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WELL-POSEDNESS AND POROSITY
OF A CERTAIN CLASS OF OPERATORS

Abstract. We prove that several fixed point problems are well-posed and study the porosity behaviour of a certain class of operators.

1. Introduction

The notion of well-posedness of a fixed point problem has evoked much interest to several mathematicians (see for example [2], [9]). In Section 2 we prove that various fixed point problems are well-posed. Several authors ([7], [8]) determined the category and porosity position of the class of contraction mappings in relation to the class of non-expansive mappings. In Section 3 we study a similar problem for a class of operators whose condition arises from Ćirić’s idea of quasi-contraction [1], which is more general than the contraction concept.

2. Well-posedness

The definition of well-posedness of a fixed point problem runs as follows.

DEFINITION 1 [2]. Let $(X, d)$ be a metric space and $f : X \rightarrow X$ be a mapping. The fixed point problem of $f$ is said to be well-posed if

(i) $f$ has a unique fixed point $x_0 \in X$,

(ii) for any sequence $\{x_n\}$ in $X$ with $d(x_n, f(x_n)) \rightarrow 0$ as $n \rightarrow \infty$ we have $d(x_0, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

Recently the well-posedness of the fixed point problem for certain type of mappings have been investigated in [9].

Unless otherwise stated, in this section $(X, d) = X$ will stand for a metric space.

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DEFINITION 2 [1]. A mapping $T : X \rightarrow X$ is called a quasi-contraction if there is a $c$, $0 < c < 1$ such that for any $x, y \in X$,
\[
d(Tx, Ty) \leq c \cdot \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}.
\]

DEFINITION 3 [1]. Let $T : X \rightarrow X$ be a mapping. $X$ is said to be $T$-orbitally complete if every Cauchy sequence which is contained in \{x, Tx, T^2x, \ldots\} for some $x \in X$ converges in $X$.

Clearly a complete metric space is $T$-orbitally complete but the converse is not true.

EXAMPLE. If $X = [0, 1)$ with the usual metric and $Tx = x/10$ for $x \in X$ then $X$ is $T$-orbitally complete but not complete.

We now recall the following theorem.

THEOREM 1 [1]. If $T : X \rightarrow X$ is a quasi-contraction and $X$ is $T$-orbitally complete then $T$ has a unique fixed point in $X$.

We prove now the following theorem.

THEOREM 2. The fixed point problem of a quasi-contraction map on a $T$-orbitally complete metric space $X$ is well-posed.

Proof. Let $T : X \rightarrow X$ be a quasi-contraction. By Theorem 1, $T$ has a unique fixed point say $x_0$ i.e. $Tx_0 = x_0$. Let $\{x_n\}$ be a sequence in $X$ such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. Now
\[
d(x_0, x_n) \leq d(x_0, Tx_n) + d(Tx_n, x_n)
\]
\[
= d(x_n, Tx_n) + d(Tx_0, Tx_n)
\]
\[
\leq d(x_n, Tx_n) + c \cdot \max\{d(x_n, Tx_n), d(x_0, Tx_0), d(x_n, Tx_0), d(Tx_n, x_0), d(x_n, x_0)\}
\]
\[
= d(x_n, Tx_n) + c \cdot \max\{d(x_0, Tx_n), d(x_n, x_0)\}, \text{ since } Tx_0 = x_0,
\]
\[
\leq d(x_n, Tx_n) + c \cdot d(x_n, x_0) + d(x_n, x_0)
\]
\[
= (1 + c)d(x_n, Tx_n) + c \cdot d(x_0, x_n).
\]

Therefore $d(x_0, x_n) \leq \frac{(1+c)}{(1-c)}d(x_0, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. This proves the theorem.

Fisher [3] defined quasi-contraction map in some other way. According to him a mapping $T : X \rightarrow X$ is a quasi-contraction if for any positive integers $p, q$ and for all $x, y \in X$,
\[
d(T^p x, T^q y) \leq c \cdot \max\{d(T^r x, T^s y), d(T^r x, T^{r' x}), d(T^s y, T^{s' y})\}
\]
where $0 < c < 1$ and whenever $0 \leq r, r' \leq p, 0 \leq s, s' \leq q$.

Fisher proved the following fixed point theorem.
THEOREM 3 [3]. If \( X \) is a complete metric space and \( T : X \to X \) is a quasi-contraction which is also continuous then \( T \) has a unique fixed point in \( X \).

The proof of the following theorem is omitted.

THEOREM 4. The fixed point problem of a continuous quasi-contraction map (due to Fisher) on a complete metric space is well-posed.

Let \( E \) be a Banach space and \( K \) be a bounded closed convex subset of \( E \) where \( \text{diam}(K) \) stands for the diameter of \( K \). A mapping \( T : K \to K \) is called contractive if there exists a decreasing function \( \phi^T : [0, \text{diam}(K)] \to [0,1] \) such that \( \phi^T(t) < 1 \) for all \( t \in (0, \text{diam}(K)] \) and \( \|Tx - Ty\| \leq \phi^T(\|x - y\|)\|x - y\| \) for all \( x, y \in K \).

The notion of contractive mappings have been studied by many authors (see for example [7]-[9] where other references can be found). In particular, it has been proved by Rakotch [6] that in a complete metric space a contractive mapping has always a unique fixed point.

The following definition of quasi-contractive mapping in a metric space is motivated from the above considerations.

Let \( X \) be bounded and let \( \alpha : [0, \text{diam}(X)] \to [0,1] \) be a non-increasing map such that \( \alpha(t) < 1 \) for \( t \neq 0 \). Then a mapping \( T : X \to X \) is called quasi-contractive if for any \( x, y \in X \)
\[
d(Tx, Ty) \leq \alpha(M(x, y))M(x, y)
\]
where \( M(x, y) = \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\} \).

The notation \( M(x, y) \) arises from Pant [5]. It remains an open problem if any quasi-contractive map in a (complete) metric space has a unique fixed point. But if it has so, the following theorem is then true.

THEOREM 5. If a quasi-contractive map \( T \) on \( X \) has a unique fixed point then the fixed point problem of \( T \) is well-posed.

Proof. Let \( x_0 \) be the fixed point of \( T \). Let \( \{x_n\} \) be a sequence in \( X \) with \( d(x_n, Tx_n) \to 0 \) as \( n \to \infty \).

If possible suppose \( d(x_0, x_n) \) does not tend to zero as \( n \to \infty \). Then there is an \( \epsilon > 0 \) such that for any \( n \in \mathbb{N} \), there is \( m > n \) with \( d(x_0, x_m) > \epsilon \). Now
\[
d(x_0, x_m) \leq d(x_0, Tx_m) + d(Tx_m, x_m)
\]
\[
= d(Tx_m, x_m) + d(Tx_0, Tx_m)
\]
\[
\leq d(x_m, Tx_m) + \alpha(M(x_0, x_m))M(x_0, x_m)
\]
\[
= d(x_m, Tx_m) + \alpha(M(x_0, x_m)) \max\{d(x_0, x_m), d(x_m, Tx_m)\}.
\]
Now $\alpha(M(x_0, x_m)) \max\{d(x_0, x_m), d(x_m, Tx_m)\}$
\[= \alpha(d(x_0, x_m))d(x_0, x_m) \quad \text{if} \quad d(x_0, x_m) \geq d(x_m, Tx_m)\]
and $= \alpha(d(x_m, Tx_m))d(x_m, Tx_m) \quad \text{if} \quad d(x_m, Tx_m) \geq d(x_0, x_m)$
\[\leq \alpha(d(x_0, x_m))d(x_m, Tx_m), \quad \text{since} \quad \alpha \text{ is non-increasing.}\]

In either case we have
\[\alpha(M(x_0, x_m)) \max\{d(x_0, x_m), d(x_m, Tx_m)\}\]
\[\leq \alpha(d(x_0, x_m))[d(x_0, x_m) + d(x_m, Tx_m)]\]
\[\leq \alpha(\varepsilon)[d(x_0, x_m) + d(x_m, Tx_m)] \quad \text{since} \quad d(x_0, x_m) > \varepsilon.\]

Therefore from above
\[d(x_0, x_m) \leq d(x_m, Tx_m) + \alpha(\varepsilon)[d(x_0, x_m) + d(x_m, Tx_m)].\]

In other words
\[d(x_m, Tx_m) \geq \frac{1 - \alpha(\varepsilon)}{1 + \alpha(\varepsilon)}d(x_0, x_m) > \frac{1 - \alpha(\varepsilon)}{1 + \alpha(\varepsilon)}\varepsilon,\]
which is against $d(x_n, Tx_n) \to 0$ as $n \to \infty$. This proves the theorem.

Let $T : X \to X$ be a mapping such that there is a $\beta$, $0 < \beta < 1/2$ such that for all $x, y \in X$
\[d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)].\]

Kannan [4] proved that if $X$ is complete then $T$ has a unique fixed point. Sometimes the above map is referred to as a Kannan map.

**THEOREM 6.** The fixed point problem for Kannan's map in a complete metric space $X$ is well-posed.

**Proof.** Let $x_0$ be the unique fixed point of $T$ i.e. $Tx_0 = x_0$. Let $\{x_n\}$ be a sequence in $X$ such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Then
\[d(x_0, x_n) \leq d(x_0, Tx_n) + d(Tx_n, x_n)\]
\[= d(x_n, Tx_n) + d(Tx_0, Tx_n)\]
\[\leq d(x_n, Tx_n) + \beta[d(x_n, Tx_n) + d(x_0, Tx_0)]\]
\[= (1 + \beta)d(x_n, Tx_n) \to 0 \quad \text{as} \quad n \to \infty.\]

This proves the theorem.

3. Porosity of certain sets of mappings

In this section we study the behaviour of certain sets of mappings from the viewpoint of porosity of sets. The definitions of porosity of sets etc. those follow are borrowed from \{[10]–[13]\}. Let $(Y, \rho)$ be a metric space. If
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Let \( y \in Y \) and \( r > 0 \) we denote by \( B(y, r) \) the ball with centre \( y \) and radius \( r \). Let \( M \subseteq Y \). Let

\[
\nu(y, r, M) = \sup\{ t > 0; \text{ there is a } z \in B(y, r) \text{ such that } B(z, t) \subseteq B(y, r) \text{ and } M \cap B(z, t) = \emptyset \}.
\]

Note that if \( M \) is dense in \( Y \) then \( \nu(y, r, M) = -\infty \). Let

\[
\begin{align*}
\bar{p}(y, M) &= \limsup_{r \to 0^+} \frac{\nu(y, r, M)}{r}, \\
p(y, M) &= \liminf_{r \to 0^+} \frac{\nu(y, r, M)}{r},
\end{align*}
\]

and if \( \bar{p}(y, M) = p(y, M) \) then we set

\[
p(y, M) = \bar{p}(y, M) = p(y, M) = \lim_{r \to 0^+} \frac{\nu(y, r, M)}{r}.
\]

The set \( M \) is said to be porous at \( y \in Y \) if \( \bar{p}(y, M) > 0 \) and \( \sigma \)-porous at \( y \in Y \) provided \( M = \bigcup_{n=1}^{\infty} M_n \) and each of the sets \( M_n \) is porous at \( y \). \( M \) is called porous or \( \sigma \)-porous in \( Y_0 \subseteq Y \) if it is so at each \( y \in Y_0 \).

The set \( M \) is said to be very porous at \( y \in Y \) if \( p(y, M) > 0 \) and very strongly porous at \( y \in Y \) if \( p(y, M) = 1 \). \( M \) is very (strongly) porous in \( Y_0 \subseteq Y \) if it is so at each \( y \in Y_0 \).

Also \( M \) is said to be uniformly very porous in \( Y_0 \subseteq Y \) if there is a \( c > 0 \) such that for each \( y \in Y_0 \) we have \( p(y, M) \geq c \).

\( M \) is said to be \( \sigma \)-very porous in \( Y_0 \subseteq Y \) if \( M = \bigcup_{n=1}^{\infty} M_n \) and each \( M_n \) is very porous at each \( y \in Y_0 \).

Again \( M \) is said to be uniformly \( \sigma \)-very strongly porous in \( Y_0 \subseteq Y \) if \( M = \bigcup_{n=1}^{\infty} M_n \) and for each \( y \in Y_0 \) and for each \( n = 1, 2, \ldots \) we have \( p(y, M_n) = 1 \).

The following lemma will be needed.

**Lemma 1** [10]. Let \((Y, \rho)\) be a metric space. Let \( M \subseteq Y \), \( M \) be an \( F_\sigma \)-set in \( Y \). Then \( M \) is uniformly \( \sigma \)-very strongly porous in \( Y/M \).

Let \( E \) be a Banach space and as before let \( K \) be a closed bounded convex subset of \( E \). Denote by \( A \) the set of all mappings \( T : K \to K \) such that

\[
\|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|\} = m_T(x, y),
\]

say for all \( x, y \in K \). We equip \( A \) with the metric \( h \) defined by \( h(S, T) = \sup\{\|Sx - Tx\|; x \in K\} \) for \( S, T \in A \). Then \((A, h)\) is clearly a complete metric space and so is of second category.

Let \( B \) be the collection of all those \( T \in A \) such that

\[
\|Tx - Ty\| \leq c(T) m_T(x, y)
\]
for all \(x, y \in K\) where \(0 < c(T) < 1\) and \(c(T)\) is a constant depending on \(T\) only. Thus \(B\) is actually a subclass of the collection of all quasi-contraction mappings \(T : K \to K\) and hence by Čirić's theorem [1] each \(T \in B\) has a unique fixed point in \(K\). Moreover \(B\) is clearly a proper subclass of \(A\).

We observe in the following theorem that \(B\) is sparse in \(A/B\). The following lemma is in this direction.

**Lemma 2.** \(B\) is a \(F_r\)-set in \((A,h)\).

**Proof.** Clearly \(B = \bigcup_{r \in \Delta} B_r\) where \(\Delta\) is an enumeration of the set of all rationals in \((0,1)\) and

\[B_r = \{T \in A; \|Tx - Ty\| \leq r.m_T(x,y) \text{ for all } x, y \in K\}.

We prove that for a fixed \(r \in \Delta\), \(B_r\) is closed. Let \(T_n \to T\) as \(n \to \infty\), where \(T_n \in B_r\) for all \(n\). Now

\[
\|Tx - Ty\| \leq \|Tx - T_n x\| + \|T_n x - T_n y\| + \|T_n y - Ty\| \\
\leq \|Tx - T_n x\| + \|T_n y - Ty\| + r.m_T(x,y) \\
\leq \|Tx - T_n x\| + \|T_n y - Ty\| \\
+ r.\max\{\|x - T_n x\|, \|T_n x - T_n y\|, \|y - Ty\|, \|Ty - T_n y\|\} \\
\leq r.\max\{\|x - T_n x\|, \|y - Ty\|\} \\
+ (1 + r)(\|Tx - T_n x\| + \|Ty - T_n y\|).
\]

Since \(T_n \to T\) as \(n \to \infty\), this shows that \(\|Tx - Ty\| \leq r.m_T(x,y)\) i.e. \(T \in B_r\). Hence \(B_r\) is closed and so \(B\) is \(F_r\).

Combining Lemma 2 with Lemma 1 we obtain

**Theorem 7.** The set \(B\) is uniformly \(\sigma\)-very strongly porous in \(A/B\).

The porosity character of \(B\) or \(A/B\) in \((A,h)\) is not clear. However we can prove the following theorem.

For \(\delta > 0\), let \(B_\delta\) be the collection of all \(T \in A\) for which there exists a constant \(c(T), 0 < c(T) < 1\) such that

\[(A) \quad \|Tx - Ty\| \leq c(T).m_T(x,y) + \delta \quad \text{for all } x, y \in K . . . .
\]

Then \(B \subset B_\delta \subset A\).

As a corollary to the next theorem, we observe that most (in the sense of category) of the mappings of \(A\) are of the form \((A)\).

**Theorem 8.** For any \(\delta > 0\), \(A/B_\delta\) is uniformly very porous in \((A,h)\).

**Proof.** Let \(T \in A\) and \(r \in (0,1]\). Fix \(\theta \in K\). Set

\[\alpha = r.(\text{diam}(K) + 1)^{-1}.2^{-m}\]
where \( m \in \mathbb{N} \) is chosen so that \( 2^{-m} < \delta/2 \). Now define \( T_\alpha \) by \( T_\alpha x = (1 - \alpha)Tx + \alpha \theta \) for all \( x \in K \). Then \( T_\alpha : K \to K \) and for all \( x, y \in K \),
\[
\| T_\alpha x - T_\alpha y \| = (1 - \alpha)\| Tx - Ty \| \leq (1 - \alpha)m_T(x, y).
\]

But
\[
\| x - Tx \| \leq \| x - T_\alpha x \| + \| T_\alpha x - Tx \|
\leq \| x - T_\alpha x \| + \alpha \| Tx - \theta \|
\leq \| x - T_\alpha x \| + \alpha \text{diam}(K).
\]

Similarly \( \| y - Ty \| \leq \| y - T_\alpha y \| + \alpha \text{diam}(K) \).

Therefore \( m_T(x, y) \leq m_{T_\alpha}(x, y) + \alpha \text{diam}(K) \). Thus
\[
\| T_\alpha x - T_\alpha y \| \leq (1 - \alpha)m_{T_\alpha}(x, y) + \alpha (1 - \alpha)\text{diam}(K)
\leq (1 - \alpha)m_{T_\alpha}(x, y) + \alpha \text{diam}(K) \leq (1 - \alpha)m_T(x, y) + \delta/2.
\]

Since \( 0 < 1 - \alpha < 1, T_\alpha \in B_\delta \). Furthermore
\[
h(T, T_\alpha) < \alpha \text{diam}(K) < r/2.
\]

Now let \( k > 1 \) be a positive integer so chosen that \( 2^{-k} < \delta/6 \) and \( S \in A \) be such that \( h(T_\alpha, S) < \beta r \) where \( \beta = 2^{-k} < \delta/6 \). Since for any \( x \in K \),
\[
\| x - T_\alpha x \| \leq \| x - Sx \| + \| Sx - T_\alpha x \| \leq \| x - Sx \| + \beta r
\]
and \( \| y - T_\alpha y \| \leq \| y - Sy \| + \beta r \), we have \( m_{T_\alpha}(x, y) \leq m_S(x, y) + \beta r \). Hence
\[
\| Sx - Sy \| \leq \| Sx - T_\alpha x \| + \| T_\alpha x - T_\alpha y \| + \| T_\alpha y - Sy \|
\leq (1 - \alpha)m_{T_\alpha}(x, y) + 2\beta r + \delta/2
\leq (1 - \alpha)m_S(x, y) + (3 - \alpha)\beta r + \delta/2
\leq (1 - \alpha)m_S(x, y) + \delta.
\]

This shows that \( \{ S \in A ; h(T_\alpha, S) \leq \beta r \} \subset B_\delta \). Also we have
\[
h(S, T) \leq h(S, T_\alpha) + h(T_\alpha, T) \leq r/2 + \beta r < r.
\]

Therefore \( \{ S \in A ; h(S, T_\alpha) \leq \beta r \} \subset \{ S \in A ; h(S, T) \leq r \} \). This proves that \( A/B_\delta \) is uniformly very porous in \( (A, h) \). The proof is complete.

Since any uniformly very porous set is porous and clearly any porous set is nowhere dense, we obtain the following corollary.

**COROLLARY.** The set \( A/B_\delta \) is nowhere dense in \( (A, h) \).

Let \( \{ \delta_n \} \) be a decreasing sequence of positive numbers tending to zero. Let \( C = \bigcap_{n=1}^{\infty} B_{\delta_n} \). Then \( B \subset C \) and we obtain the following theorem from Theorem 8 and the above Corollary.

**THEOREM 9.** There exists a set \( C \) containing \( B \) which is residual and such that \( A/C \) is \( \sigma \)-very porous in \( (A, h) \).
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