,
(d) the inequality (1) holds for all $x, y$ in $X$, where $F \in F$. Then $S, T, I, J$ have a unique common fixed point.

Proof. Suppose $x_0$ an arbitrary point in $X$. Then, since (a) holds, we can define inductively a sequence

$$\{Sx_0, Tx_1, Sx_2, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots\}$$

such that $Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$ Using inequality (1), we have successively

$$F(d(Sx_{2n}, Tx_{2n+1}), d(Ix_{2n}, Sx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(Ix_{2n+1}, Sx_{2n})) \leq 0,$$

$$F(d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Sx_{2n+1}), 0) \leq 0.$$ 

By $(F_a)$, we have

$$d(Sx_{2n}, Tx_{2n+1}) \leq h \cdot d(Tx_{2n-1}, Sx_{2n}).$$

Similarly, by $(F_b)$, we have

$$d(Sx_{2n}, Tx_{2n-1}) \leq h \cdot d(Sx_{2n-2}, Tx_{2n-1})$$

and so

$$d(Sx_{2n}, Tx_{2n-1}) \leq (h)^2 d(Sx_0, Tx_1) \quad \text{for } n = 0, 1, 2, \ldots.$$ 

By a routine calculation it follows that (2) is a Cauchy sequence. Since $X$ is complete, the sequence (2) converges to a point $z$ in $X$. Hence, $z$ is also the limit of the subsequences $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ of (2).

Let us now suppose that $I$ is continuous, so that the sequence $\{ISx_{2n}\}$ converges to $Iz$. We have

$$d(ISx_{2n}, Iz) \leq d(ISx_{2n}, ISx_{2n}) + d(ISx_{2n}, Iz).$$

Since $I$ is continuous and $S$ and $I$ are compatible, letting $n$ tend to infinity, we state that the sequence $\{ISx_{2n}\}$ also converges to $Iz$. Using (1), we have

$$F(d(ISx_{2n}, Tx_{2n+1}), d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, ISx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(Jx_{2n+1}, Tx_{2n+1}), d(Jx_{2n+1}, ISx_{2n})) \leq 0.$$
Letting \( n \) tend to infinity we have, by the continuity of \( F \),

\[
F(d(Iz, z), d(Iz, z), 0, 0, d(Iz, z), d(z, z)) \leq 0,
\]
a contradiction to \((F_3)\), if \( d(Iz, z) \neq 0 \). Thus \( Iz = z \). Further, by (1), we have

\[
F(d(Sz, Tx_{2n+1}), d(Iz, Jx_{2n+1}), d(Iz, Sz),
\]
\[
d(Jx_{2n+1}, Tx_{2n+1}), d(Iz, Tx_{2n+1}), d(Jx_{2n+1}, Sz)) \leq 0
\]
and letting \( n \) tend to infinity we get

\[
F(d(Sz, z), 0, d(z, Sz), 0, 0, d(z, Sz)) \leq 0
\]
which implies, by \((F_3)\), that \( z = Sz \). This means that \( z \) is in the range of \( S \) and, since \( S(X) \subset J(X) \), there exists a point \( u \) in \( X \) such that \( Ju = z \).

Using (1), we have successively

\[
F(d(Sz, Tu), d(Iz, Ju), d(Iz, Sz), d(Ju, Tu), d(Iz, Tu), d(Ju, Sz))
\]
\[
= F(d(z, Tu), 0, 0, d(z, Tu), d(z, Tu), 0) \leq 0
\]
which implies by \((F_3)\), that \( z = Tu \).

Since \( Ju = Tu = z \), by Lemma 2, it follows that \( TJu = JTu \) and so \( Tz = TJu = JTu = Jz \). Thus, from (1) we have

\[
F(d(Sz, Tz), d(Iz, Jz), d(Iz, Sz), d(Jz, Tz), d(Iz, Tz), d(Jz, Sz))
\]
\[
= F(d(z, Tz), d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) \leq 0,
\]
a contradiction to \((F_3)\), if \( z \neq Tz \). Thus \( z = Tz = Jz \). We have therefore proved that \( z \) is a common fixed point of \( S, T, I, J \). The same result holds, if we assume that \( J \) is continuous instead of \( I \).

Now suppose that \( S \) is continuous. Then the sequence \( \{SIx_{2n}\} \) converges to \( Sz \). We have

\[
d(ISx_{2n}, Sz) \leq d(ISx_{2n}, SIZx_{2n}) + d(SIZx_{2n}, Sz).
\]

Since \( S \) is continuous and \( S \) and \( T \) are compatible, letting \( n \) tend to infinity, we state that \( \{ISx_{2n}\} \) converges to \( Sz \). Using the inequality (1), we have

\[
F(d(S^2x_{2n}, Tx_{2n+1}), d(ISx_{2n}, Jx_{2n+1}), d(ISx_{2n}, S^2x_{2n}),
\]
\[
d(Jx_{2n+1}, Tx_{2n+1}), d(ISx_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, S^2x_{2n})) \leq 0.
\]
Letting \( n \) tend to infinity, we have, by continuity of \( F \),

\[
F(d(Sz, z), d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)) \leq 0,
\]
a contradiction to \((F_3)\) if \( z \neq Sz \). Thus \( z = Sz \). This means that \( z \) is in the range of \( S \) and, since \( S(X) \subset J(X) \), there exists a point \( v \) in \( X \) such that \( Jv = z \). Thus, by (1), we have
\[ F(d(s^2x_2^n, Tv), d(ISx_2^n, Jv), d(ISx_2^n, S^2x_2^n), \\
   d(Jv, Tv), d(ISx_2^n, Tv), d(Jv, S^2x_2^n)) \leq 0. \]

Letting \( n \) tend to infinity we get
\[ F(d(z, Tv), 0, 0, d(z, Tv), d(z, Tv), 0) \leq 0 \]
and, by \((F_a)\), it follows that \( z = Tv \). Since \( Jv = Tv = z \), by Lemma 2, it follows that \( Tz = TJv = JTv = Jz \).

Thus, from (1) we have
\[ F(d(Sx_2^n, Tz), d(Ix_2^n, Jz), d(Ix_2^n, Sz_2^n), \\
   d(Jz, Tz), d(Ix_2^n, Tz), d(Jz, Sz_2^n)) < 0. \]

Letting \( n \) tend to infinity, we obtain
\[ F(d(z, Tz), d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) < 0 \]
and, by \((F_b)\), it follows that \( z = Sw = Tw \). Since \( Sw = Tw = z \), by Lemma 2, it follows that \( z = Sz = Stw = Iz \) and thus \( z = Iz \). We have therefore proved that \( z \) is a common fixed point of \( S, T, I \) and \( J \).

The same result holds, if we assume that \( T \) is continuous instead of \( S \). By Theorem 1, \( z \) is the unique common fixed point of \( S, T, I, J \).

For a function \( f : (X, d) \to (X, d) \) we denote \( F_f = \{ x \in X : x = f(x) \} \).

**Theorem 3.** Let \( I, J, S, T \) be mappings from a metric space \( (X, d) \) into itself. If the inequality (1) holds for all \( x, y \) in \( X \) then \((F_I \cap F_J) \cap FS = (F_I \cap F_J) \cap F_T\).

**Proof.** Let \( x \in (F_I \cap F_J) \cap FS \). Then, by (1), we have
\[ F(d(Sx, Tx), d(Ix, Jx), d(Ix, Sx), d(Jx, Tx), d(Ix, Tx), d(Jx, Sx)) \\
   = F(d(x, Tx), 0, 0, d(x, Tx), d(x, Tx), 0)) \leq 0 \]
which implies, by \((F_a)\), that \( x = Tx \). Thus \((F_I \cap F_J) \cap FS \subset (F_I \cap F_J) \cap F_T\).

Similarly, we have by \((F_b)\), that \((F_I \cap F_J) \cap FT \subset (F_I \cap F_J) \cap FS\).

The Theorems 2 and 3 imply the following one.

**Theorem 4.** Let \( I, J \) and \( \{T_i\}_{i \in N} \) be mappings from a complete metric space into itself such that
(a) \( T_2(X) \subset I(X) \) and \( T_1(X) \subset J(X) \),
(b) one of \( I, J, T_1, \) and \( T_2 \) is continuous,
(c) the pairs \((T_1, I)\) and \((T_2, J)\) are compatible,
(d) the inequality
\[ F(d(T_i x, T_{i+1} y), d(I x, J y), d(I x, T_i x), \]
\[ d(J y, T_{i+1} y), d(I x, T_{i+1} y), d(J y, T_i x)) \leq 0 \]
holds for each \( x, y \) in \( X \), \( \forall i \in \mathbb{N}^\ast \) and \( F \in \mathcal{F} \). Then \( I, J, \) and \( \{T_i\}_{i \in \mathbb{N}^\ast} \) have a unique common fixed point.

References


DEPARTMENT OF MATHEMATICS AND PHYSICS
UNIVERSITY OF BACĂU
5500- BACĂU, ROMANIA

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