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QUALITATIVE BEHAVIOUR OF AN INTEGRAL EQUATION RELATED TO SOME EPIDEMIC MODEL

Abstract. Sufficient conditions for the uniqueness, global existence and for the convergence to zero when $t \to \infty$ of solutions of an integral equation related to an epidemic model are proved. The existence result is proved by applying the Banach fixed point theorem and for the proof of the convergence result a new type of integral inequality is used.

1. Introduction

G. Gripenberg studied in the paper [3] the qualitative behaviour of solutions of the equation

\[
x(t) = k \left( p(t) - \int_0^t A(t-s)x(s)ds \right) \left( f(t) - \int_0^t a(t-s)x(s)ds \right),
\]

which arises in the study of the spread of an infectious disease that does not induce the permanent immunity. The existence of a nonnegative, continuous and bounded solution of the equation (1) is proved there. B. G. Pachpatte [10] proved a result on the convergence to zero when $t \to \infty$ of solutions of this equation using the integral inequalities approach and the comparison method.

We shall study the following integral equation

\[
x(t) = \left( g_1(t) + \int_0^t A_1(t-s)F_1(s,x(s))ds \right) \ldots
\]

\[
\ldots \left( g_p(t) + \int_0^t A_p(t-s)F_p(s,x(s))ds \right).
\]

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First we prove a sufficient condition for the global existence and uniqueness of continuous solutions of the equation (2) and then we prove a sufficient condition for the convergence of solutions to zero when $t \to \infty$. In the proof of the convergence result we apply a new type of integral inequality derived in the present paper without using the comparison method. This our result is different from that proved by B. G. Pachpatte in [10]. In its proof we use some ideas from the papers [5]–[9].

2. Existence result

Using the Banach fixed point theorem we shall prove the following theorem on global existence and uniqueness of continuous solutions of the equation (2).

**Theorem 1.** Let $g_i(t), A_i(t), i = 1, 2, \ldots, p$ be continuous real-valued functions on $R_+ = (0, \infty)$ and $F_i(t,u), i = 1, 2, \ldots, p$ be continuous real-valued functions on $R_+ \times R$. Assume that $\delta = \max_{1 \leq i \leq p} \int_0^\infty |A_i(s)|ds < \infty$ and for every $i \in \{1, 2, \ldots, p\}$ the following conditions are satisfied:

(a) There is a constant $L > 0$ such that

$$|F_i(t,x) - F_i(t,y)| \leq L|x - y|$$

for all $(t, x), (t, y) \in R_+ \times R$.

(b) There is a constant $Q > 0$ and a positive number $m$ such that

$$|F_i(t,x)| \leq Q|x|^m$$

for all $(t, x) \in R_+ \times R$.

(c) There is a constant $M > 0$ such that

$$(S + \delta Q M^m)^p \leq M,$$

where $S = \max\{|g_i(t)| : i = 1, 2, \ldots, p, t \in R_+\} < \infty$.

Then there exists a unique continuous solution $x(t)$ of the equation (2) defined on $R_+$ with $|x(t)| \leq M$ for all $t \in R_+$.

**Proof.** Let $h$ be an arbitrary positive number and $C_h = C(I_h, R)$ be the space of real-valued continuous functions defined on the interval $I_h = (0, h)$ with the metric

$$d_r(f, g) = \max_{t \in I_h} \{e^{-rt}|f(t) - g(t)|\}, f, g \in C_h,$$

where the number $r > 0$ will be specified later. Let

$$C_h(M) = \{x \in C_h : |x(t)| \leq M, t \in I_h\}.$$
The set $C_h(M)$ is a close subset of $C_h$ and thus it is a complete metric space with the metric $d_r$. Let us define the following maps:

$$F_i : C_h(M) \to C(I_h, R), \quad i = 1, 2, \ldots, p,$$

$$F_i(x)(t) = g_i(t) + \int_0^t A_i(t - s)F_i(s, x(s))ds,$$

$$F : C_h(M) \to C(I_h, R),$$

$$F(x)(t) = F_1(x)(t)F_2(x)(t) \ldots F_p(x)(t).$$

Obviously, if $x \in C_h(M)$ then form the conditions (a) - (c) it follows that

$$|F_i(x)(t)| \leq S + \delta Q M^m, \quad i = 1, 2, \ldots, p, t \in I_h$$

and this yields

$$|F(x)(t)| \leq (S + \delta Q M^m)^p \leq M.$$ 

This means that the map $F$ maps the set $C_h(M)$ into itself. Now it suffices to prove the contractivity of the map $F$.

If $x, y \in C_h(M)$ then

$$|F(x)(t) - F(y)(t)|$$

$$= |F_1(x)(t) \ldots F_p(x)(t) - F_1(y)(t) \ldots F_p(y)(t)|$$

$$= |[F_1(x)(t)F_2(x)(t) \ldots F_p(x)(t) - F_1(y)(t)F_2(x)(t) \ldots F_p(x)(t)]$$

$$+ [F_1(y)(t)F_2(x)(t) \ldots F_p(x)(t) - F_1(y)(t)F_2(y)(t)F_3(x)(t) \ldots F_p(x)(t)]$$

$$+ \ldots +$$

$$+ [F_1(y)(t) \ldots F_{p-1}(y)(t)F_p(x)(t) - F_1(y)(t)F_2(y)(t) \ldots F_p(y)(t)]|$$

$$= |[F_1(x)(t) - F_1(y)(t)]F_2(x)(t) \ldots F_p(x)(t)$$

$$+ [F_2(x)(t) - F_2(y)(t)]F_1(y)(t)F_3(x)(t) \ldots F_p(x)(t) + \ldots$$

$$+ [F_p(x)(t) - F_p(y)(t)]F_1(y)(t) \ldots F_{p-1}(y)(t)]|$$

$$\leq (S + \delta Q M^m)^{p-1} \sum_{i=1}^p |F_i(x)(t) - F_i(y)(t)|.$$ 

Since

$$|F_i(x)(t) - F_i(y)(t)| \leq L \int_0^t |A_i(t - s)||x(s) - y(s)|ds$$

$$\leq L \delta \int_0^t e^{rs}e^{-rs}||x(s) - y(s)||ds \leq \{L \delta \frac{1}{r}(e^{rt} - 1)\}d_r(x, y), i = 1, 2, \ldots, p,$$
we have
\[ |\mathcal{F}(x)(t) - \mathcal{F}(y)(t)| \leq \left\{ p(S + \delta Q M^m)^{p-1} L \delta \frac{1}{r} (e^{rt} - 1) \right\} d_r(x, y) \]
and from this inequality we obtain
\[ d_r(\mathcal{F}(x), \mathcal{F}(y)) \leq p(S + \delta Q M^m)^{p-1} L \delta \frac{1}{r} (1 - e^{-rh}) d_r(x, y). \]
If we choose \( r > p(S + \delta Q M^m)^{p-1} L \delta \) then
\[ d_r(\mathcal{F}(x), \mathcal{F}(y)) \leq (1 - e^{-rh}) d_r(x, y), \]
i.e. the map \( \mathcal{F} \) is contractive and the existence and uniqueness of the fixed point of \( \mathcal{F} \) follows from the Banach fixed point theorem. Since the number \( h > 0 \) is arbitrary, we have proved the existence and uniqueness of the global continuous solution of the equation (2).

REMARK. In the proof we have used the norm \( ||f||_r = \max_{t \in I_h} \{e^{-rt}|f(t)|\} \), called the Bielecki norm, which enables us to prove the contractivity of the map \( \mathcal{F} \) on any compact interval \( I_h = (0, h) \), where the number \( r > 0 \) is chosen sufficiently large. We found this trick in the very well written textbook [2] by L. Gorniewicz and R. J. Ingarden, where it is applied in the proof of the Picard existence theorem for the initial valued problem of ordinary differential equations.

3. Convergence result

The following convergence theorem is related to the result by B. G. Pachpatte [10, Theorem 3] concerning the equation (1).

**Theorem 2.** Let \( g_i(t), A_i(t) \) and \( F_i(t, u) \) be continuous, real-valued functions on \( R_+ = (0, \infty) \) and on \( R_+ \times R \), respectively. Assume that

\[ |g_i(t)| \leq c_i e^{-rt}, \quad t \geq 0, \tag{3} \]
\[ |A_i(t - s)| \leq \alpha_i(s) e^{-r(t-s)}, \quad 0 \leq s \leq t, \tag{4} \]
where \( c_i \geq 0 \) are constants, \( \alpha_i : R_+ \to R_+ \) are continuous, nonnegative functions \((i = 1, 2, \ldots, p)\) and \( r > 0 \). Let the functions \( F_1, F_2, \ldots, F_p \) satisfy the condition (b) of Theorem 1. If \( x : R_+ \to R \) is a continuous solution of the equation (2) defined on \( R_+ \), then there exists a constant \( T > 0 \) such that
\[ |x(t)| \leq Te^{-rt}, \quad t \geq 0, \]
provided
\[ L = (mp - 1)(2p^{-1}c^p)^{mp-1} 2^{p-1} L_1 L_2 < 1, \quad \text{if } mp > 1, \]
\[ L_1 L_2 < \infty, \quad \text{if } mp \leq 1, \]
where
\[ c = \max\{c_1, c_2, \ldots, c_p\}, \quad L_1 = \left( \int_0^\infty \sqrt{\beta(s)^q} ds \right)^\frac{q}{q'}, \quad L_2 = \int_0^\infty \sqrt{\beta(s)^p} ds, \]

\[ \alpha(t) = \max\{\alpha_1(t), \alpha_2(t), \ldots, \alpha_p(t)\}, \quad \beta(t) = \alpha(t)e^{r(1-m)t} \]
and \( q > 1 \) is such that \( \frac{1}{p} + \frac{1}{q} = 1. \)

**Lemma.** Let \( y(t), f_1(t), f_2(t), \ldots, f_p(t) \) be real-valued nonnegative, continuous functions defined on \( \mathbb{R}_+ \), \( c_i, i = 1, 2, \ldots, p \) be nonnegative constants, \( m > 0 \) and
\[
y(t) \leq \left( c_1 + \int_0^t f_1(s)y(s)^m ds \right) \cdots \left( c_p + \int_0^t f_p(s)y(s)^m ds \right), \quad t \geq 0.
\]
Assume that
\[
H = (mp - 1)(2^{p-1}c^p)^{mp-1}2^{p-1}H_1H_2 < 1, \quad \text{if } mp > 1
\]
and \( H_1H_2 < \infty, \) if \( mp \leq 1, \) where
\[
H_1 = \left( \int_0^\infty \sqrt{f(s)^q} ds \right)^\frac{q}{q'} \quad \text{and} \quad H_2 = \int_0^\infty \sqrt{f(s)^p} ds,
\]
\[ c = \max\{c_1, c_2, \ldots, c_p\}, \quad f(t) = \max\{f_1(t), f_2(t), \ldots, f_p(t)\}\]
and \( q > 1 \) is such that \( \frac{1}{p} + \frac{1}{q} = 1. \) Then there exists a constant \( R > 0 \) such that
\[ y(t) \leq R, \quad t \geq 0. \]

**Proof.** Obviously the inequality (5) yields
\[ y(t) \leq \left( c + \int_0^t f(s)y(s)^m ds \right)^p, \quad t \geq 0, \]
where \( c, f \) are as in Lemma. Applying the Hölder inequality we obtain
\[
y(t) \leq \left[ c + \left( \int_0^t \sqrt{f(s)^q} ds \right)^\frac{1}{q'} \left( \int_0^t \sqrt{f(s)^p y(s)^{pm}} ds \right)^\frac{1}{p'} \right]^p,
\]
where \( q > 1 \) is as in Lemma. Since the elementary inequality \((a + b)^p \leq 2^{p-1}(a^p + b^p)\) holds for all \( a \geq 0, b \geq 0, \) we obtain
\[
y(t) \leq 2^{p-1}c^p + 2^{p-1} \left( \int_0^t \sqrt{f(s)^q} ds \right)^\frac{q}{q'} \int_0^t \sqrt{f(s)^p y(s)^{pm}} ds
\]
\[ \leq 2^{p-1}c^p + 2^{p-1} \left( \int_0^\infty \sqrt{f(s)^q} ds \right)^\frac{q}{q'} \int_0^t \sqrt{f(s)^p y(s)^{pm}} ds. \]
By the Bihari lemma (see [1], or [4, Theorem 1.3.8]) in the case \( mp \neq 1 \) and by the Gronwall lemma in the case \( mp = 1 \)
\[
y(t) \leq R, \ t \in R_+,
\]
where
\[
R = \frac{2^{p-1}c^p}{(1 - H)^{mp - 1}}, \text{ if } mp > 1,
\]
\[
R = 2^{p-1}c^p \exp \left[ 2^{p-1}H_1H_2 \right], \text{ if } mp = 1,
\]
\[
R = \left[ (2^{p-1}c^p)^{1-mp} + (1 - mp)2^{p-1}2^{p-1}H_1H_2 \right]^{1-mp}, \text{ if } mp < 1,
\]
and \( H, H_1, H_2 \) are defined in Lemma.

**Proof of Theorem 2.** The conditions (3), (4) and the equation (2) yield
\[
|x(t)| \leq \left( c_1 e^{-rt} + e^{-rt} \int_0^t Q\alpha_1(s)e^{r_s}|x(s)|^m ds \right) \ldots
\]
\[
\left( c_p e^{-rt} + e^{-rt} \int_0^t Q\alpha_p(s)e^{r_s}|x(s)|^m ds \right).
\]
From this inequality it follows that
\[
v(t) \leq \left( c + \int_0^t Q\beta(s)v(s)^m \right)^p,
\]
where \( \beta \) and \( c \) are as in Theorem 1 and \( v(t) = |x(t)|e^{rt} \). By Lemma there exists a constant \( T > 0 \) such that \( v(t) \leq T \) for all \( t \in (0, \infty) \), i.e. \( |x(t)| \leq Te^{-rt}, t \geq 0 \).

**References**


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