1. Introduction

Let $S$ and $T$ be two self mappings of a metric space $(X, d)$ Sessa [2] defines $S$ and $T$ to be weakly commuting if $d(STx, TSx) \leq d(Tx, Sx)$ for all $x$ in $X$. Jungck [1] defines $S$ and $T$ to be compatible if $\lim_{n \to \infty} (STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = x$ for some $x$ in $X$. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but neither implications is reversible [3, Ex. 1] and [1, Ex. 2.2].

**Lemma 1** [4]. Let $f$ and $g$ be two self mappings of the set $X = \{x, y\}$ with any metric $d$. If the range of $g$ contains the range of $f$, then the following statements are equivalent:

1) $f$ and $g$ commute,
2) $f$ and $g$ weakly commute,
3) $f$ and $g$ are compatible.

By Lemma 1, we suppose that $X$ contains at least three points.

**Lemma 2** [1]. Let $f$ and $g$ be compatible self mappings on a metric space $(X, d)$. If $f(t) = g(t)$, then $fg(t) = gf(t)$.

The purpose of this paper is to prove some fixed point theorems for compatible mappings satisfying an implicit relation.

2. Implicit relations

Let $\mathcal{F}$ be the set of all real continuous functions $F(t_1, \ldots, t_6) : R^6_+ \to R$ satisfying the following conditions:
F₁: F is non-increasing in variables t₅ and t₆,
F₂: there exists h ∈ (0, 1) such that for every u, v ≥ 0 with
(Fₐ): F(u, v, v, u, u + v, 0) ≤ 0 or
(F₇): F(u, v, u, v, 0, u + v) ≤ 0
we have u ≤ h · v.
F₃: F(u, u, 0, 0, u, u) > 0, ∀u > 0.

EXAMPLE 1. F(t₁, ..., t₆) = t₁ - k · max{t₂, t₃, t₄, ½(t₅ + t₆)}, where k ∈ (0, 1). F₁: Obviously.
(Fₐ): Let be u > 0 and F(u, v, v, u, u + v, 0) = u - k · max {v, v, u, ½(u + v)} ≤ 0. If u ≥ v, then u ≤ k · u < u, a contradiction. Thus u < v and u ≤ k · v = h · v, where h = k ∈ (0, 1).
(F₇): Let be u > 0 and F(u, v, u, v, 0, u + v) ≤ 0, then u ≤ h · v. If u = 0, then u ≤ h · v.
F₃: F(u, u, 0, 0, u, u) = u - k · u = (1 - k)u > 0, ∀u > 0.

EXAMPLE 2. F(t₁, ..., t₆) = t₁² - c₁ · max{t₂², t₃², t₄²} - c₂ · max{t₅t₆, t₅t₆} - c₃t₅t₆, where c₁ > 0, c₂ ≥ 0, c₁ + 2c₂ < 1, and c₁ + c₃ < 1.
F₁: Obviously.
(Fₐ): Let be u > 0 and F(u, v, u, v, 0, u + v) = u² - c₁ · max{u², v²} - c₂v(u + v) ≤ 0. If u ≥ v, then u²(1 - c₁ - 2c₂) ≤ 0 which implies c₁ + 2c₂ ≥ 1, a contradiction. Thus u < v and u ≤ √(c₁ + 2c₂)v = hv, where h = √c₁ + 2c₂ < 1.
(F₇): Let be u > 0 and F(u, v, u, v, 0, u + v) = u² - u(av + bu + cv) < 0. Then u ≤ (⁴⁺civ) v = h₂v, where h₂ = ½⁺civ < 1. Therefore, u ≤ h, v where h = max{h₁, h₂}.
If u = 0 then u ≤ hv.
F₃: F(u, u, 0, 0, u, u) = u²(1 - (c₁ + c₃)) > 0, ∀u > 0.

EXAMPLE 3. F(t₁, ..., t₆) = t₁² - t₁(2t₂ + bt₃ + ct₄) - dt₅t₆, where a > 0, b, c, d ≥ 0, a + b + c < 1 and a + d < 1.
F₁: Obviously.
(Fₐ): Let be u > 0 and F(u, v, u, v, 0, u + v) = u² - u(av + bv + cu) ≤ 0. Then u ≤ (⁴⁺civ) · v = h₁ · v, where h₁ = ½⁺civ < 1.
(F₇): Let be u > 0 and F(u, v, u, v, 0, u + v) = u² - u(av + bu + cv) ≤ 0. Then u ≤ (⁴⁺civ) v = h₂v, where h₂ = ½⁺civ < 1. Therefore, u ≤ h, v where h = max{h₁, h₂}.
If u = 0 then u ≤ hv.
F₃: F(u, u, 0, 0, u, u) = u²(1 - (a + d)) > 0, ∀u > 0.

EXAMPLE 4. F(t₁, ..., t₆) = t₁³ - at₁t₂ - bt₁t₃t₄ - ct₂t₆ - dt₅t₆², where a > 0, b, c, d ≥ 0, a + b < 1 and a + c + d < 1.
F₁: Obviously.
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(Fa): Let be \( u > 0 \) and \( F(u, v, v, u, u + v, 0) = u^3 - au^2v - bu^2v \leq 0 \), then \( u \leq (a + b)v = hv \), where \( h = a + b < 1 \).

(Fb): Let be \( u > 0 \) and \( F(u, v, u, v, 0, u + v) \leq 0 \), then \( u \leq hv \). If \( u = 0 \), then \( u \leq hv \).

\( F_3: F(u, u, 0, 0, u, u) = u^3(1 - (a + d + c)) > 0, \forall u > 0. \)

**Example 5.** \( F(t_1, \ldots, t_6) = t_1^3 - c \cdot \frac{t_2 t_4 + t_2 t_4^2}{t_2 + t_4 + 1} \), where \( c \in (0, 1) \).

\( F_1: \) Obviously.

\( (F_a): \) Let be \( u > 0 \) and \( F(u, v, v, u, u + v, 0) = u^3 - \frac{cu^2v}{u+2v+1} \leq 0 \) which implies \( u \leq \frac{cv}{2v+u+1} \). But \( \frac{cv}{2v+u+1} \leq cv \) is equivalent to \( u + v + 1 > 0 \) an evident relation. Thus \( u \leq cv = hv \), where \( h = c < 1 \).

\( (F_b): \) Let be \( u > 0 \) and \( F(u, v, u, v, 0, u + v) \leq 0 \), then \( u \leq hv \). If \( u = 0 \) then \( u \leq hv \).

\( F_3: F(u, u, 0, 0, u, u)u^3 - \frac{cu^4}{v+1} = u^3 \cdot \frac{(1-c)u+1}{u+1} > 0, \forall u > 0. \)

**Remark.** There exists a function \( F \in F \) which is increasing in variables \( t_3 \) or \( t_4 \).

**Example 6.** \( F(t_1, t_2, \ldots, t_6) = t_1^2 - a t_2^2 - \frac{b t_3 t_6}{t_2 + t_4 + 1} \), where \( a > 0, b \geq 0 \) and \( a + b < 1 \).

\( F_1: \) Obviously.

\( (F_a): \) Let be \( u > 0 \) and \( F(u, v, v, u, u + v, 0) = u^2 - av^2 \leq 0 \) which implies \( u \leq a^\frac{1}{2}v = vh \), where \( h = a^\frac{1}{2} < 1 \).

\( (F_b): \) Let be \( u > 0 \) be and \( F(u, v, u, v, 0, u + v) \leq 0 \), then \( u \leq hv \). If \( u = 0 \), then \( u \leq hv \).

\( F_3: F(u, u, 0, 0, u, u) = u^2(1 - a - b) > 0, \forall u > 0. \)

3. Common fixed point theorems

**Theorem 1.** Let \( (X, d) \) be a metric space and \( S, T, I, J : (X, d) \rightarrow (X, d) \) four mappings satisfying the inequality

(1) \( F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \leq 0 \)

for all \( x, y \) in \( X \), where \( F \) satisfies property \( (F_3) \). Then \( S, T, I, J \) have at most one common fixed point.

**Proof.** Suppose that \( S, T, I, J \) have two common fixed points \( z \) and \( z' \) with \( z \neq z' \). Then by (1) we have

\[
F(d(Sz, Tz'), d(Iz, Jz'), d(Iz, Sz), d(Jz', Tz'), d(Iz, Tz'), d(Jz', Sz)) = F(d(z, z'), d(z, z'), 0, 0, d(z, z'), d(z, z')) \leq 0,
\]

a contradiction to \( (F_3) \).
THEOREM 2. Let $S, T, I, J$ be mappings from a complete metric space $(X, d)$ into itself satisfying the conditions:

(a) $S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$,
(b) one of $S, T, I, J$ is continuous,
(c) $S$ and $I$ as well as $T$ and $J$ are compatible,
(d) the inequality (1) holds for all $x, y$ in $X$, where $F \in \mathcal{F}$. Then $S, T, I, J$ have a unique common fixed point.

Proof. Suppose $x_0$ an arbitrary point in $X$. Then, since (a) holds, we can define inductively a sequence

$$\{Sx_0, Tx_1, Sx_2, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots\}$$

such that $Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$. Using inequality (1), we have successively

$$F(d(Sx_{2n}, Tx_{2n+1}), d(Ix_{2n+1}, Sx_{2n}),$$
$$d(Jx_{2n+1}, Tx_{2n+1}), d(Ix_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, Sx_{2n}) \leq 0,$$
$$F(d(Sx_{2n}, Tx_{2n+1}), d(Ix_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}),$$
$$d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}), 0) \leq 0.$$

By (F_a), we have

$$d(Sx_{2n}, Tx_{2n+1}) \leq h \cdot d(Tx_{2n-1}, Sx_{2n}).$$

Similarly, by (F_b), we have

$$d(Sx_{2n}, Tx_{2n-1}) \leq h \cdot d(Sx_{2n-2}, Tx_{2n-1})$$

and so

$$d(Sx_{2n}, Tx_{2n-1}) \leq (h)^2 d(Sx_0, Tx_1) \quad \text{for } n = 0, 1, 2, \ldots.$$ 

By a routine calculation it follows that (2) is a Cauchy sequence. Since $X$ is complete, the sequence (2) converges to a point $z$ in $X$. Hence, $z$ is also the limit of the subsequences $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ of (2).

Let us now suppose that $I$ is continuous, so that the sequence $\{ISx_{2n}\}$ converges to $Iz$. We have

$$d(ISx_{2n}, Iz) \leq d(ISx_{2n}, ISx_{2n}) + d(ISx_{2n}, Iz).$$

Since $I$ is continuous and $S$ and $I$ are compatible, letting $n$ tend to infinity, we state that the sequence $\{ISx_{2n}\}$ also converges to $Iz$. Using (1), we have

$$F(d(ISx_{2n}, Tx_{2n+1}), d(I^2x_{2n}Jx_{2n+1}), d(I^2x_{2n}, ISx_{2n}),$$
$$d(Jx_{2n+1}, Tx_{2n+1}), d(I^2x_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, ISx_{2n})) \leq 0.$$
Letting $n$ tend to infinity we have, by the continuity of $F$,

$$F(d(Iz, z), d(Iz, z), 0, 0, d(Iz, z), d(z, z)) \leq 0,$$

a contradiction to $(F_3)$, if $d(Iz, z) \neq 0$. Thus $Iz = z$. Further, by (1), we have

$$F(d(Sz, Tx_{2n+1}), d(Iz, Jx_{2n+1}), d(Iz, Sz),$$

$$d(Jx_{2n+1}, Tx_{2n+1}), d(Iz, Tx_{2n+1}), d(Jx_{2n+1}, Sz)) \leq 0$$

and letting $n$ tend to infinity we get

$$F(d(Sz, z), 0, d(z, Sz), 0, 0, d(z, Sz)) \leq 0$$

which implies, by $(F_b)$, that $z = Sz$. This means that $z$ is in the range of $S$ and, since $S(X) \subset J(X)$, there exists a point $v$ in $X$ such that $Ju = z$. Using (1), we have successively

$$F(d(Sz, Tu), d(Iz, Ju), d(Iz, Sz), d(Ju, Tu), d(Iz, Tu), d(Ju, Sz))$$

$$= F(d(z, Tu), 0, 0, d(z, Tu), d(z, Tu), 0) \leq 0$$

which implies by $(F_a)$, that $z = Tu$.

Since $Ju = Tu = z$, by Lemma 2, it follows that $TJu = JTu$ and so $Tz = TJu = JTu = Jz$. Thus, from (1) we have

$$F(d(Sz, Tz), d(Iz, Jz), d(Iz, Sz), d(Jz, Tz), d(Iz, Tz), d(Jz, Sz))$$

$$= F(d(z, Tz), d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) \leq 0,$$

a contradiction to $(F_3)$, if $z \neq Tz$. Thus $z = Tz = Jz$. We have therefore proved that $z$ is a common fixed point of $S, T, I, J$. The same result holds, if we assume that $J$ is continuous instead of $I$.

Now suppose that $S$ is continuous. Then the sequence $\{Sx_{2n}\}$ converges to $Sz$. We have

$$d(ISx_{2n}, Sz) \leq d(ISx_{2n}, Sx_{2n}) + d(Sx_{2n}, Sz).$$

Since $S$ is continuous and $S$ and $T$ are compatible, letting $n$ tend to infinity, we state that $\{Sx_{2n}\}$ converges to $Sz$. Using the inequality (1), we have

$$F(d(S^2x_{2n}, Tx_{2n+1}), d(ISx_{2n}, Jx_{2n+1}), d(ISx_{2n}, S^2x_{2n}),$$

$$d(Jx_{2n+1}, Tx_{2n+1}), d(ISx_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, S^2x_{2n})) \leq 0.$$

Letting $n$ tend to infinity, we have, by continuity of $F$,

$$F(d(Sz, z), d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)) \leq 0,$$

a contradiction to $(F_3)$ if $z \neq Sz$. Thus $z = Sz$. This means that $z$ is in the range of $S$ and, since $S(X) \subset J(X)$, there exists a point $v$ in $X$ such that $Ju = z$. Thus, by (1), we have
Letting \( n \) tend to infinity we get
\[
F(d(z, Tv), 0, 0, d(z, Tv), d(z, Tv), 0) < 0
\]
and, by \((F_a)\), it follows that \( z = Tv \). Since \( Tv = Jv \), by Lemma 2, it follows that \( Tz = TJu = JTv = Jz \). Thus, from (1) we have
\[
F(d(Sx_{2n}, Tz), d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}),
\quad d(Jz, Tz), d(Ix_{2n}, Tz), d(Jz, Sx_{2n})) 
\leq 0.
\]
Letting \( n \) tend to infinity, we obtain
\[
F(d(z, Tz), d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) < 0
\]
and, by \((F_b)\), we have \( z = Sw = Iw \). Since \( Sw = Iw = z \), by Lemma 2, it follows that \( z = Sz = SIw = Iz \) and thus \( z = Iz \). We have therefore proved that \( z \) is a common fixed point of \( S, T, I \) and \( J \).

The same result holds, if we assume that \( T \) is continuous instead of \( S \).

By Theorem 1, \( z \) is the unique common fixed point of \( S, T, I, J \).

For a function \( f : (X, d) \to (X, d) \) we denote \( F_f = \{ x \in X : x = f(x) \} \).

**Theorem 3.** Let \( I, J, S, T \) be mappings from a metric space \((X, d)\) into itself. If the inequality (1) holds for all \( x, y \) in \( X \) then \((F_I \cap F_J) \cap F_S = (F_I \cap F_J) \cap F_T\).

**Proof.** Let \( x \in (F_I \cap F_J) \cap F_S \). Then, by (1), we have
\[
F(d(Sx, Tx), d(Ix, Jx), d(Ix, Sx), d(Jx, Tx), d(Ix, Tx), d(Jx, Sx))
\quad = F(d(x, Tx), 0, 0, d(x, Tx), d(x, Tx), 0)) \leq 0
\]
which implies, by \((F_a)\), that \( x = Tx \). Thus \((F_I \cap F_J) \cap F_S \subset (F_I \cap F_J) \cap F_T\).

Similarly, we have by \((F_b)\), that \((F_I \cap F_J) \cap F_T \subset (F_I \cap F_J) \cap F_S\).

The Theorems 2 and 3 imply the following one.

**Theorem 4.** Let \( I, J \) and \( \{T_i\}_{i \in \mathbb{N}} \) be mappings from a complete metric space into itself such that
(a) \( T_2(X) \subset I(X) \) and \( T_1(X) \subset J(X) \),
(b) one of \( I, J, T_1 \) and \( T_2 \) is continuous,
(c) the pairs \((T_1, I)\) and \((T_2, J)\) are compatible,
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(d) the inequality

$$F(d(T_i x, T_{i+1} y), d(I x, J y), d(I x, T_i x),
   d(J y, T_{i+1} y), d(I x, T_{i+1} y), d(J y, T_i x)) \leq 0$$

holds for each $x, y$ in $X$, $\forall i \in \mathbb{N}^*$ and $F \in \mathcal{F}$. Then $I, J$, and $\{T_i\}_{i \in \mathbb{N}^*}$ have a unique common fixed point.

References


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